

Question 1

A square plate occupies $0 < x < 1$, $0 < y < 1$. Its steady temperature satisfies

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(1, y) = u(x, 0) = 0, \quad u(x, 1) = \sin(\pi x) + \frac{1}{2} \sin(3\pi x).$$

The conductivity is one, so the outward heat flux is $-\partial_n u$.

Tasks

1. Construct the harmonic temperature by separation of variables and verify all four boundary values.
2. At height $y = 1/2$, determine the attenuation of each boundary mode and their amplitude ratio. Explain which spatial feature penetrates farther into the plate.
3. Compute the total outward heat flux on each edge and verify global balance. Use outward normals consistently.
4. Sketch temperature profiles at $y = 1/4, 1/2, 1$. Prove that the temperature is strictly positive in the interior despite the sign changes of the third sine mode.

Question 1 – Solution

Strategy. Separate the two imposed spatial frequencies and track their normal derivatives as well as their temperatures.

Step 1: Extend each boundary mode. The zero side values give $\sin(n\pi x)$; the corresponding vertical equation is $Y'' - (n\pi)^2 Y = 0$. The bottom value selects $\sinh(n\pi y)$ and the top value normalizes it:

$$u = \frac{\sinh(\pi y)}{\sinh \pi} \sin(\pi x) + \frac{1}{2} \frac{\sinh(3\pi y)}{\sinh(3\pi)} \sin(3\pi x).$$

Each term has zero Laplacian. Substitution gives all four traces, including compatible corner values. The maximum principle gives uniqueness among continuous harmonic extensions.

Step 2: Compare penetration of the modes. Since $\sinh(a/2)/\sinh a = 1/(2 \cosh(a/2))$, the first and third modes are attenuated by $1/(2 \cosh(\pi/2))$ and $1/(2 \cosh(3\pi/2))$. Their third-to-first amplitude ratio is

$$\frac{\cosh(\pi/2)}{2 \cosh(3\pi/2)}.$$

It is smaller than the boundary ratio $1/2$: the higher spatial frequency loses amplitude faster away from the driven edge.

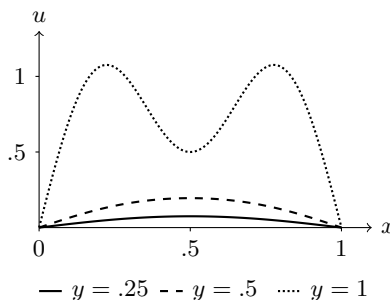
Step 3: Balance the four edge fluxes. Integrating $-\partial_n u$ along each unit edge gives

$$F_{\text{top}} = -2 \coth \pi - \coth(3\pi), \quad F_{\text{bottom}} = 2 \operatorname{csch} \pi + \operatorname{csch}(3\pi),$$

$$F_{\text{left}} = F_{\text{right}} = \tanh(\pi/2) + \frac{1}{2} \tanh(3\pi/2).$$

For example, at the bottom the outward normal is $(0, -1)$, so the outward heat flux is u_y , not $-u_y$. The identity $\coth a - \operatorname{csch} a = \tanh(a/2)$ proves that the four totals sum to zero.

Step 4: Interpret the profiles and their sign. Writing $s = \sin(\pi x) \in [0, 1]$, the top value is $s + \frac{1}{2}(3s - 4s^3) = s(5/2 - 2s^2) \geq 0$, and it is positive for $0 < x < 1$. The weak maximum principle gives $u \geq 0$; the strong maximum principle gives $u > 0$ inside because the boundary data are not constant. Individual modes need not be positive.



Question 2

Let u be harmonic on the unit square, with zero values on the left, right and bottom edges and

$$u(x, 1) = x(1 - x).$$

You may use Fourier sine convergence for a smooth function vanishing at both endpoints and Green's first identity.

Tasks

1. Derive the sine coefficients of the top boundary function and construct the complete harmonic extension.
2. Justify continuity up to every edge and harmonicity in the interior. Explain why boundary convergence and interior differentiation require different estimates.
3. Prove the pointwise bounds $0 \leq u(x, y) \leq y/4$. Determine whether equality can occur at an interior point.
4. Compute the Dirichlet energy $\mathcal{E} = \int_0^1 \int_0^1 |\nabla u|^2 dx dy$ as an explicitly convergent positive series. Justify the limiting operation.

Question 2 – Solution

Strategy. Use the boundary Fourier coefficients for construction, a harmonic barrier for bounds, and orthogonality for energy.

Step 1: Compute the boundary expansion. Two integrations by parts give

$$b_n = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = \frac{4(1 - (-1)^n)}{(n\pi)^3}.$$

Thus $b_n = 8/(n\pi)^3$ for odd n , and $b_n = 0$ for even n . The solution is

$$u(x, y) = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{8}{(n\pi)^3} \frac{\sinh(n\pi y)}{\sinh(n\pi)} \sin(n\pi x).$$

Step 2: Establish the appropriate convergence. The hyperbolic ratio is between zero and one for $0 \leq y \leq 1$. Since $\sum |b_n| < \infty$, the series converges uniformly on the closed square. Its top trace is the sine series of $x(1-x)$, and its other traces vanish. On any strip $y \leq 1 - \epsilon$, derivatives of every fixed order are bounded by a polynomial in n times $e^{-n\pi\epsilon}$. They converge uniformly there, so the Laplacian vanishes term by term. Uniform convergence of the original series alone would not justify taking two derivatives.

Step 3: Compare with a harmonic barrier. The boundary values are nonnegative. The function $h(x, y) = y/4$ is harmonic, is nonnegative on the sides, vanishes on the bottom, and dominates $x(1-x) \leq 1/4$ on the top. Applying the maximum principle to u and $u - h$ gives

$$0 \leq u(x, y) \leq y/4.$$

Both inequalities are strict in the interior: either equality there would make the corresponding harmonic difference constant, contrary to its boundary values.

Step 4: Evaluate energy without losing boundary terms. For a finite partial sum, Green's identity leaves only the top contribution $\int_0^1 u(x, 1) u_y(x, 1) dx$. Sine orthogonality gives

$$\mathcal{E}_N = \frac{1}{2} \sum_{\substack{n \leq N \\ n \text{ odd}}} b_n^2 n\pi \coth(n\pi).$$

The same identity applied to a difference of partial sums proves their gradients are Cauchy in L^2 , since the summands are $O(n^{-5})$. The uniform limit has that weak gradient and is the harmonic field already constructed. Therefore

$$\mathcal{E} = \frac{32}{\pi^5} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\coth(n\pi)}{n^5}.$$

Every term is positive; truncating this energy series gives a lower bound.

Question 3

Use centered coordinates $X = x - \frac{1}{2}$, $Y = y - \frac{1}{2}$ on the unit square. A harmonic temperature has the following boundary data:

$$\begin{aligned} u(x, 0) &= X^2 - X - \frac{1}{4}, & u(x, 1) &= X^2 + X - \frac{1}{4}, \\ u(0, y) &= \frac{1}{4} - Y - Y^2, & u(1, y) &= \frac{1}{4} + Y - Y^2. \end{aligned}$$

Seek a harmonic polynomial of degree at most two.

Tasks

1. Recover the polynomial from the four traces. Check their corner compatibility and prove uniqueness among all continuous harmonic solutions.
2. Find and classify every interior critical point. Explain why its existence does not contradict the maximum principle.
3. Find the exact global maximum and minimum on the closed square and all points where they occur.
4. Draw the zero-temperature lines and label the signs of the regions they separate. Verify that the total outward conductive flux, with conductivity one, is zero.

Question 3 – Solution

Strategy. Boundary polynomials can identify an exact harmonic field, but its interior stationary point must be classified rather than assumed to be an extremum.

Step 1: Reconstruct and verify the polynomial. A general quadratic $aX^2 + bXY + cY^2 + dX + eY + f$ is harmonic precisely when $c = -a$. The top and bottom traces force $a = 1$, $b = 2$, $d = e = 0$, and $f = 0$. The side traces then agree:

$$u = X^2 + 2XY - Y^2.$$

At the corners its values are $1/2, -1/2, -1/2, 1/2$, in the order $(0,0), (1,0), (0,1), (1,1)$, whether computed from horizontal or vertical edges. The difference of two continuous harmonic solutions has zero boundary data; its maximum and minimum are zero.

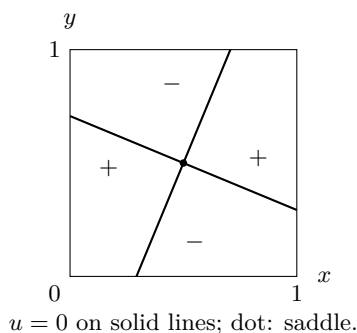
Step 2: Classify the critical point. The gradient is $(2X+2Y, 2X-2Y)$, so its sole zero is $(X, Y) = (0, 0)$. The Hessian has eigenvalues $2\sqrt{2}$ and $-2\sqrt{2}$, hence the center is a saddle. In particular $u(X, 0) = X^2$ while $u(0, Y) = -Y^2$. The maximum principle prohibits a nonconstant harmonic field's interior maximum or minimum, not a saddle.

Step 3: Locate the actual extrema. On the bottom, $X^2 - X - 1/4$ decreases on $[-1/2, 1/2]$, with range $[-1/2, 1/2]$; the top has the same range. On the left, $1/4 - Y - Y^2$ decreases over the allowed interval, also with that range, and the right behaves symmetrically. Therefore

$$\max u = \frac{1}{2} \text{ at } (0,0), (1,1), \quad \min u = -\frac{1}{2} \text{ at } (1,0), (0,1).$$

No interior point attains either value by the strong maximum principle.

Step 4: Draw the zero set and check flux. Solving $X^2 + 2XY - Y^2 = 0$ gives the two lines $Y = (1 \pm \sqrt{2})X$, clipped to the square. The horizontal regions are positive and the vertical ones negative. Direct integration of $-\partial_n u$ gives flux -1 on each vertical edge and $+1$ on each horizontal edge; the sum is zero, also following from $\Delta u = 0$.



Question 4

Inside the unit disk, seek the bounded harmonic extension of the boundary function

$$g(\theta) = \begin{cases} 1, & -\pi/2 < \theta < \pi/2, \\ 0, & \pi/2 < \theta < 3\pi/2. \end{cases}$$

Angles are interpreted modulo 2π . Values assigned at the two jump points do not affect the Poisson integral

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} g(\phi) d\phi, \quad 0 \leq r < 1.$$

You may use that this kernel is positive, integrates to 2π , and has Fourier expansion $1 + 2 \sum_{n \geq 1} r^n \cos(n\alpha)$.

Tasks

1. Compute the boundary Fourier coefficients and the resulting harmonic series.
2. Prove $0 < u < 1$ in the disk and find the center value. Explain the uniqueness class for these discontinuous data.
3. Determine the radial boundary limits on both open semicircles and at the two jump points. Explain why no continuous extension to the entire closed disk exists.
4. Find a closed expression for $u(x, 0)$ along the horizontal diameter and sketch it, including its limiting endpoint values.

Question 4 – Solution

Strategy. The Poisson kernel supplies a bounded harmonic extension without imposing impossible continuity at jumps.

Step 1: Compute and damp the Fourier modes. The average is $1/2$. Symmetry makes every sine coefficient zero, while

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(n\phi) d\phi = \frac{2 \sin(n\pi/2)}{n\pi}.$$

The kernel expansion therefore gives

$$u(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \sum_{k \geq 0} \frac{(-1)^k r^{2k+1} \cos((2k+1)\theta)}{2k+1}.$$

On every smaller closed disk all derivatives converge uniformly by geometric decay. Each polar harmonic is the real part of $(x + iy)^{2k+1}$.

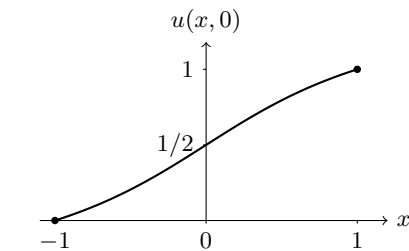
Step 2: Use positivity and specify uniqueness. Both semicircles have positive length, and the kernel is strictly positive inside. Thus their harmonic weights lie strictly between zero and one, giving $0 < u < 1$ and $u(0) = 1/2$. Uniqueness holds among bounded harmonic functions with these radial traces almost everywhere. Indeed, for the difference w , the Poisson formula on a circle of radius $R < 1$ expresses a fixed interior value through $w(R, \phi)$. As $R \uparrow 1$, boundedness and the a.e. zero trace let dominated convergence give $w = 0$. No common continuous trace at the jumps is required.

Step 3: Distinguish the boundary limits. At a point strictly within either semicircle the kernel is an approximate identity: away from any fixed angular neighborhood it tends uniformly to zero, while its total mass is fixed. The radial limit is therefore the local constant, one or zero. At $\theta = \pm\pi/2$ every displayed cosine term vanishes, so $u(r, \pm\pi/2) = 1/2$ for all $r < 1$. Neighboring boundary limits disagree; assigning $1/2$ at the jumps cannot make the closed-disk extension continuous.

Step 4: Sum the diameter profile. For $-1 < x < 1$, integrate the geometric series for $1/(1+x^2)$ from zero to x to obtain $\arctan x = \sum_{k \geq 0} (-1)^k x^{2k+1}/(2k+1)$. Hence

$$u(x, 0) = \frac{1}{2} + \frac{2}{\pi} \arctan x.$$

Its derivative is $2/[\pi(1+x^2)] > 0$; its limits at $x = -1, 1$ are zero and one. These endpoints lie on the open constant-data arcs, not at the jumps.



Endpoint dots show boundary limits.

Question 5

A disk of radius one has prescribed boundary temperature

$$g(\theta) = 2 + \cos \theta + \frac{1}{2} \cos(2\theta).$$

Let u be its continuous harmonic extension. For $r > 0$, the polar Laplacian is $u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}$.

Tasks

1. Construct u in polar and Cartesian coordinates. Explain why excluding singular radial terms is essential at the center.
2. Determine the exact boundary range and the locations of its extrema. Deduce sharp strict bounds for interior temperature.
3. Calculate the center temperature and the angular mean on every concentric circle. Explain their relation to the boundary average.
4. Find $\partial_n u$ on the boundary, verify zero total outward heat flux for conductivity one, and compute $\int_{\text{disk}} |\nabla u|^2 dA$.

Question 5 – Solution

Strategy. Regular disk modes preserve the mean, damp angular oscillations and diagonalize the boundary energy calculation.

Step 1: Retain only regular radial factors. For mode $n \geq 1$, separation gives r^n and r^{-n} ; the mean mode gives a constant and $\log r$. The singular factors cannot occur in a harmonic function on the full disk. Thus

$$u = 2 + r \cos \theta + \frac{1}{2} r^2 \cos(2\theta) = 2 + x + \frac{1}{2}(x^2 - y^2).$$

The Cartesian polynomial verifies $\Delta u = 1 - 1 = 0$ even at the center, and setting $r = 1$ gives the prescribed data.

Step 2: Find the sharp range. Put $c = \cos \theta$. Then $g = 3/2 + c + c^2 = (c + 1/2)^2 + 5/4$, with $-1 \leq c \leq 1$. The minimum $5/4$ occurs at $\theta = 2\pi/3, 4\pi/3$; the maximum $7/2$ occurs at $\theta = 0$ modulo 2π . Consequently

$$\frac{5}{4} < u(x, y) < \frac{7}{2} \quad \text{inside the disk.}$$

The inequalities are strict by the strong maximum principle. The constants cannot be improved over the whole interior, since approaching the stated boundary points approaches those values.

Step 3: Identify the preserved mean. Both cosine terms have zero angular integral, so

$$\frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) d\theta = 2 = u(0) \quad (0 < r \leq 1).$$

This explicitly realizes the mean-value property. The center equals the boundary average, although most boundary points and most interior points have different temperatures.

Step 4: Compute flux and energy. At $r = 1$, the outward derivative is $u_r = \cos \theta + \cos(2\theta)$. Its integral is zero, hence so is the total conductive flux $-\int u_r d\theta$. Green's identity and trigonometric orthogonality give

$$\int_{\text{disk}} |\nabla u|^2 dA = \int_0^{2\pi} g(\theta) u_r(1, \theta) d\theta = \pi + \frac{1}{2}\pi = \boxed{\frac{3}{2}\pi}.$$

Zero net heat flux does not imply zero gradient or zero energy: heat enters through some boundary portions and leaves through others.

Question 6

A conducting annulus $a < r < R$ has constant inner and outer temperatures T_i, T_o , where $0 < a < R$ and $T_i \neq T_o$. Conductivity is one. Seek a radial harmonic field. Total outward heat flux means the integral of $-\partial_n u$ over the indicated boundary circle.

Tasks

1. Derive the temperature and prove it is the unique continuous harmonic solution of these boundary data, including among nonradial competitors.
2. Compute both total boundary fluxes with the annulus's outward normals. Identify its conductance, defined as the magnitude of transmitted total heat flux divided by $|T_i - T_o|$.
3. Compute the Dirichlet energy and describe its relation to the temperature difference and conductance.
4. Hold R, T_i, T_o fixed and let $a \downarrow 0$. Determine the local limit of temperature away from the origin, the energy limit and the largest gradient. Explain the failure of uniform convergence to T_o . Sketch the profiles for $R = 1, T_i = 1, T_o = 0$ and $a = 0.1, 0.3, 0.5$.

Question 6 – Solution

Strategy. The logarithmic radial mode is admissible on an annulus, but its small-hole limit is not uniform near the shrinking inner boundary.

Step 1: Solve the radial equation. The equation $(ru_r)' = 0$ gives $u = A \log r + B$. Writing $D = \log(R/a) > 0$ and $\Delta T = T_i - T_o$, the boundary data give

$$u(r) = T_o + \Delta T \frac{\log(R/r)}{D}.$$

Its radial Laplacian is zero and its two traces are correct. Any other continuous harmonic solution differs by a harmonic function with zero boundary data; the maximum principle on the annulus proves uniqueness.

Step 2: Respect the inner normal. Here $u_r = -\Delta T/(rD)$. At $r = R$ the outward normal points toward increasing r , whereas at $r = a$ it points toward decreasing r . Thus

$$F_{\text{outer}} = \frac{2\pi\Delta T}{D}, \quad F_{\text{inner}} = -\frac{2\pi\Delta T}{D}, \quad G = \frac{2\pi}{D}.$$

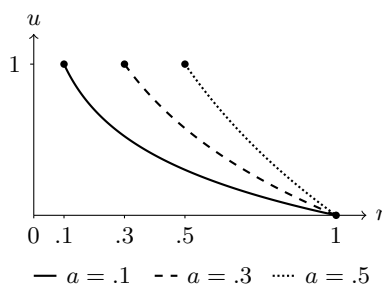
The sum is zero. When $\Delta T > 0$, heat enters the annulus at the inner boundary and leaves at the outer boundary.

Step 3: Integrate the energy. Direct radial integration yields

$$\mathcal{E} = 2\pi \int_a^R \frac{(\Delta T)^2}{r^2 D^2} r dr = \frac{2\pi(\Delta T)^2}{D} = G(\Delta T)^2.$$

Equivalently, Green's identity gives $\mathcal{E} = -T_i F_{\text{inner}} - T_o F_{\text{outer}}$, with the same sign and value.

Step 4: Separate local and uniform limits. As $a \downarrow 0$, $D \rightarrow \infty$. On each fixed $r_0 \leq r \leq R$ the field tends uniformly to T_o , and $\mathcal{E} \rightarrow 0$. However $\sup_{a \leq r \leq R} |u - T_o| = |\Delta T|$ for every a . Also $\max |\nabla u| = |\Delta T|/[a \log(R/a)] \rightarrow \infty$ at the inner circle. The concentration near a shrinking boundary is compatible with vanishing integrated energy.



Question 7

On the unit disk consider the Neumann problem

$$\Delta u = 0, \quad \partial_n u = 1 + 3 \cos \theta + 4 \sin(2\theta) \quad \text{on } r = 1.$$

Here $\partial_n u$ is the outward normal derivative, not the outward conductive heat flux.

Tasks

1. Decide whether a classical solution exists. Derive the necessary compatibility condition and evaluate it for these data.
2. Subtract a constant c from the prescribed derivative to make the data compatible. Find c and construct a regular harmonic solution whose area average on the disk is zero.
3. Prove uniqueness under that normalization, and state precisely the nonuniqueness without it.
4. For the corrected solution, compute its boundary trace, Dirichlet energy and net conductive heat flux. Check the energy independently by integrating the Cartesian gradient.

Question 7 – Solution

Strategy. Test flux compatibility before separation; then divide each nonconstant disk mode by its normal-derivative multiplier.

Step 1: Test the solvability condition. The divergence theorem gives

$$0 = \int_{\text{disk}} \Delta u \, dA = \int_{\partial \text{disk}} \partial_n u \, ds.$$

The prescribed derivative integrates to 2π , since both oscillatory terms integrate to zero. It violates a necessary condition, so the original problem has no classical solution.

Step 2: Correct the mean and construct the field. The corrected integral is $2\pi(1 - c)$, so the only constant correction is $c = 1$. Since $\partial_r(r^n) = n$ at $r = 1$, the regular solution is

$$u = 3r \cos \theta + 2r^2 \sin(2\theta) + C = 3x + 4xy + C.$$

The nonconstant terms have zero area average by symmetry. Thus normalization gives $C = 0$, and

$$\boxed{u(x, y) = 3x + 4xy.}$$

Its Cartesian Laplacian vanishes, and its outward derivative is exactly $3 \cos \theta + 4 \sin(2\theta)$.

Step 3: Establish uniqueness up to a constant. For two solutions, their difference w has $\Delta w = 0$ and $\partial_n w = 0$. Green's identity gives $\int |\nabla w|^2 = \int_{\partial \text{disk}} w \partial_n w = 0$. Hence $\nabla w = 0$, so w is constant on the connected disk. The zero-area-mean condition forces that constant to vanish. Without a normalization, every constant translate is another solution and these are all possibilities.

Step 4: Check energy in two coordinate systems. The boundary trace is $3 \cos \theta + 2 \sin(2\theta)$. Orthogonality gives

$$\mathcal{E} = \int_0^{2\pi} [3 \cos \theta + 2 \sin(2\theta)][3 \cos \theta + 4 \sin(2\theta)] \, d\theta = \boxed{17\pi}.$$

Independently, $|\nabla u|^2 = (3 + 4y)^2 + 16x^2 = 9 + 24y + 16r^2$. Its disk integral is $9\pi + 0 + 16(\pi/2) = 17\pi$. The corrected normal derivative integrates to zero, so its negative, the conductive flux, has zero total as required.

Question 8

A unit square has insulated vertical sides and prescribed horizontal temperatures:

$$\Delta u = 0, \quad u_x(0, y) = u_x(1, y) = 0, \quad u(x, 0) = 0, \quad u(x, 1) = 1 + \cos(2\pi x).$$

Seek a solution smooth up to the boundary. Conductivity is one.

Tasks

1. Construct the solution by separation, treating the zero spatial eigenvalue explicitly.
2. Verify every datum and prove uniqueness for this mixed boundary problem using Green's identity.
3. Compute the horizontal average at each height and the integrated outward heat flux on each edge. Explain why zero side flux does not force constant temperature.
4. Calculate the Dirichlet energy. Prove $0 < u < 2$ at every interior point, justifying how the insulated sides are handled.

Question 8 – Solution

Strategy. Neumann side conditions select cosine modes, including a zero mode whose vertical factor is linear rather than hyperbolic.

Step 1: Keep the zero mode. The side eigenfunctions are $1, \cos(n\pi x)$ with eigenvalues $0, (n\pi)^2$. The mean mode satisfies $Y'' = 0$ with bottom value zero and top value one, so it is y . The second cosine mode gives

$$u(x, y) = y + \frac{\sinh(2\pi y)}{\sinh(2\pi)} \cos(2\pi x).$$

Dropping the zero mode would lose the entire prescribed mean temperature.

Step 2: Verify and prove mixed uniqueness. The displayed terms are harmonic. The sine factor in u_x vanishes at both vertical sides; the bottom and top traces are exactly zero and $1 + \cos(2\pi x)$. For the difference w of two solutions, Green's identity yields $\int |\nabla w|^2 = \int_{\partial\Omega} w \partial_n w = 0$: $w = 0$ on the horizontal edges and $\partial_n w = 0$ on the vertical ones. Thus w is constant, and its bottom value forces $w = 0$.

Step 3: Track the mean and edge fluxes. The cosine has zero horizontal integral, so $\bar{u}(y) = \int_0^1 u(x, y) dx = y$. The integrated outward conductive fluxes are

$$F_{\text{top}} = -1, \quad F_{\text{bottom}} = 1, \quad F_{\text{left}} = F_{\text{right}} = 0.$$

They balance. Insulation forbids heat crossing the vertical sides; it does not forbid a vertical gradient or heat transfer between the horizontal edges.

Step 4: Compute energy and bound the field. Only the top contributes to $\int_{\partial\Omega} u \partial_n u$. At the top, $u_y = 1 + 2\pi \coth(2\pi) \cos(2\pi x)$, so orthogonality gives

$$\mathcal{E} = 1 + \pi \coth(2\pi).$$

For $0 < y < 1$, strict convexity of $\sinh(2\pi y)$ gives $0 < \sinh(2\pi y)/\sinh(2\pi) < y$. Therefore $0 < y - \sinh(2\pi y)/\sinh(2\pi) \leq u \leq y + \sinh(2\pi y)/\sinh(2\pi) < 2y < 2$. This direct estimate avoids incorrectly treating the unspecified side temperatures as Dirichlet data.

Question 9

Let u be regular and harmonic on the unit disk. Its boundary temperature is known to have the form

$$g(\theta) = A + B \cos(N\theta), \quad N \geq 1,$$

where N is known and A, B are unknown. The temperature is measured on the circle $r = \rho$, where $0 < \rho < 1$.

Tasks

1. Derive exact recovery formulas for A, B from the measured angular mean and cosine coefficient. Specify the consistency conditions on the other measured Fourier coefficients.
2. For $\rho = 1/2$, $N = 6$, and measured trace $2 + 0.03 \cos(6\theta)$, recover the boundary data and the full field.
3. If the measured N th cosine coefficient has error at most η , find the sharp error bound for the recovered B . Derive the largest allowable mode number when the desired per-coefficient error is at most $\varepsilon > \eta$. Evaluate it for $\rho = 1/2, \eta = 10^{-4}, \varepsilon = 10^{-2}$.
4. Contrast forward boundary stability with inverse instability when arbitrarily high modes are allowed. Give an explicit sequence showing why small interior-circle errors need not imply small boundary errors.

Question 9 – Solution

Strategy. Harmonic continuation damps a mode by ρ^N ; inversion must divide by exactly that small factor.

Step 1: Recover the two parameters. The regular harmonic extension is $u = A + Br^N \cos(N\theta)$. For the measured trace $h(\theta) = u(\rho, \theta)$, define

$$m = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) d\theta, \quad c_N = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos(N\theta) d\theta.$$

Orthogonality gives

$$A = m, \quad B = \rho^{-N} c_N.$$

Consistency with the stated model requires every sine coefficient and every nonconstant cosine coefficient except c_N to vanish. Since $\rho > 0$, the two parameters are unique.

Step 2: Reconstruct the given field. Here $\rho^N = 1/64$, $m = 2$, and $c_6 = 0.03 = 3/100$, so $B = 48/25 = 1.92$. Thus

$$g(\theta) = 2 + \frac{48}{25} \cos(6\theta), \quad u(r, \theta) = 2 + \frac{48}{25} r^6 \cos(6\theta).$$

Substitution at $r = 1/2$ reproduces the measured coefficient $3/100$. The polar mode is a Cartesian harmonic polynomial and is regular at the center.

Step 3: Quantify inverse amplification. A coefficient error e , $|e| \leq \eta$, produces error $\rho^{-N}e$ in B . Consequently $|\delta B| \leq \eta \rho^{-N}$, with equality when $|e| = \eta$. The desired bound is met exactly when

$$N \leq \frac{\log(\varepsilon/\eta)}{\log(1/\rho)}, \quad N_{\max} = \left\lfloor \frac{\log(\varepsilon/\eta)}{\log(1/\rho)} \right\rfloor.$$

If this integer is zero, no positive mode qualifies. The given numbers give $N_{\max} = 6$: mode six has error bound 0.0064, whereas mode seven has 0.0128. This is a per-coefficient criterion, not a bound on a sum of many mode errors.

Step 4: Separate forward and inverse stability. For continuous boundary perturbations, the maximum principle gives $\|u_1 - u_2\|_{\infty, \text{disk}} \leq \|g_1 - g_2\|_{\infty, \partial \text{disk}}$. In contrast, take $g_n(\theta) = \cos(n\theta)$ and $u_n = r^n \cos(n\theta)$. Then $\|g_n\|_{\infty} = 1$ but $\|u_n(\rho, \cdot)\|_{\infty} = \rho^n \rightarrow 0$. Thus no fixed bound can control boundary sup-norm errors by measured-circle sup-norm errors across all modes, although each exact finite-mode inversion is unique.

Question 10

Let u be harmonic on the unit square Ω , continuous on its closure, with prescribed boundary values g . A proposed approximation v is twice continuously differentiable on the closed square and satisfies

$$|v - g| \leq \varepsilon \quad \text{on } \partial\Omega, \quad |\Delta v| \leq \delta \quad \text{in } \Omega, \quad \varepsilon, \delta \geq 0.$$

You may use the weak maximum principle for subharmonic functions: if $\Delta w \geq 0$, then $\max_{\overline{\Omega}} w \leq \max_{\partial\Omega} w$.

Tasks

1. Construct a quadratic nonnegative barrier ψ with $-\Delta\psi = 1$ and maximum $1/8$ on the square.
2. Prove $|u - v| \leq \varepsilon + \delta\psi \leq \varepsilon + \delta/8$. Check the signs of both comparison functions.
3. Test the estimate for $g = x + y$ and $v = x + y + ax(1-x)y(1-y)$, where $a \neq 0$. Determine the exact boundary error, residual bound and actual maximum error, and compare with the certificate.
4. Explain why checking only the PDE residual cannot certify the answer. Show that the coefficient of ε in the estimate is sharp, and state whether this example proves sharpness of $1/8$.

Question 10 – Solution

Strategy. Convert a bounded residual into an explicit comparison barrier, while retaining a separate bound for incorrect boundary values.

Step 1: Construct the barrier. Set

$$\psi(x, y) = \frac{x(1-x) + y(1-y)}{4}.$$

It is nonnegative on the closed square and satisfies $\Delta\psi = -1$. Each quadratic is at most $1/4$, so $\max\psi = 1/8$ at $(1/2, 1/2)$. The barrier need not vanish on every edge; its nonnegative boundary values are sufficient for the comparison.

Step 2: Compare the error in both directions. Put $e = u - v$, so $\Delta e = -\Delta v$. For $w_+ = e - \varepsilon - \delta\psi$ and $w_- = -e - \varepsilon - \delta\psi$,

$$\Delta w_+ = -\Delta v + \delta \geq 0, \quad \Delta w_- = \Delta v + \delta \geq 0.$$

On the boundary, $|e| = |g - v| \leq \varepsilon$ and $\psi \geq 0$, so both $w_{\pm} \leq 0$. The stated maximum principle yields

$$|u - v| \leq \varepsilon + \delta\psi(x, y) \leq \varepsilon + \delta/8.$$

This is an analytic bound over the entire square, not an inference from sampled residuals.

Step 3: Compare the certificate with the exact error. The exact harmonic solution for $g = x + y$ is $u = x + y$. The added factor vanishes on every edge, so $\varepsilon = 0$. Direct differentiation gives

$$\Delta v = -2a[x(1-x) + y(1-y)], \quad \|\Delta v\|_{\infty} = |a|.$$

Thus the certificate gives $\|u - v\|_{\infty} \leq |a|/8$. The exact error is $|a|x(1-x)y(1-y)|$, with maximum $|a|/16$ at the center. The certificate is valid and overestimates this example's maximum by a factor of two.

Step 4: Identify what must be checked and what is sharp. The field $v = u + M$ has zero residual for every constant M , yet its error is $|M|$ everywhere. Without a boundary-error check, an arbitrarily inaccurate temperature can pass the PDE test. Taking $|M| = \varepsilon$ and $\delta = 0$ attains the bound $\|u - v\|_{\infty} = \varepsilon$, proving that coefficient one is sharp. The polynomial example does not attain the residual coefficient $1/8$ and does not prove it sharp. The barrier supplies a rigorous convenient certificate without a claim of optimality.