

Question 1

A string on $0 < x < 1$ has unit tension and unit mass per length, fixed endpoints, and initial data

$$u_{tt} = u_{xx}, \quad u(0, t) = u(1, t) = 0, \quad u(x, 0) = \sin(\pi x) + \frac{1}{2} \sin(2\pi x), \quad u_t(x, 0) = 2\pi \sin(3\pi x).$$

Its energy is $E(t) = \frac{1}{2} \int_0^1 (u_t^2 + u_x^2) dx$.

Tasks

1. Construct the complete modal solution, including the correct coefficient of the velocity-generated mode.
2. Verify the PDE, both endpoint conditions and both initial data. Determine whether the entire string can ever have zero displacement.
3. Compute the energy carried by each active mode and the total energy. Explain why cross terms do not contribute.
4. Find the smallest positive period of the full state (u, u_t) . Prove minimality rather than merely giving a return time.

Question 1 – Solution

Strategy. Initial displacement and initial velocity set different temporal coefficients; spatial orthogonality separates both energy and state recurrence.

Step 1: Normalize the three temporal modes. Fixed ends select $\sin(n\pi x)$ with frequencies $\omega_n = n\pi$. A modal initial velocity g_n gives coefficient g_n/ω_n multiplying $\sin(\omega_n t)$. Therefore

$$u = \sin(\pi x) \cos(\pi t) + \frac{1}{2} \sin(2\pi x) \cos(2\pi t) + \frac{2}{3} \sin(3\pi x) \sin(3\pi t).$$

The last coefficient is $2/3$, not 2π .

Step 2: Verify the state and test for a flat snapshot. Each term has equal second derivatives in x and t ; each spatial sine vanishes at both endpoints. At $t = 0$ the first two terms give the stated displacement. Differentiation gives $2\pi \sin(3\pi x)$ as the initial velocity. If $u(\cdot, t)$ were identically zero, orthogonality would force both $\cos(\pi t) = 0$ and $\cos(2\pi t) = 0$. But the first condition gives $t = k + 1/2$, at which $\cos(2\pi t) = -1$. Thus the string is never entirely flat.

Step 3: Add the orthogonal modal energies. For $q_n(t) \sin(n\pi x)$, the energy is $E_n = \frac{1}{4}[q_n'^2 + (n\pi)^2 q_n^2]$. Distinct sines are orthogonal in the kinetic term, and distinct cosines in the elastic term, so no cross terms remain. Evaluating at $t = 0$ gives

$$E_1 = \frac{\pi^2}{4}, \quad E_2 = \frac{\pi^2}{4}, \quad E_3 = \pi^2, \quad E = \frac{3\pi^2}{2}.$$

Each modal energy is constant because $q_n'' + (n\pi)^2 q_n = 0$.

Step 4: Find the fundamental state period. All three temporal modes return with their derivatives after time 2. Conversely, a return of the full state forces its nonzero first mode to satisfy $\cos(\pi P) = 1$ and $\sin(\pi P) = 0$. Thus P must be a positive even integer. The smallest is

$$P = 2.$$

A state recurrence requires displacement and velocity together; isolated zeros of one modal displacement do not establish a period.

Question 2

A uniform string of length L , tension $T > 0$ and density $\mu > 0$ has fixed endpoints and wave speed $c = \sqrt{T/\mu}$. It is plucked to height $h > 0$ at $x = a$, where $0 < a < L$, then released from rest:

$$f(x) = \begin{cases} hx/a, & 0 \leq x \leq a, \\ h(L-x)/(L-a), & a \leq x \leq L, \end{cases} \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0.$$

Use the finite-energy solution of $u_{tt} = c^2 u_{xx}$.

Tasks

1. Derive the sine coefficients and the full solution. State in what sense the initial data and PDE are satisfied near the moving corners.
2. Set $a = L/3$. Identify precisely which modes are absent and compute the initial, hence conserved, energy.
3. Prove $u(x, L/c) = -f(L-x)$ and $u(x, 2L/c) = f(x)$. Include the velocity state at these times.
4. Let F be the odd, $2L$ -periodic extension of f . Use it to obtain exact piecewise-linear profiles at $t = 0, L/(2c), L/c$ for $L = c = h = 1$, $a = 1/3$, and sketch them without truncating a Fourier series.

Question 2 – Solution

Strategy. Fourier coefficients reveal the missing harmonics, while reflected traveling profiles preserve the corners exactly.

Step 1: Compute the pluck coefficients. Integrating separately on $[0, a]$ and $[a, L]$ gives

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx = \frac{2hL^2}{a(L-a)n^2\pi^2} \sin \frac{n\pi a}{L}.$$

Thus

$$u(x, t) = \sum_{n \geq 1} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}.$$

The $O(n^{-2})$ coefficients give uniform convergence of displacement. The initial velocity is zero in L^2 , and the solution has finite conserved energy. It solves the PDE weakly and classically away from the propagating slope corners; no globally C^2 solution is claimed.

Step 2: Identify the spectrum and energy. For $a = L/3$, $b_n = 9h \sin(n\pi/3)/(n^2\pi^2)$, so exactly the positive multiples of three are absent. Initially all energy is elastic:

$$E = \frac{T}{2} \int_0^L f'^2 dx = \frac{Th^2L}{2a(L-a)} = \boxed{\frac{9Th^2}{4L}} \quad (a = L/3).$$

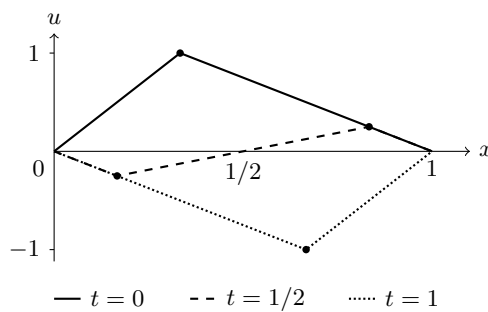
The finite-energy wave evolution conserves this value.

Step 3: Verify the reflected and full returns. At $t = L/c$, each cosine is $(-1)^n$, and $\sin(n\pi(L-x)/L) = (-1)^{n+1} \sin(n\pi x/L)$. Hence $u(x, L/c) = -f(L-x)$. At $2L/c$ every cosine equals one, so $u = f$. The modal velocities vanish at both times, giving zero velocity in L^2 .

Step 4: Construct exact corner-preserving snapshots. The reflected d'Alembert form is $u(x, t) = \frac{1}{2}[F(x-ct) + F(x+ct)]$. Oddness and periodicity enforce both endpoints and recover both initial data. For the stated numbers, the half-time profile is

$$u(x, \tfrac{1}{2}) = \begin{cases} -3x/2, & 0 \leq x \leq 1/6, \\ 3x/4 - 3/8, & 1/6 \leq x \leq 5/6, \\ 3(1-x)/2, & 5/6 \leq x \leq 1. \end{cases}$$

The other two profiles are $f(x)$ and $-f(1-x)$; their corners are plotted at their exact positions.



Question 3

A unit string has fixed endpoints and unit tension and density. It starts straight and is struck so that its initial velocity is

$$u(x, 0) = 0, \quad g(x) = u_t(x, 0) = \begin{cases} V, & 1/3 < x < 2/3, \\ 0, & 0 < x < 1/3 \text{ or } 2/3 < x < 1, \end{cases} \quad V > 0.$$

Values assigned at the two jump points do not change the finite-energy solution.

Tasks

1. Derive the velocity sine coefficients and construct the complete fixed-end solution of $u_{tt} = u_{xx}$.
2. Explain the symmetry and identify exactly which modes vanish. State the appropriate initial trace and regularity claims.
3. Find an exact formula for the midpoint displacement for $0 \leq t \leq 1/2$. Identify when its velocity first changes and explain why the displacement then stays positive.
4. Compute the conserved energy directly from the initial data. Derive the corresponding modal Parseval identity, including every normalization factor.

Question 3 – Solution

Strategy. A velocity profile must be integrated in time; its jumps do not permit a globally classical interpretation of the initial state.

Step 1: Convert the strike to temporal sine amplitudes. The velocity coefficients are

$$g_n = 2V \int_{1/3}^{2/3} \sin(n\pi x) dx = \frac{2V}{n\pi} \left(\cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3} \right).$$

Because the initial displacement is zero,

$$u(x, t) = \sum_{n \geq 1} \frac{g_n}{n\pi} \sin(n\pi t) \sin(n\pi x).$$

The additional division by $n\pi$ is essential to recover $u_t(\cdot, 0) = g$.

Step 2: Interpret the symmetry and traces. The strike satisfies $g(1-x) = g(x)$, so only the symmetric, odd-indexed sine modes can occur. The displayed coefficient vanishes for every even n and is nonzero for every odd n : the cosine difference for odd n is either 1 or -2 . Displacement converges uniformly since its coefficients are $O(n^{-2})$. Velocity recovers g in L^2 ; its initial sine series takes the average $V/2$ at either jump. The PDE holds weakly everywhere and classically away from the reflected characteristic fronts.

Step 3: Compute the midpoint before reflected arrivals. For $0 \leq t \leq 1/2$, the interval $[1/2-t, 1/2+t]$ remains within $[0, 1]$. The velocity-only d'Alembert formula therefore gives

$$u(\tfrac{1}{2}, t) = \frac{1}{2} \int_{1/2-t}^{1/2+t} g(s) ds = \begin{cases} Vt, & 0 \leq t \leq 1/6, \\ V/6, & 1/6 \leq t \leq 1/2. \end{cases}$$

At $t = 1/6$, both edges of the moving integration interval pass the strike edges. The one-sided midpoint velocities change from V to zero. A zero subsequent velocity does not undo the displacement already accumulated.

Step 4: Match physical and modal energies. Initially $u_x = 0$, so

$$E = \frac{1}{2} \int_0^1 g^2 dx = \frac{V^2}{6}.$$

Parseval with $\int_0^1 \sin^2(n\pi x) dx = 1/2$ gives $\int g^2 = \frac{1}{2} \sum g_n^2$, hence $E = \frac{1}{4} \sum g_n^2$. Equivalently,

$$\sum_{n \geq 1} \frac{[\cos(n\pi/3) - \cos(2n\pi/3)]^2}{n^2} = \frac{\pi^2}{6}.$$

Each mode trades kinetic and elastic energy while preserving its contribution $g_n^2/4$.

Question 4

A string of length L has tension $T > 0$, density $\mu > 0$ and speed $c = \sqrt{T/\mu}$. Its left end is fixed and its right end is free for $t > 0$:

$$u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0, \quad u_x(L, t) = 0.$$

At release its displacement is $f(x) = hx/L$, $h > 0$, and its velocity is zero. Interpret this as a finite-energy problem, since the initial slope need not satisfy the newly released end condition.

Tasks

1. Derive the mixed-end eigenfunctions, frequencies and coefficients of f . Construct the solution.
2. Explain the initial corner incompatibility. Prove that the full state reverses sign after $2L/c$ and has smallest positive period $4L/c$.
3. Show that at $t = L/c$ the string is flat but is not at rest. Find its velocity in the open interval and verify the conserved energy.
4. For $L = c = h = 1$, sketch exact profiles at $t = 0, 1/2, 1$. Use the odd reflection at the fixed end and even reflection at the free end, and mark the moving corner.

Question 4 – Solution

Strategy. A free end selects half-integer modes; the release launches a corner rather than making the initial slope compatible retroactively.

Step 1: Construct the mixed-mode solution. The conditions $X(0) = 0$, $X'(L) = 0$ give $k_n = (2n+1)\pi/(2L)$, $n \geq 0$, and $\omega_n = ck_n$. Since $\int_0^L \sin^2(k_n x) dx = L/2$,

$$b_n = \frac{2}{L} \int_0^L \frac{hx}{L} \sin(k_n x) dx = \frac{8h(-1)^n}{(2n+1)^2 \pi^2}, \quad \boxed{u = \sum_{n \geq 0} b_n \sin(k_n x) \cos(\omega_n t)}.$$

Step 2: State the trace and recurrence precisely. Initially $f'(L) = h/L \neq 0$, so a solution with continuous u_x at the release corner cannot satisfy the new free-end condition there. The displayed finite-energy solution recovers f uniformly and zero initial velocity in L^2 ; its boundary condition has the usual weak/natural meaning at wavefront events. Every active frequency is an odd multiple of $\pi c/(2L)$. Thus both u and u_t change sign after $2L/c$ and return after $4L/c$. The nonzero lowest mode requires that full period, proving minimality.

Step 3: Resolve the flat but moving state. At $t = L/c$ every temporal cosine vanishes. For $0 < x < L$,

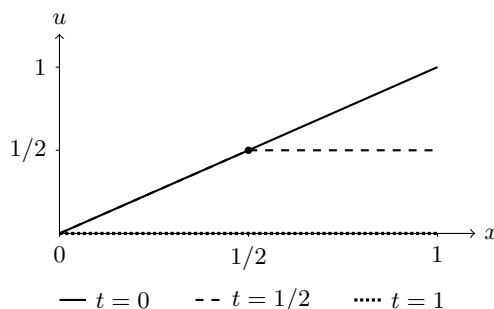
$$u_t(x, L/c) = -\frac{4hc}{L\pi} \sum_{n \geq 0} \frac{\sin((2n+1)\pi x/(2L))}{2n+1} = \boxed{-hc/L}.$$

Here $\sum_{n \geq 0} \sin((2n+1)z)/(2n+1) = \pi/4$ for $0 < z < \pi$, the sine series of a constant on that interval. The fixed-end velocity remains zero; the interior velocity trace is an L^2 statement, not corner continuity. Initially $E = Th^2/(2L)$. At this flat snapshot, the kinetic energy is $\mu L(hc/L)^2/2 = Th^2/(2L)$, so the energy is unchanged.

Step 4: Follow the released-end corner. Extend f oddly across zero and evenly across L , with period $4L$, and take the average of its two translates. For the stated units and $0 \leq t \leq 1$ this gives

$$\boxed{u(x, t) = \min(x, 1-t), \quad 0 \leq x \leq 1.}$$

For $0 < t < 1$, the corner is at $x = 1-t$: the left part retains slope one and the right part is flat and moving downward. The snapshots below use exact line segments.



Question 5

A unit string with fixed ends, unit density and unit tension experiences uniform viscous drag:

$$u_{tt} + 2\beta u_t = u_{xx}, \quad \beta = 2\pi, \quad u(x, 0) = \sin(\pi x) + \sin(2\pi x) + \sin(3\pi x), \quad u_t(x, 0) = 0.$$

Tasks

1. Derive the modal equations and classify each of the three active modes as over-, critically or under-damped.
2. Construct the full solution, satisfying both initial conditions for every mode.
3. Derive the energy dissipation law for $E = \frac{1}{2} \int_0^1 (u_t^2 + u_x^2) dx$. Explain why energy is nonincreasing even when a modal displacement oscillates.
4. Find the leading long-time displacement profile and its exact exponential decay rate. More generally, state when all positive string modes are underdamped.

Question 5 – Solution

Strategy. The same drag rate competes with different natural frequencies, so one string can exhibit all three damping regimes at once.

Step 1: Classify the modal roots. With $u = \sum q_n(t) \sin(n\pi x)$, each coefficient satisfies $q_n'' + 2\beta q_n' + (n\pi)^2 q_n = 0$. For $\beta = 2\pi$, the discriminant sign is the sign of $4 - n^2$. Thus mode one is overdamped, mode two is critical, and mode three is underdamped.

Step 2: Impose both initial conditions. The required coefficient functions, all with $q_n(0) = 1$ and $q_n'(0) = 0$, are

$$\begin{aligned} q_1 &= e^{-2\pi t} \left[\cosh(\sqrt{3}\pi t) + \frac{2}{\sqrt{3}} \sinh(\sqrt{3}\pi t) \right], \\ q_2 &= e^{-2\pi t} (1 + 2\pi t), \\ q_3 &= e^{-2\pi t} \left[\cos(\sqrt{5}\pi t) + \frac{2}{\sqrt{5}} \sin(\sqrt{5}\pi t) \right]. \end{aligned}$$

Then $u = \sum_{n=1}^3 q_n(t) \sin(n\pi x)$ satisfies the damped PDE and both endpoints. The added sinh, linear and sine terms are necessary to make the initial velocities zero.

Step 3: Account for dissipated energy. Differentiate E and integrate $u_x u_{xt}$ by parts:

$$E' = [u_x u_t]_0^1 + \int_0^1 u_t (u_{tt} - u_{xx}) dx = \left[-2\beta \int_0^1 u_t^2 dx \leq 0 \right].$$

The boundary term is zero because fixed ends have zero velocity. Oscillation changes how energy is partitioned; drag still removes energy whenever the velocity is not identically zero.

Step 4: Identify the slow surviving mode. Expanding q_1 into two exponentials, its slower coefficient is $C = (1 + 2/\sqrt{3})/2$ and its slower decay rate is $\gamma = (2 - \sqrt{3})\pi$. The other two modes decay as $e^{-2\pi t}$ times bounded or linear factors. Therefore

$$e^{\gamma t} u(x, t) \longrightarrow C \sin(\pi x) \quad \text{uniformly for } 0 \leq x \leq 1.$$

For a nonnegative general drag parameter, mode n is underdamped when $\beta < n\pi$; all positive modes are underdamped exactly when $0 \leq \beta < \pi$. The threshold compares drag with the lowest frequency, not with the highest one.

Question 6

A unit string with unit tension and density has fixed ends and is initially at rest. A spatially sinusoidal force is held constant for a chosen duration $\tau > 0$:

$$u_{tt} - u_{xx} = F_0 \sin(\pi x) \chi_{(0,\tau)}(t), \quad u(x, 0) = u_t(x, 0) = 0, \quad F_0 > 0.$$

The indicator is one during the pulse and zero otherwise; isolated switch-time values do not affect the motion.

Tasks

1. Derive the modal displacement during and after the pulse, imposing continuity of displacement and velocity at switch-off.
2. Determine every positive duration that leaves the string exactly at rest after switch-off. Prove that both state variables vanish for those durations.
3. Compute the residual energy as a function of τ and verify it equals the net work of the applied force.
4. Find the durations that maximize residual energy. Plot its normalized dependence on pulse duration and explain how a force of one sign can do zero net work over a nontrivial pulse.

Question 6 – Solution

Strategy. Switch-off is a delayed negative step in forcing; controlling pulse duration can cancel the entire excited state.

Step 1: Superpose the two forcing steps. Only the first spatial mode is driven. Writing $u = q(t) \sin(\pi x)$ gives $q'' + \pi^2 q = F_0 \chi_{(0,\tau)}$ with zero initial data. Hence

$$q(t) = \frac{F_0}{\pi^2} \begin{cases} 1 - \cos(\pi t), & 0 \leq t \leq \tau, \\ \cos(\pi(t - \tau)) - \cos(\pi t), & t \geq \tau. \end{cases}$$

The two expressions and their first derivatives agree at $t = \tau$; only acceleration can jump. They verify the forcing on each open time interval.

Step 2: Find exact cancellation durations. For $t \geq \tau$, the cosine difference becomes

$$q(t) = \frac{2F_0}{\pi^2} \sin \frac{\pi\tau}{2} \sin(\pi(t - \tau/2)).$$

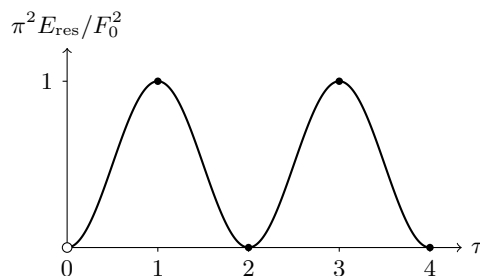
Its displacement and velocity vanish identically precisely when $\sin(\pi\tau/2) = 0$, giving $\tau = 2m, m = 1, 2, \dots$. Equivalently, the switch-off state $q(\tau) = F_0(1 - \cos \pi\tau)/\pi^2$, $q'(\tau) = F_0 \sin \pi\tau/\pi$ is zero exactly for those durations.

Step 3: Evaluate residual energy and work. After switch-off,

$$E_{\text{res}} = \frac{1}{4}(q'^2 + \pi^2 q^2) = \frac{F_0^2}{\pi^2} \sin^2 \frac{\pi\tau}{2}.$$

During forcing, the power is $\int_0^1 F_0 \sin(\pi x) u_t dx = F_0 q'/2$. Its integral from zero to τ is $F_0 q(\tau)/2 = F_0^2(1 - \cos \pi\tau)/(2\pi^2)$, the same energy.

Step 4: Interpret the pulse-duration curve. The maximum is F_0^2/π^2 , attained for $\tau = 2m + 1, m \geq 0$. For an even-integer duration, positive work during motion in the force direction is canceled by negative work while the string moves against the force. A positive force need not supply positive power at every instant.



Open dot: the excluded zero-duration limit.

Question 7

A unit fixed-end string with speed one is known to involve only modes $1, \dots, N$:

$$u(x, t) = \sum_{n=1}^N \left[f_n \cos(n\pi t) + \frac{g_n}{n\pi} \sin(n\pi t) \right] \sin(n\pi x).$$

A displacement sensor at $x = s \in (0, 1)$ records $h_s(t) = u(s, t)$ for $0 \leq t \leq 2$.

Tasks

1. Derive recovery formulas for f_n, g_n from the trace whenever the n th mode is visible.
2. For a rational sensor position $s = p/q$ in lowest terms, identify exactly the invisible modes. Explain why continuous-time sampling cannot recover those modes.
3. Use sensors at $s = 1/2$ and $s = 1/3$. Determine their common invisible modes and the largest N for which the entire stated finite-mode state is uniquely recoverable.
4. Prove that an irrational sensor sees every individual mode but can be arbitrarily poorly conditioned as N grows. You may use the pigeonhole fact that for every integer $Q \geq 1$ there are integers $1 \leq n \leq Q, k$ with $|ns - k| \leq 1/Q$.

Question 7 – Solution

Strategy. Time orthogonality separates frequencies, but it cannot restore a spatial mode whose node lies at the sensor.

Step 1: Project the temporal record. On $[0, 2]$, distinct integer-frequency sines and cosines are orthogonal and each squared norm is one. Consequently

$$f_n = \frac{\int_0^2 h_s(t) \cos(n\pi t) dt}{\sin(n\pi s)}, \quad g_n = \frac{n\pi \int_0^2 h_s(t) \sin(n\pi t) dt}{\sin(n\pi s)}$$

whenever $\sin(n\pi s) \neq 0$. These recover both initial displacement and velocity coefficients.

Step 2: Identify rational-position blindness. For $s = p/q$ in lowest terms, $\sin(n\pi s) = 0$ exactly when q divides n . Every such mode contributes zero to $h_s(t)$ for every time and for every choice of its two coefficients. Thus even a perfect continuous trace cannot distinguish states that differ only in those modes.

Step 3: Combine the two sensors. The midpoint misses the even modes; the one-third sensor misses multiples of three. Their common invisible modes are exactly the multiples of six. For $N \leq 5$, each mode has a nonzero factor at at least one sensor, so the formulas recover the entire state. For $N \geq 6$, arbitrary sixth-mode displacement or velocity can be added without changing either record. The largest fully identifiable cutoff is therefore

$$N = 5.$$

Step 4: Distinguish visibility from stability. If s is irrational, no ns is an integer, so each denominator is nonzero. But the supplied approximation gives $|\sin(n\pi s)| = |\sin(\pi(ns - k))| \leq \pi/Q$. As $Q \rightarrow \infty$, the selected indices cannot stay in a finite set: each such index has a fixed positive distance from the integers. Hence arbitrarily large modes have arbitrarily small sensor factors. An error e_n in the cosine projection becomes $e_n/\sin(n\pi s)$ in f_n ; the sine projection has the additional factor $n\pi$ for g_n . Exact visibility at all modes does not provide a uniform noise bound.

Question 8

A uniform string of length L , tension $T > 0$ and density $\mu > 0$ is fixed at both ends and carries a point mass $M > 0$ at its midpoint. Away from that mass, $u_{tt} = c^2 u_{xx}$ with $c = \sqrt{T/\mu}$. Displacement is continuous at $x = L/2$, and the mass obeys

$$Mu_{tt}(L/2, t) = T[u_x(L/2+, t) - u_x(L/2-, t)].$$

Tasks

1. For a normal mode $u = \phi(x) \cos(\omega t)$, derive the interface condition and classify antisymmetric modes about the midpoint.
2. Derive the frequency equation for symmetric modes using $z = \omega L/(2c)$ and $\alpha = M/(\mu L)$. Prove there is exactly one positive symmetric root in each interval $(j\pi, j\pi + \pi/2)$, $j = 0, 1, \dots$
3. Find the limiting lowest symmetric frequency as $M \downarrow 0$ and its leading behavior as $M \rightarrow \infty$. Interpret the latter as an effective spring-mass system.
4. Write the total conserved energy, including the attached mass, and prove conservation by checking the interface terms.

Question 8 – Solution

Strategy. The midpoint mass changes the dynamic matching condition, but modes with a node there do not move the added mass.

Step 1: Separate and use symmetry. On each half, $\phi'' + k^2\phi = 0$, $k = \omega/c$, and the mass equation becomes

$$T(\phi'_+ - \phi'_-) + M\omega^2\phi(L/2) = 0.$$

Antisymmetry gives $\phi(L/2) = 0$ and matching slopes. The fixed endpoints then yield

$$\phi(x) = A \sin(2m\pi x/L), \quad \omega = 2m\pi c/L, \quad m = 1, 2, \dots$$

These frequencies are independent of M .

Step 2: Derive and locate the symmetric frequencies. A symmetric shape has $\phi = A \sin(kx)$ on the left and $\phi = A \sin(k(L-x))$ on the right. Its slope jump is $-2Ak \cos z$ and midpoint value is $A \sin z$. Substitution gives

$$2Tk \cos z = Mc^2 k^2 \sin z, \quad \cot z = \alpha z, \quad \alpha = \frac{M}{\mu L}.$$

On $(j\pi, j\pi + \pi/2)$, $\cot z - \alpha z$ decreases strictly from positive infinity to a negative value. It has exactly one root. On the intervening intervals the cotangent is negative, so no positive root occurs there. Values $\sin z = 0$ do not solve the original matching equation. There is no nonzero static mode with both endpoints fixed.

Step 3: Interpret the lowest root. For the root $z_0 \in (0, \pi/2)$, $\alpha \downarrow 0$ gives $z_0 \rightarrow \pi/2$, recovering $\omega_0 \rightarrow \pi c/L$. As $\alpha \rightarrow \infty$, $z_0 \rightarrow 0$ and $z \cot z \rightarrow 1$ gives $\alpha z_0^2 \rightarrow 1$. Thus

$$\omega_0 \sim 2\sqrt{\frac{T}{ML}} \quad (M \rightarrow \infty).$$

A small midpoint deflection d gives each half-string slope magnitude $2d/L$; their restoring forces add to $4Td/L$. The effective stiffness is $4T/L$, consistent with the displayed mass-spring frequency.

Step 4: Include the concentrated kinetic energy. For piecewise smooth motion, the energy is

$$E = \frac{1}{2} \int_0^L (\mu u_t^2 + T u_x^2) dx + \frac{M}{2} u_t(L/2, t)^2.$$

Integration by parts on each half leaves $T u_t(L/2)(u_x^- - u_x^+)$; the fixed-end contributions vanish. The point-mass term differentiates to $M u_t(L/2) u_{tt}(L/2) = T u_t(L/2)(u_x^+ - u_x^-)$. They cancel, so $\boxed{E' = 0}$. Omitting the mass term would falsely suggest loss or gain of energy at the interface.

Question 9

A string of length L has tension $T > 0$, density $\mu > 0$ and speed $c = \sqrt{T/\mu}$. It is initially straight and at rest in the open interval. For $t > 0$, its left end stays fixed while its right end moves upward at constant speed $V > 0$:

$$u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0, \quad u(L, t) = Vt.$$

Only times $0 \leq t \leq 2L/c$ are considered. Define $r_+ = \max(r, 0)$.

Tasks

1. Construct the incident wave before it reaches the left end. Locate its front and verify the boundary and initial data in their appropriate senses.
2. Add the fixed-end reflected wave and obtain one formula valid through $t = 2L/c$. Verify both endpoint displacements and explain the time restriction.
3. Compute the total string energy on each side of the first reflection time and verify the boundary power balance.
4. Explain the release-corner and wavefront regularity. For $L = c = V = 1$, sketch exact profiles at $t = 1/2, 1, 3/2, 2$ and describe the interior state at the last time.

Question 9 – Solution

Strategy. A prescribed end motion launches a ramp front; reflection reverses its displacement sign and transfers the injected energy into a growing slope.

Step 1: Launch the incoming ramp. For $0 \leq t \leq L/c$,

$$u_{\text{in}}(x, t) = V(t - (L - x)/c)_+.$$

It vanishes ahead of the front $x = L - ct$ and equals Vt at $x = L$. Every function of $t + x/c$ satisfies the wave equation distributionally; here it is classical away from the front. At $t = 0$, displacement is zero and velocity is zero at every interior point.

Step 2: Reflect at the fixed left end. The reflected ramp must cancel the incoming displacement at $x = 0$:

$$u = V(t - (L - x)/c)_+ - V(t - (L + x)/c)_+.$$

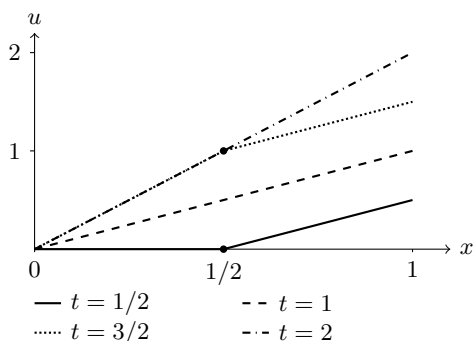
The left trace is identically zero. At $x = L$, the second term is zero through $t = 2L/c$, giving $u(L, t) = Vt$. After that time it would change the right trace, so additional reflections would be needed. The formula is not asserted for later times.

Step 3: Balance energy and injected power. Before the first reflection, the moving portion has length ct , with $u_t = V$ and $u_x = V/c$. Since $T/c^2 = \mu$, its energy is $E = \mu c V^2 t$. For $L/c < t < 2L/c$, the reflected left portion has length $ct - L$, velocity zero and slope $2V/c$; the right portion has length $2L - ct$, velocity V and slope V/c . Hence again

$$E = 2\mu V^2(ct - L) + \mu V^2(2L - ct) = \boxed{\mu c V^2 t}.$$

For $0 < t < 2L/c$, the right-end slope is V/c , so the input power is $Tu_x(L, t)u_t(L, t) = TV^2/c = \mu c V^2 = E'$.

Step 4: Interpret the corner and the final snapshot. The initial interior velocity is zero, but the driven endpoint velocity jumps to V : no continuous velocity at that release corner is possible. Displacement stays continuous across each front, while first derivatives jump; the weak wave equation remains valid. At $t = 2L/c$, $u = 2Vx/c$ and the open-interval velocity is zero; energy is entirely elastic, $2\mu LV^2$. The endpoint still moves, so this is not a smooth state of rest that persists afterward.



Question 10

A unit fixed-end string with unit tension and density is released from rest with $u(x, 0) = x(1 - x)$. Let u_N retain only sine modes with index $n \leq N$, where $N \geq 1$ is an integer, from the exact finite-energy solution. Define the error energy

$$E_{\text{err}}(t) = \frac{1}{2} \int_0^1 [(u_t - u_{N,t})^2 + (u_x - u_{N,x})^2] dx.$$

Tasks

1. Derive the Fourier solution and its total energy. State the regularity caveat at the initial boundary corners.
2. Derive an exact series for E_{err} and prove the all-time estimate $E_{\text{err}} \leq 16/(3\pi^4 N^3)$.
3. Prove the all-time displacement estimate $\|u - u_N\|_{\infty} \leq 4/(\pi^3 N^2)$. Find a sufficient integer N for error at most 10^{-3} everywhere.
4. Find a sufficient integer N for relative energy error at most 10^{-4} . Explain why neither estimate depends on elapsed time and why wave evolution supplies no heat-like smoothing factor.

Question 10 – Solution

Strategy. Orthogonality gives a conserved tail energy, while absolute Fourier tails give a different uniform displacement certificate.

Step 1: Construct the solution and normalize energy. Two integrations by parts give

$$b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = \frac{4(1-(-1)^n)}{(n\pi)^3}.$$

Thus

$$u(x, t) = \frac{8}{\pi^3} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x) \cos(n\pi t)}{n^3}, \quad E = \frac{1}{2} \int_0^1 (1-2x)^2 dx = \frac{1}{6}.$$

Displacement and first-derivative series converge uniformly, and the PDE has a finite-energy interpretation. A globally C^2 corner solution is impossible: fixed endpoints would have $u_{tt} = 0$ there, while the initial second spatial derivative is -2 .

Step 2: Bound the conserved energy tail. Each omitted mode contributes $(n\pi)^2 b_n^2/4$, independent of time. Orthogonality therefore gives

$$E_{\text{err}} = \frac{16}{\pi^4} \sum_{\substack{n > N \\ n \text{ odd}}} \frac{1}{n^4} \leq \frac{16}{\pi^4} \sum_{n > N} \frac{1}{n^4} \leq \boxed{\frac{16}{3\pi^4 N^3}}.$$

The last inequality is the decreasing-function integral bound $\sum_{n > N} n^{-4} \leq \int_N^\infty s^{-4} ds$.

Step 3: Certify displacement independently. Using $|\sin|, |\cos| \leq 1$ and the same tail comparison,

$$\|u - u_N\|_\infty \leq \frac{8}{\pi^3} \sum_{n > N} \frac{1}{n^3} \leq \frac{4}{\pi^3 N^2}.$$

This holds simultaneously for every x and every t . For tolerance $\delta = 10^{-3}$, it suffices to take $N \geq \sqrt{4/(\pi^3 \delta)} \approx 11.36$, so $\boxed{N = 12}$ suffices. The bound includes even indices for convenience; it is sufficient, not a claim of the smallest possible retained set.

Step 4: Choose an energy-based cutoff. Since $E = 1/6$, $E_{\text{err}}/E \leq 32/(\pi^4 N^3)$. For tolerance $\eta = 10^{-4}$, it suffices that

$$N \geq \left(\frac{32}{\pi^4 \eta} \right)^{1/3} \approx 14.87, \quad \boxed{N = 15}.$$

Each wave mode preserves its energy and merely oscillates. There is no factor decaying with $n^2 t$ as in diffusion, so waiting does not reduce the truncation's energy error. These two cutoffs measure different errors and need not agree.