

Question 1

Three problems use the interval $0 < x < \pi$ with zero values at $x = 0, \pi$:

$$H_t = \frac{1}{2}H_{xx}, \quad H(x, 0) = \sin(2x); \quad W_{tt} = 9W_{xx}, \quad W(x, 0) = \sin(2x), \quad W_t(x, 0) = 3\sin(2x).$$

The third is $P_{xx} + P_{yy} = 0$ on $0 < x < \pi$, $0 < y < 1$, with $P(x, 0) = 0$ and $P(x, 1) = \sin(2x)$, as well as the zero side values.

Tasks

1. Derive the common spatial eigenproblem using $-X'' = \lambda X$. Explain why neither $\lambda = 0$ nor $\lambda < 0$ gives a nonzero Dirichlet eigenfunction.
2. Derive the remaining ordinary differential equation for each PDE and solve all three specified problems.
3. Verify every original datum, including the wave velocity. Explain why the three remaining factors require different kinds or numbers of conditions.
4. A proposed rule says a positive spatial eigenvalue always produces an exponential decay factor. Diagnose the rule and state the correct behavior for these three problems.

Question 1 – Solution

Strategy. Keep one spatial sign convention, then let the original PDE determine whether the remaining equation is first-order decay, oscillation or a spatial boundary problem.

Step 1: Fix the spatial spectrum. The conditions are $X(0) = X(\pi) = 0$. At $\lambda = 0$, the affine solution must vanish. For $\lambda = -\sigma^2 < 0$, $X = A \sinh(\sigma x)$ after the left condition, and the right condition forces $A = 0$. For $\lambda > 0$, the nonzero possibilities are

$$\lambda_n = n^2, \quad X_n(x) = \sin(nx), \quad n = 1, 2, \dots$$

All supplied data occupy the $n = 2$ mode.

Step 2: Derive the three remaining equations. Substitution gives $a' = -\lambda a/2$ for heat, $b'' = -9\lambda b$ for the wave, and $d'' = \lambda d$ for Laplace's equation. At $\lambda = 4$, the specified solutions are

$$H = e^{-2t} \sin(2x), \quad W = (\cos(6t) + \tfrac{1}{2} \sin(6t)) \sin(2x), \quad P = \frac{\sinh(2y)}{\sinh 2} \sin(2x).$$

Step 3: Check the original conditions. Each spatial sine vanishes at both ends. At $t = 0$, the two time-dependent fields have displacement $\sin(2x)$, while $W_t(x, 0) = 6(\frac{1}{2}) \sin(2x) = 3 \sin(2x)$. The harmonic field is zero at $y = 0$ and equals $\sin(2x)$ at $y = 1$. Direct second derivatives verify all three PDEs. The heat factor needs one initial value; the wave factor needs position and velocity; the harmonic factor uses two values on different spatial edges.

Step 4: Correct the proposed rule. A positive eigenvalue produces heat decay $e^{-\lambda t/2}$, wave frequency $3\sqrt{\lambda}$, and hyperbolic spatial factors for the harmonic extension. The last factor increases with y because it connects a zero bottom trace to a nonzero top trace; y is not time. The sign of the spatial eigenvalue alone does not identify the remaining factor without the PDE and its data.

Question 2

Impose insulated side conditions: $H_x = W_x = P_x = 0$ at $x = 0, \pi$. Consider

$$H_t = H_{xx}, \quad H(x, 0) = 2 + \cos x; \quad W_{tt} = W_{xx}, \quad W(x, 0) = 2 + \cos x, \quad W_t(x, 0) = 3 + 2 \cos x.$$

On $0 < x < \pi$, $0 < y < 1$, let P be harmonic with $P(x, 0) = 0$ and $P(x, 1) = 2 + \cos x$.

Tasks

1. Derive the Neumann eigenfunctions, including the zero eigenvalue. State the norm of the constant eigenfunction and of each positive cosine mode.
2. Construct all three solutions, treating their zero modes explicitly.
3. Compute the horizontal mean in each problem and explain exactly which data would be lost if the zero mode were omitted.
4. Compute the wave energy $\frac{1}{2} \int_0^\pi (W_t^2 + W_x^2) dx$. Explain why a constant energy does not imply bounded wave displacement in this setting.

Question 2 – Solution

Strategy. The spatial constant is an actual eigenfunction for insulated sides; its remaining equation changes with the type of PDE.

Step 1: Retain the zero eigenvalue. The problem $-X'' = \lambda X$, $X'(0) = X'(\pi) = 0$ has $X_0 = 1$, $\lambda_0 = 0$, and $X_n = \cos(nx)$, $\lambda_n = n^2$ for $n \geq 1$. Their squared $L^2(0, \pi)$ norms are π and $\pi/2$, respectively. The mean coefficient therefore uses $1/\pi$, not the positive-mode factor $2/\pi$.

Step 2: Solve the zero and first-mode equations. For $\lambda = 0$, the remaining equations are $a' = 0$, $b'' = 0$ and $d'' = 0$. Using all the supplied data gives

$$\begin{aligned} H(x, t) &= 2 + e^{-t} \cos x, \\ W(x, t) &= 2 + 3t + (\cos t + 2 \sin t) \cos x, \\ P(x, y) &= 2y + \frac{\sinh y}{\sinh 1} \cos x. \end{aligned}$$

The cosine derivatives vanish at the side boundaries. The scalar factors verify the PDEs, both wave initial data and the two harmonic edge traces.

Step 3: Compare the three means. Since $\int_0^\pi \cos x \, dx = 0$, their horizontal means are

$$\overline{H} = 2, \quad \overline{W} = 2 + 3t, \quad \overline{P} = 2y.$$

Omitting the constant spatial eigenfunction would lose the heat initial mean, both the wave initial mean and mean velocity, and the harmonic top mean. The zero mode preserves a heat mean, permits a freely drifting wave mean, and linearly joins two spatial mean values in the harmonic problem.

Step 4: Separate drift energy from modal energy. Put $q = \cos t + 2 \sin t$. Orthogonality gives

$$E = \frac{1}{2} \left[9\pi + \frac{\pi}{2} (q'^2 + q^2) \right] = \frac{9\pi}{2} + \frac{5\pi}{4} = \boxed{\frac{23\pi}{4}}.$$

The spatially constant displacement $2 + 3t$ contributes no elastic energy, although its velocity contributes a fixed kinetic energy. Thus W becomes unbounded through its mean while its energy remains constant. A displacement bound from this energy alone would require control of that mean.

Question 3

Consider the separated spatial problem

$$-X'' = \lambda X, \quad 0 < x < L, \quad X(0) = 0, \quad X'(L) + hX(L) = 0,$$

where $L > 0$ and $h \in \mathbb{R}$. The same boundary conditions are imposed on the heat equation $u_t = \kappa u_{xx}$ and the wave equation $v_{tt} = c^2 v_{xx}$, with $\kappa, c > 0$.

Tasks

1. Derive the energy identity for λ . Use $X(L)^2 \leq L \int_0^L X'^2 dx$ to identify when every eigenvalue is positive and when a zero eigenvalue occurs.
2. For $h < -1/L$, derive the negative-eigenvalue equation and prove there is exactly one negative eigenvalue.
3. Choose $L = 1$ and $h = -\coth 1$. Construct a heat solution and a wave solution with initial profile $\sinh x$, taking zero initial wave velocity.
4. Explain why these solutions invalidate a rule that separated heat modes always decay and separated wave modes always oscillate. Identify the boundary term responsible.

Question 3 – Solution

Strategy. The admissible spatial spectrum depends on the boundary operator, including the sign of a Robin coefficient.

Step 1: Include the Robin boundary term. Multiplication by X and integration by parts give

$$\lambda \int_0^L X^2 dx = \int_0^L X'^2 dx + hX(L)^2.$$

For $h \geq 0$ the right side is positive for nonzero X with $X(0) = 0$. For $-1/L < h < 0$, the supplied trace inequality bounds it below by $(1 + hL) \int X'^2 > 0$. At $h = -1/L$ it is nonnegative, with equality precisely for an affine function through zero; $X = x$ is the zero mode. Conversely, solving $X'' = 0$ shows a nonzero zero mode occurs only at $h = -1/L$.

Step 2: Find the negative mode. For $\lambda = -\sigma^2$, $\sigma > 0$, the left condition gives $X = A \sinh(\sigma x)$. The right condition is

$$\sigma \coth(\sigma L) = -h.$$

The left side increases strictly from $1/L$ to infinity: with $z = \sigma L$, the derivative of $z \coth z$ is $(\sinh z \cosh z - z)/\sinh^2 z > 0$. Thus there is exactly one positive σ when $-h > 1/L$, and none otherwise. This counts all negative eigenvalues because every such eigenfunction must have the displayed hyperbolic form.

Step 3: Build the two growing examples. For the specified L, h , $\sigma = 1$ and $\lambda = -1$. Hence

$$u(x, t) = e^{\kappa t} \sinh x, \quad v(x, t) = \cosh(ct) \sinh x.$$

Their spatial second derivatives equal the fields; their time derivatives satisfy the respective PDEs. The left traces are zero, and at $x = 1$ the Robin expression is proportional to $\cosh 1 - \coth 1 \sinh 1 = 0$. The initial profiles are $\sinh x$ and the initial wave velocity is zero.

Step 4: Locate the failure of the rule. Heat factors satisfy $a' = -\kappa \lambda a$; wave factors satisfy $b'' + c^2 \lambda b = 0$. A negative eigenvalue therefore gives growth rather than decay or pure oscillation. In the spatial energy identity, $hX(L)^2$ is negative here. The usual nonnegative energy form for dissipative or fixed boundaries cannot be assumed for this Robin condition. The counterexample changes the spectrum, not the separation calculation.

Question 4

On $0 < x < 1$, compare $H_t = H_{xx}$ and $W_{tt} = W_{xx}$ with the moving boundary values

$$H(0, t) = W(0, t) = t, \quad H(1, t) = W(1, t) = 2t.$$

Take $H(x, 0) = W(x, 0) = 0$ and $W_t(x, 0) = 1 + x$.

Tasks

1. For general boundary values $A(t), B(t)$, derive the transformed PDE and initial data after subtracting $\ell(x, t) = (1 - x)A(t) + xB(t)$ in each problem.
2. For the specified data, decide whether the affine lifting itself solves each original PDE.
3. Construct the complete heat solution using a stationary correction to its transformed forcing and a sine expansion. Derive the required coefficients.
4. Verify the reconstructed original data and find the long-time heat offset relative to the moving line. Contrast it with the wave field and state the heat release-corner regularity caveat.

Question 4 – Solution

Strategy. Homogenizing boundary values creates a source involving the first or second time derivative of the lifting, depending on the PDE.

Step 1: Transform the PDE and every initial datum. Since $\ell_{xx} = 0$, the zero-endpoint fields $v = H - \ell$ and $z = W - \ell$ obey

$$v_t - v_{xx} = -\ell_t, \quad z_{tt} - z_{xx} = -\ell_{tt}.$$

Their initial displacements are the respective original profiles minus $\ell(x, 0)$; the wave initial velocity is the original velocity minus $\ell_t(x, 0)$. Transforming only the endpoints would omit essential source and initial terms.

Step 2: Test the actual moving line. Here $\ell = t(1 + x)$, so $\ell_t = 1 + x$ but $\ell_{tt} = \ell_{xx} = 0$. It fails the heat PDE but solves the wave PDE. The wave's transformed displacement and velocity are both zero, giving

$$W(x, t) = t(1 + x).$$

The heat correction has $v_t - v_{xx} = -(1 + x)$ and $v(x, 0) = 0$.

Step 3: Solve the forced heat correction. Seek $P'' = 1 + x$ with $P(0) = P(1) = 0$. Integration gives

$$P(x) = \frac{x^2}{2} + \frac{x^3}{6} - \frac{2x}{3} = \frac{x(x-1)(x+4)}{6}.$$

The sine coefficients of $1 + x$ are $a_n = 2[1 - 2(-1)^n]/(n\pi)$. Twice integrating P'' against a sine gives $p_n = -a_n/(n\pi)^2$. Thus

$$H = t(1 + x) + P(x) + \sum_{n \geq 1} \frac{2[1 - 2(-1)^n]}{(n\pi)^3} e^{-(n\pi)^2 t} \sin(n\pi x).$$

Step 4: Verify and interpret the offset. At both ends, P and every sine vanish. At $t = 0$ the summable sine coefficients reconstruct $-P$ uniformly, giving zero initial temperature. For $t > 0$, Gaussian damping permits differentiated convergence; substitution verifies $H_t = H_{xx}$. The initial flat field has $H_{xx} = 0$ while the boundary time derivatives are one and two, so joint classical corner smoothness is not asserted. Finally $H - \ell \rightarrow P < 0$ on $0 < x < 1$, uniformly on the closed interval, whereas $W - \ell = 0$ at all times. An identical boundary lifting can have different physical consequences in the two PDEs.

Question 5

On the square $0 < x, y < \pi$, let

$$U_t = U_{xx} + U_{yy}, \quad U = 0 \text{ on every edge}, \quad U(x, y, 0) = \sin x \sin(2y) + 2 \sin(2x) \sin y.$$

Tasks

1. Derive the product eigenfunctions and explain why the two active products have the same decay rate. Construct U .
2. Determine whether U is separable as $T(t)F(x, y)$ and whether its nonzero spatial profile can be written as $X(x)Y(y)$. Prove both conclusions.
3. Find the interior zero-temperature curve, its limiting edge endpoints and the signs on either side. Sketch that geometry.
4. Replace the initial coefficients by unknown A, B . Explain why one fixed-point time trace cannot distinguish them, but traces at $(\pi/2, \pi/4)$ and $(\pi/4, \pi/2)$ can. Give exact recovery formulas.

Question 5 – Solution

Strategy. A repeated eigenvalue allows a common time factor without forcing the spatial combination to be a single coordinate product.

Step 1: Identify the repeated eigenvalue. Zero edges give $\sin(mx)\sin(ny)$ with eigenvalues $m^2 + n^2$. The pairs (1, 2) and (2, 1) both give five. Therefore

$$U = e^{-5t} [\sin x \sin(2y) + 2 \sin(2x) \sin y].$$

Every edge and the initial field are correct, and applying the Laplacian multiplies the bracket by -5 .

Step 2: Distinguish two meanings of separation. The displayed formula is exactly one product $T(t)F(x, y)$. It is not one product $T(t)X(x)Y(y)$. Indeed, the spatial slice at $y = \pi/2$ is $2 \sin(2x)$, whereas at $y = \pi/4$ it is $\sin x + \sqrt{2} \sin(2x)$. These two nonzero slices are not proportional by sine orthogonality, as they would have to be for $X(x)Y(y)$.

Step 3: Determine the nodal geometry. Factor the bracket as $2 \sin x \sin y (\cos y + 2 \cos x)$. The first two factors are positive inside the square, so the interior zero set is

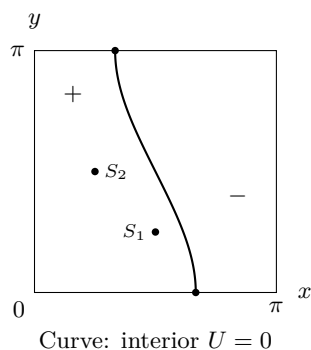
$$\cos y = -2 \cos x, \quad \pi/3 < x < 2\pi/3.$$

Its limiting endpoints are $(\pi/3, \pi)$ and $(2\pi/3, 0)$. At fixed y , the field is positive for $x < \arccos(-\cos y/2)$ and negative to its right. The zero curve is unchanged by the positive decay factor.

Step 4: Resolve the equal-rate inverse problem. At any one sensor, the entire trace has the form $e^{-5t}(A\alpha + B\beta)$ with fixed spatial factors α, β ; it supplies only one scalar equation. At the two specified sensors the unwanted product vanishes and the wanted product equals one. Hence

$$A = e^{5t}U(\pi/2, \pi/4, t), \quad B = e^{5t}U(\pi/4, \pi/2, t).$$

Distinct time samples at one point cannot separate equal decay rates; spatially independent observations can.



Question 6

Consider diffusion with variable spatial coefficient:

$$u_t = (x^2 u_x)_x, \quad 1 < x < e, \quad u(1, t) = u(e, t) = 0,$$

$$u(x, 0) = x^{-1/2} [\sin(\pi \log x) + 2 \sin(2\pi \log x)].$$

Use $s = \log x$ and $U(s, t) = u(e^s, t)$.

Tasks

1. Derive the transformed PDE and find a substitution $U = e^{-s/2}V$ that removes its first spatial derivative.
2. Derive the original spatial eigenfunctions, eigenvalues and their $L^2(1, e; dx)$ norms. Construct the complete solution.
3. Derive the coefficient formula for general initial data in these eigenfunctions. Explain why ordinary unweighted sine projection in s of $u(e^s, 0)$ is incorrect.
4. Verify the original PDE and data and derive the energy identity $\frac{1}{2}(d/dt) \int_1^e u^2 dx = - \int_1^e x^2 u_x^2 dx$. Check it on each separated eigenfunction.

Question 6 – Solution

Strategy. Changing coordinates changes the differential operator and the integration measure; a further amplitude factor restores ordinary sine orthogonality.

Step 1: Transform the operator, not just the variable. The chain rule gives $u_x = U_s/x$ and $(x^2 u_x)_x = U_{ss} + U_s$. With $U = e^{-s/2}V$, direct differentiation yields

$$V_t = V_{ss} - \frac{1}{4}V, \quad 0 < s < 1, \quad V(0, t) = V(1, t) = 0.$$

Ignoring U_s would shift every eigenvalue incorrectly.

Step 2: Recover the spectrum and field. The sine modes give

$$X_n(x) = x^{-1/2} \sin(n\pi \log x), \quad \lambda_n = (n\pi)^2 + \frac{1}{4}.$$

Since $dx = e^s ds$, the two $x^{-1/2}$ factors cancel the measure in a product:

$$\int_1^e X_m X_n dx = \int_0^1 \sin(m\pi s) \sin(n\pi s) ds.$$

Thus the squared norms are $1/2$, with zero cross products. The solution is

$$u = e^{-\lambda_1 t} X_1 + 2e^{-\lambda_2 t} X_2.$$

Step 3: Use the correct projection. For initial data f ,

$$b_n = 2 \int_1^e f(x) x^{-1/2} \sin(n\pi \log x) dx = 2 \int_0^1 e^{s/2} f(e^s) \sin(n\pi s) ds.$$

It is the transformed field $V(s, 0) = e^{s/2} f(e^s)$ that has ordinary sine coefficients. For example, the ungauged sines are not orthogonal in dx :

$$\int_1^e \sin(\pi \log x) \sin(2\pi \log x) dx = -\frac{4\pi^2(e+1)}{(1+\pi^2)(1+9\pi^2)} \neq 0.$$

Step 4: Verify diffusion and dissipation. Each X_n satisfies $-(x^2 X_n')' = \lambda_n X_n$ and vanishes at both endpoints. The temporal factors therefore verify the original PDE and exactly recover the supplied initial coefficients. Integration by parts gives $\frac{1}{2}(d/dt) \int u^2 = [u x^2 u_x]_1^e - \int x^2 u_x^2 = -\int x^2 u_x^2$. On a mode $a e^{-\lambda_n t} X_n$, both sides equal $-\lambda_n a^2 e^{-2\lambda_n t}/2$, since $\int x^2 X_n'^2 = \lambda_n \int X_n^2 = \lambda_n/2$.

Question 7

On $0 < x < \pi$, compare the heat equation $H_t = H_{xx}$ with the wave equation $W_{tt} = W_{xx}$. Both have zero endpoint displacement for positive time and the initial interior profile $f(x) = 1$; the wave has zero initial velocity. Use L^2 convergence for the initial displacement. For the wave, use the reflected bounded function obtained from the odd, 2π -periodic extension F of f , not a presumed finite-energy solution.

Tasks

1. Derive the common sine coefficients and the heat and wave series, stating the appropriate interpretation of each.
2. Use $W(x, t) = \frac{1}{2}[F(x - t) + F(x + t)]$ to determine the wave profile for $0 < t < \pi/2$, including values at its two fronts.
3. Explain the contrast with heat smoothing, and why neither field can converge uniformly to 1 over the whole open interval as $t \downarrow 0$. Sketch both profiles at $t = \pi/4$.
4. Compute the energy of the first N wave modes and show why this problem has no finite-energy wave solution with those Dirichlet initial data. Explain what remains valid in the reflected interpretation.

Question 7 – Solution

Strategy. The same initial Fourier coefficients can lead to a smooth heat field and a discontinuous wave field; convergence alone does not guarantee finite energy.

Step 1: Compute and evolve the coefficients. The coefficients are $b_n = 2(1 - (-1)^n)/(n\pi)$, so

$$H = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{e^{-n^2 t} \sin(nx)}{n}, \quad W = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt) \sin(nx)}{n}.$$

Both recover f in L^2 as $t \downarrow 0$. Heat derivatives converge uniformly for every $t \geq \tau > 0$. The wave series represents the reflected function in L^2 and takes average values at jumps; its zero initial time derivative is understood in distributions.

Step 2: Track the two reflected fronts. The extension is $F(s) = \text{sgn}(\sin s)$ away from multiples of π , with value zero at those multiples. For $0 < t < \pi/2$,

$$W(x, t) = \begin{cases} 0, & 0 < x < t \text{ or } \pi - t < x < \pi, \\ 1, & t < x < \pi - t, \\ 1/2, & x = t \text{ or } x = \pi - t. \end{cases}$$

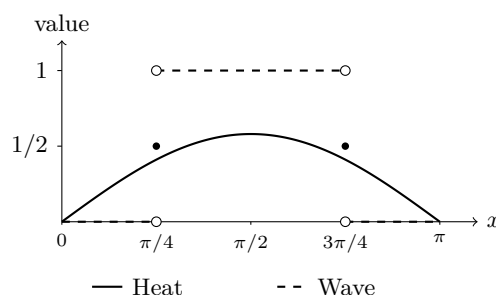
Each endpoint trace is zero. Reflection has produced two moving jumps, not a gradually smoothing profile.

Step 3: Compare smoothing and initial convergence. For $t > 0$, the heat field is smooth and $0 < H < 1$ in the interior, by diffusion and the maximum principle. The wave instead has the step profile above. For any positive time approaching zero, the wave's zero strips give sup-norm error one; the heat's continuous zero endpoint trace makes its supremum error also one over the open interval. Thus L^2 initial convergence does not imply uniform convergence across the incompatible endpoint layer.

Step 4: Test energy rather than assume it. The partial wave energy is

$$E_N = \frac{\pi}{4} \sum_{n \leq N} n^2 b_n^2 = \frac{4}{\pi} \#\{n \leq N : n \text{ odd}\} \rightarrow \infty.$$

No finite-energy Dirichlet wave has this initial profile; it is not in the zero-trace energy space. The reflected bounded field still solves the wave equation distributionally, with the stated initial displacement and zero initial velocity in the corresponding weak sense. At a moving jump, its derivatives need not be ordinary square-integrable functions.



Question 8

On $0 < x < \pi$ with zero endpoint values, consider the stationary problem

$$-u'' - 4u = f(x), \quad f(x) = \sin x + a \sin(2x) + \sin(3x), \quad a \in \mathbb{R},$$

and the associated evolution $v_t = v_{xx} + 4v + f$, initially $v(x, 0) = 0$.

Tasks

1. Derive the necessary stationary solvability condition by testing against the kernel. Show that it is sufficient for these data.
2. Find every stationary solution when one exists, and identify the one orthogonal to the kernel.
3. Solve the evolution from rest. Distinguish growth caused by a kernel forcing from growth caused by a negative eigenvalue of the stationary operator.
4. Replace the zero initial state by $b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x)$. Determine exactly when the evolution stays bounded and give its limiting stationary field.

Question 8 – Solution

Strategy. A separated coefficient equation can be singular or unstable; stationarity, uniqueness and attraction are separate questions.

Step 1: Test the stationary kernel. For $\mathcal{L} = -d^2/dx^2 - 4$, the eigenvalues on $\sin(nx)$ are $n^2 - 4$. The kernel is spanned by $\sin(2x)$. Integration by parts, using both zero endpoints, gives $0 = \int_0^\pi (\mathcal{L}u) \sin(2x) dx = \int_0^\pi f \sin(2x) dx = a\pi/2$. Thus $a = 0$ is necessary.

Step 2: Solve the compatible stationary equations. When $a = 0$, divide the first and third coefficients by $1 - 4 = -3$ and $9 - 4 = 5$:

$$u = -\frac{1}{3} \sin x + \frac{1}{5} \sin(3x) + C \sin(2x), \quad C \in \mathbb{R}.$$

Direct substitution proves sufficiency. Orthogonality to the kernel forces $C = 0$, and without it the stationary field is not unique.

Step 3: Evolve from rest. The modal equations are $q'_1 = 3q_1 + 1$, $q'_2 = a$, $q'_3 = -5q_3 + 1$. Their zero-initial-value solutions give

$$v = \frac{e^{3t} - 1}{3} \sin x + at \sin(2x) + \frac{1 - e^{-5t}}{5} \sin(3x).$$

Nonzero a drives linear growth in the kernel mode. Separately, the first mode grows exponentially because $\mathcal{L}$ has eigenvalue -3 . Therefore even compatible forcing does not make the stationary family an attractor from rest.

Step 4: Identify the bounded initial states. For the prescribed three-mode initial state,

$$q_1 = -\frac{1}{3} + (b_1 + \frac{1}{3})e^{3t}, \quad q_2 = b_2 + at, \quad q_3 = \frac{1}{5} + (b_3 - \frac{1}{5})e^{-5t}.$$

Orthogonality prevents cancellation between spatial modes. Boundedness holds exactly when

$$a = 0, \quad b_1 = -1/3.$$

The coefficients b_2, b_3 are unrestricted. Then the uniform limit is $-\frac{1}{3} \sin x + b_2 \sin(2x) + \frac{1}{5} \sin(3x)$. Eliminating the unstable component and satisfying the kernel compatibility are both necessary.

Question 9

Let $u_t = u_{xx}$ on $0 < x < \pi$, with zero endpoint values and $u(x, 0) = \sin x + \sin(2x)$. A proposed solution strategy assumes that the complete answer is one product $X(x)T(t)$ because the PDE is separable.

Tasks

1. Derive the exact solution and explain why separability of the PDE does not force every solution to be a single product.
2. Prove that a nonzero product solution must use a spatial eigenfunction. Explain how the argument handles zeros of X or T .
3. Test the exact solution with the two-by-two sample determinant at $x_1 = \pi/3$, $x_2 = 2\pi/3$, $t_1 = 0$, $t_2 = \log 2$. Compute it and use it to rule out one-product factorization.
4. Find the exact uniform error when $e^t u(x, t)$ is approximated by $\sin x$, and a sufficient time for that error to be at most $\varepsilon \in (0, 1)$. Sketch normalized profiles for second-mode ratios 1, 1/2, 1/10 and zero.

Question 9 – Solution

Strategy. An eigenfunction expansion can become nearly one mode without ever being exactly one product at a finite time.

Step 1: Superpose the two decays. The eigenvalues are one and four, so

$$u = e^{-t} \sin x + e^{-4t} \sin(2x).$$

The PDE and every datum follow term by term. Separation constructs building blocks; linearity allows their sums, whose distinct temporal factors need not combine into one product.

Step 2: Derive the restriction on a product. If $u = XT$ is a nonzero classical product, then $XT' = X''T$. Choose a time with $T \neq 0$ and points where $X \neq 0$. The ratios satisfy $T'/T = X''/X = -\lambda$, independent of both variables. The differential identity $X'' = -\lambda X$ extends across spatial zeros by continuity; the temporal ODE $T' = -\lambda T$ extends likewise. Zero boundary values give the Dirichlet eigenproblem. The trivial zero product is separate. Division at zeros is unnecessary and cannot create extra eigenfunctions.

Step 3: Use a factorization invariant. Every product has determinant

$$u(x_1, t_1)u(x_2, t_2) - u(x_1, t_2)u(x_2, t_1) = 0.$$

For the stated points, the four entries are

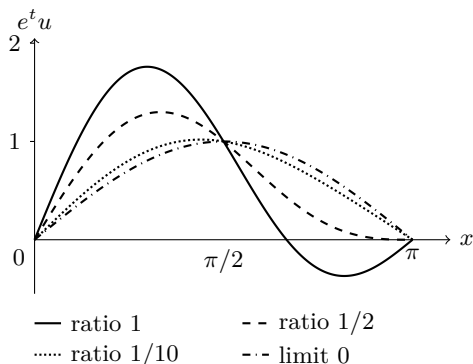
$$u(x_1, 0) = \sqrt{3}, \quad u(x_2, 0) = 0, \quad u(x_1, \log 2) = \frac{9\sqrt{3}}{32}, \quad u(x_2, \log 2) = \frac{7\sqrt{3}}{32}.$$

Their determinant is $\boxed{21/32 \neq 0}$, proving that the exact field is not one product.

Step 4: Quantify approximate separation. The normalized field is $e^t u = \sin x + e^{-3t} \sin(2x)$, so

$$\|e^t u - \sin x\|_\infty = e^{-3t}, \quad t \geq \frac{1}{3} \log(1/\varepsilon) \implies \|e^t u - \sin x\|_\infty \leq \varepsilon.$$

Equality in the norm follows at a point where $|\sin(2x)| = 1$. The plotted ratios correspond to $t = 0, (\log 2)/3, (\log 10)/3$ and the limiting profile. Approximation by a surviving mode does not retroactively justify a single-product initial ansatz.



Question 10

Let $f(x) = \sum_{n \geq 1} a_n \sin(nx)$ on $0 \leq x \leq \pi$, where $|a_n| \leq n^{-2}$. Use it as initial displacement for zero-endpoint heat and wave problems $H_t = H_{xx}$, $W_{tt} = W_{xx}$, with zero initial wave velocity. Also use it as the top trace of a harmonic rectangle of height $b > 0$, with zero side and bottom values. In each field retain modes $n \leq N$, $N \geq 1$.

Tasks

1. Write the three complete expansions and justify the original data and the relevant convergence claims.
2. Derive uniform truncation bounds for heat at $t \geq \tau > 0$, for the wave at all times, and for the harmonic field at $0 \leq y \leq b - d$, $0 < d < b$.
3. For $\tau = 0.1$, $b = 1$, $d = 0.25$ and error tolerance 10^{-2} , find the smallest integer certified by each of your displayed bounds. Plot the three bound functions and mark these sufficient cutoffs.
4. Explain why the different cutoffs reflect different mechanisms, and why waiting alone does not create an all-time smoothing estimate for the wave.

Question 10 – Solution

Strategy. Apply the same coefficient envelope to three different modal factors, keeping the location where each error bound is valid.

Step 1: Construct the fields and their traces. The three factors give

$$H = \sum a_n e^{-n^2 t} \sin(nx), \quad W = \sum a_n \cos(nt) \sin(nx), \quad P = \sum a_n \frac{\sinh(ny)}{\sinh(nb)} \sin(nx).$$

The summable coefficient envelope ensures uniform displacement and boundary-series convergence. Heat derivatives converge away from $t = 0$, and harmonic derivatives away from the top edge. The wave has finite energy since $\sum n^2 a_n^2 < \infty$; its data and PDE have the corresponding energy/weak meaning without an automatic higher-smoothness claim.

Step 2: Bound the three tails. Using $\sum_{n>N} n^{-2} \leq 1/N$ gives

$$B_H(N) = \frac{e^{-(N+1)^2 \tau}}{N}, \quad B_W(N) = \frac{1}{N}.$$

For $y \leq b - d$, write the hyperbolic ratio as $e^{-n(b-y)}(1 - e^{-2ny})/(1 - e^{-2nb})$. It is at most $e^{-nd}/(1 - e^{-2b})$, so

$$B_P(N) = \frac{e^{-(N+1)d}}{N(1 - e^{-2b})}.$$

These respectively bound the heat error for $t \geq \tau$, the wave displacement error at all times, and the harmonic error in the stated lower strip.

Step 3: Choose certified integer cutoffs. Each bound decreases with N . With the specified parameters,

	preceding bound	accepted bound	N
H	$B_H(4) \approx .02052$	$B_H(5) \approx .005465$	5
W	$B_W(99) \approx .01010$	$B_W(100) = .01000$	100
P	$B_P(9) \approx .01055$	$B_P(10) \approx .007394$	10

These are the smallest integers certified by these bounds, not necessarily the fewest modes required by particular data. The graph uses continuous bound envelopes on logarithmic axes.

Step 4: Interpret the filtering mechanisms. Heat damps high modes by $e^{-n^2 t}$; distance from the driven harmonic edge gives spatial attenuation of order e^{-nd} . Wave modes retain their amplitudes and only oscillate: at every $t = 2m\pi$, the entire displacement returns to f . Consequently waiting cannot supply a uniform future decay factor for every wave tail. The heat and harmonic improvements require positive waiting time or positive distance; they cannot be silently extended to $t = 0$ or to the top trace.

