

Question 1

Let $\kappa, L > 0$ and $A, B, C \in \mathbb{R}$. Solve

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad u(0, t) = A, \quad u(L, t) = B,$$

with $u(x, 0) = A + (B - A)x/L + C \sin(\pi x/L)$. Define the rightward diffusive flux as $j = -\kappa u_x$; the outward flux at the right endpoint is $j(L, t)$.

Tasks

1. Determine the unique stationary temperature and explain why it need not have zero flux.
2. Subtract that stationary field, solve the transformed initial-boundary problem, and verify the reconstructed solution against every original datum.
3. Take $L = \kappa = 1$, $A = 1$, $B = 3$, $C = 2$. Determine when the right reservoir receives heat from the rod and when it supplies heat to the rod. Find the exact switching time.
4. Compute total heat and its flux balance in this example. Sketch profiles at $t = 0, 0.08, 0.3$, together with the stationary field, and explain why the hotter endpoint can initially receive heat.

Question 1 – Solution

Strategy. Separate the maintained stationary gradient from the transient temperature excess.

Step 1: Find the stationary conduction state. Stationarity requires $S'' = 0$, and the two boundary values give

$$S(x) = A + (B - A)x/L, \quad j_S = -\kappa(B - A)/L.$$

This state is unique because a linear function is determined by two endpoint values. Unless $A = B$, it carries a constant nonzero flux. Stationary temperature means zero local storage rate, not an absence of heat transport.

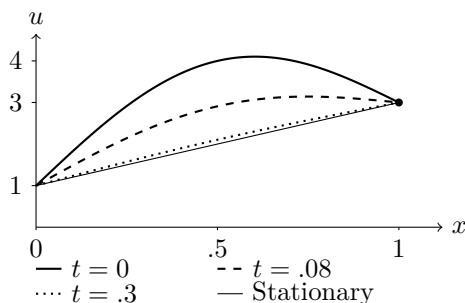
Step 2: Solve the homogeneous-boundary transient. With $v = u - S$, $v_t = \kappa v_{xx}$, $v(0, t) = v(L, t) = 0$ and $v(x, 0) = C \sin(\pi x/L)$. Thus, writing $\lambda = \kappa\pi^2/L^2$,

$$u(x, t) = S(x) + Ce^{-\lambda t} \sin(\pi x/L).$$

Since $S'' = 0$, differentiation verifies the PDE. The sine vanishes at both ends, and at $t = 0$ the original initial field is recovered exactly.

Step 3: Determine the direction of endpoint heat flow. For the given numbers, $u_x(1, t) = 2 - 2\pi e^{-\pi^2 t}$, so $j(1, t) = -2 + 2\pi e^{-\pi^2 t}$. The right reservoir receives heat when $j > 0$, that is $0 \leq t < t_* := \log \pi / \pi^2$. At t_* the right flux vanishes; for $t > t_*$ it supplies heat to the rod. The eventual rightward flux is -2 .

Step 4: Verify the global balance and the profiles. Here $H(t) = \int_0^1 u dx = 2 + (4/\pi)e^{-\pi^2 t}$, so $H' = -4\pi e^{-\pi^2 t}$. Directly $j(0, t) = -2 - 2\pi e^{-\pi^2 t}$ and $j(1, t) = -2 + 2\pi e^{-\pi^2 t}$, giving $H' = j(0, t) - j(1, t)$. Initially the interior exceeds the right endpoint temperature, so heat can leave through that endpoint despite its being hotter than the left one.



Question 2

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ and is initially at zero temperature in $L^2(0, 1)$. At $t = 0$ the left endpoint is raised to one and held there, while the right remains zero: $u(0, t) = 1$, $u(1, t) = 0$ for $t > 0$. You may use Fourier sine convergence, Parseval, and $\int_0^\infty e^{-\pi^2 s^2} ds = 1/(2\sqrt{\pi})$.

Tasks

1. Subtract the stationary field, compute the sine coefficients of the transformed initial data and construct the full solution.
2. Justify the PDE for positive time and the L^2 initial trace. State the initial limit at interior points and at the left endpoint; explain why uniform convergence to the zero initial field is impossible.
3. Derive a convergent total-heat formula and prove that total heat increases for every positive time, although the source-free PDE has no interior forcing.
4. Find the rightward flux entering at $x = 0$ and its leading behavior as $t \downarrow 0$. Use a Riemann-sum limit to justify the singular rate rather than evaluating a divergent series at $t = 0$.

Question 2 – Solution

Strategy. The sudden boundary change creates an initial corner mismatch; subtracting the stationary profile isolates its decaying sine expansion.

Step 1: Construct the boundary step response. The stationary field is $S = 1 - x$. For $v = u - S$, the endpoints are zero and $v(x, 0) = x - 1$. Integration gives $2 \int_0^1 (x - 1) \sin(n\pi x) dx = -2/(n\pi)$. Hence

$$u(x, t) = 1 - x - \frac{2}{\pi} \sum_{n \geq 1} \frac{e^{-n^2 \pi^2 t} \sin(n\pi x)}{n}.$$

Step 2: Verify the appropriate initial trace. For $t \geq \tau > 0$, Gaussian decay makes every finitely differentiated transient series uniformly convergent. It satisfies the heat equation and zero endpoint values; adding S gives the required boundary data. The square-summable coefficients and Parseval prove $v \rightarrow x - 1$ in L^2 , hence $u \rightarrow 0$ in L^2 . At each interior point, the Fourier limit is zero after reconstruction; Gaussian damping preserves that limit by summation by parts. At $x = 0$, the positive-time value is always one, so its limit is one. Uniform convergence to zero on the closed interval is therefore impossible. No corner-continuous solution is claimed.

Step 3: Integrate the boundary-driven heating. For positive time,

$$H(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-n^2 \pi^2 t}}{n^2}, \quad H'(t) = 4 \sum_{n \text{ odd}} e^{-n^2 \pi^2 t} > 0.$$

The initial norm trace gives $H(0+) = 0$ and the final heat is $1/2$. Differentiating the spatial series gives $H' = [u_x]_0^1 = j(0, t) - j(1, t)$, where $j = -u_x$. The endpoint supplies heat; no interior source is needed.

Step 4: Quantify the singular initial input. At the left endpoint,

$$j(0, t) = 1 + 2 \sum_{n \geq 1} e^{-n^2 \pi^2 t}.$$

With $h = \sqrt{t}$, the decreasing, integrable Gaussian gives $h \sum_{n \geq 1} e^{-\pi^2 (nh)^2} \rightarrow \int_0^\infty e^{-\pi^2 s^2} ds$ by upper/lower integral bounds. Thus $\sqrt{t} j(0, t) \rightarrow 1/\sqrt{\pi}$, or $j(0, t) \sim 1/\sqrt{\pi t}$. The singular rate is integrable in time and is consistent with vanishing initial heat. Substituting $t = 0$ into the differentiated series would be invalid.

Question 3

For $\kappa, L, Q > 0$, consider uniform internal heating with unequal fixed endpoints:

$$u_t = \kappa u_{xx} + Q, \quad u(0, t) = A, \quad u(L, t) = B, \quad u(x, 0) = \ell(x) := A + (B - A)x/L.$$

You may use the sine expansion of a smooth zero-endpoint function.

Tasks

1. Determine the stationary field and find exactly when its maximum lies strictly inside the rod. Locate that maximum.
2. Subtract the stationary field, compute the transient coefficients and construct the solution satisfying the original linear initial profile.
3. Verify the reconstructed PDE and data, explain its initial and positive-time regularity, and check the stationary heat-flux balance against the source integral.
4. For $L = \kappa = 1$, $A = 1$, $B = 2$, $Q = 4$, determine the stationary maximum and sketch the initial field, the profiles at $t = 0.02, 0.1$, and the stationary field. Explain why the maximum may exceed both boundary values.

Question 3 – Solution

Strategy. Subtract a stationary solution of the forced equation, not just the line joining its boundary values.

Step 1: Solve the stationary forced problem. The equation $\kappa S'' + Q = 0$ and the endpoint data give

$$S = \ell + \frac{Q}{2\kappa}x(L-x).$$

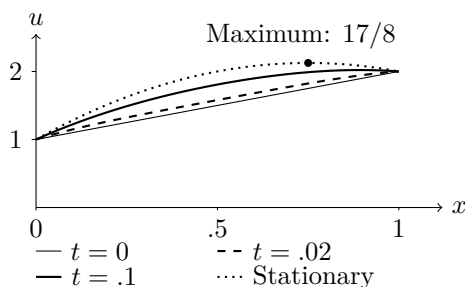
It is strictly concave, with critical point $x_* = L/2 + \kappa(B-A)/(QL)$. This is strictly interior exactly when $|B-A| < QL^2/(2\kappa)$. If equality holds, the maximum is at an endpoint; otherwise the maximum is at the endpoint in the direction of the larger value.

Step 2: Construct the transient from the actual initial difference. For $v = u - S$, the PDE and boundaries are homogeneous and $v(x, 0) = -Qx(L-x)/(2\kappa)$. Two integrations by parts give the nonzero coefficients $-4QL^2/(\kappa\pi^3n^3)$ for odd n . Therefore

$$u = S - \frac{4QL^2}{\kappa\pi^3} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L)}{n^3}.$$

Step 3: Check the PDE, trace and maintained fluxes. The transient solves the homogeneous heat equation; $\kappa S'' + Q = 0$ then verifies the forced equation for u . The transient vanishes at the ends. Summable coefficients and Fourier convergence recover $u(\cdot, 0) = \ell$ uniformly. Gaussian decay permits differentiation for $t > 0$. The initial linear profile has $\kappa\ell'' + Q = Q$, inconsistent with a zero endpoint time derivative at an initial corner, so full corner smoothness is not asserted. For $j_S = -\kappa S'$, one has $j_S(L) - j_S(0) = QL$. Hence the steady balance $0 = j_S(0) - j_S(L) + QL$ accounts for all generated heat.

Step 4: Interpret the internally heated profile. In the example, $S = 1 + x + 2x(1-x)$, $x_* = 3/4$ and $S(x_*) = 17/8$. The final maximum exceeds both boundary values because heat is generated inside the rod. The source-free maximum principle cannot be applied unchanged to this forced equation. The transient decays uniformly toward the plotted stationary field.



Question 4

On $0 < x < 1$, let the storage and conductivity vary so that

$$\frac{u_t}{1+x} = ((1+x)u_x)_x, \quad u(0,t) = 1, \quad u(1,t) = 3.$$

Define $j = -(1+x)u_x$, $\ell = \log 2$, and let the initial field be

$$f(x) = 1 + \frac{2 \log(1+x)}{\ell} + \sin\left(\frac{\pi \log(1+x)}{\ell}\right).$$

Tasks

1. Derive the stationary temperature and stationary flux. Explain why the straight line $1 + 2x$ is not stationary for this equation.
2. Use $s = \log(1+x)$ to transform the entire PDE and its endpoints. Construct and verify the full solution for the given initial field.
3. Compute the stored quantity $H(t) = \int_0^1 u(x,t)/(1+x) dx$ and verify $H' = j(0,t) - j(1,t)$.
4. For the error $v = u - S$, derive the weighted energy identity. Evaluate its weighted squared norm for this solution and explain why the weight cannot simply be omitted from the identity.

Question 4 – Solution

Strategy. The stationary gradient and the natural storage norm both follow from the variable coefficients.

Step 1: Integrate constant stationary flux. The stationary equation is $((1+x)S')' = 0$, so $(1+x)S' = C$. The endpoint difference gives $C = 2/\ell$, hence

$$S = 1 + 2 \log(1+x)/\ell, \quad j_S = -2/\ell.$$

For $1+2x$, the quantity $((1+x)u_x)' = 2$ is nonzero, so it is not stationary. Equal endpoint values do not make different interior profiles solve the same PDE.

Step 2: Transform and solve the full problem. Write $U(s, t) = u(e^s - 1, t)$ on $0 < s < \ell$. Then $u_x = U_s/(1+x)$ and $((1+x)u_x)_x = U_{ss}/(1+x)$. The transformed equation is $U_t = U_{ss}$, with endpoints one and three. Subtracting $1 + 2s/\ell$ gives one sine mode. With $\lambda = \pi^2/\ell^2$,

$$u = S + e^{-\lambda t} \sin(\pi \log(1+x)/\ell).$$

The chain-rule identities verify the original PDE, and the sine gives zero endpoint correction and exactly the prescribed initial field.

Step 3: Check storage against boundary flux. Since $dx/(1+x) = ds$,

$$H(t) = 2\ell + \frac{2\ell}{\pi} e^{-\lambda t}, \quad H' = -\frac{2\pi}{\ell} e^{-\lambda t}.$$

Also $j = -U_s = -2/\ell - (\pi/\ell)e^{-\lambda t} \cos(\pi s/\ell)$. Its left value minus its right value equals the displayed derivative. The unweighted integral of u is not the stored quantity in this model.

Step 4: Derive the weighted error dissipation. Multiplying $v_t/(1+x) = ((1+x)v_x)_x$ by v and integrating, with zero error at both ends, gives

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \frac{v^2}{1+x} dx = - \int_0^1 (1+x) v_x^2 dx.$$

For this mode the squared norm is $(\ell/2)e^{-2\lambda t}$ and the right-hand integral is $(\pi^2/(2\ell))e^{-2\lambda t}$, agreeing with the identity. The weight is the storage coefficient; dropping it changes the energy calculation.

Question 5

Let both endpoint temperatures rise at a constant rate $r > 0$:

$$u_t = \kappa u_{xx}, \quad 0 < x < L, \quad u(0, t) = A + rt, \quad u(L, t) = B + rt,$$

where $\kappa, L > 0$ and $u(x, 0) = \ell(x) := A + (B - A)x/L$. You may use the sine expansion of $x(L - x)$.

Tasks

1. Subtract the instantaneous boundary line $\ell(x) + rt$ and derive the PDE and initial data for the remaining field. Explain why that line alone is not a solution.
2. Find a time-independent zero-endpoint correction $P(x)$ so that $\ell + rt + P$ solves the PDE. Add the transient needed to recover the original initial field.
3. Verify the reconstructed solution, justify the initial trace and positive-time differentiations, and determine its limiting offset relative to the moving boundary line.
4. Compute the limiting midpoint lag, the limiting mean lag and the eventual total-heat growth rate. Explain why there is no stationary equilibrium even though the offset approaches a fixed shape.

Question 5 – Solution

Strategy. A boundary lifting with nonzero time derivative creates a source in the transformed equation; a persistent lag balances that source.

Step 1: Keep the source created by the lifting. For $w = u - \ell - rt$,

$$w_t = \kappa w_{xx} - r, \quad w(0, t) = w(L, t) = 0, \quad w(x, 0) = 0.$$

The line $\ell + rt$ has time derivative r and zero second spatial derivative, so it fails the original source-free PDE when $r > 0$.

Step 2: Find the lag profile and the initial correction. A fixed P must satisfy $\kappa P'' = r$, $P(0) = P(L) = 0$, giving $P = -rx(L-x)/(2\kappa)$. The remaining transient has initial data $-P$. Its odd sine coefficients are $4rL^2/(\kappa\pi^3n^3)$. Thus

$$u = \ell + rt + P + \frac{4rL^2}{\kappa\pi^3} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L)}{n^3}.$$

Step 3: Verify all data and the moving-frame limit. The field $\ell + rt + P$ satisfies $u_t = \kappa u_{xx} = r$; its added transient satisfies the homogeneous heat equation. All correction terms vanish at the endpoints. At $t = 0$, the summable sine expansion equals $-P$ uniformly and recovers ℓ . For $t \geq \tau > 0$, the exponential transient permits all termwise derivatives. Initial corner smoothness is not implied: the boundary time derivative is r , whereas $\kappa \ell'' = 0$. As $t \rightarrow \infty$, the transient tends uniformly to zero, so $u - \ell - rt \rightarrow P$ uniformly.

Step 4: Quantify the persistent lag and heat input. The midpoint offset is $P(L/2) = -rL^2/(8\kappa)$, and the mean offset is $L^{-1} \int P = -rL^2/(12\kappa)$. Their negatives are the positive lags behind the boundary line. Since $P'(L) - P'(0) = rL/\kappa$, the eventual flux balance is $H' = \kappa[u_x]_0^L \rightarrow rL$, also obtained by differentiating the integrated solution. Boundary values grow indefinitely; it is the offset, not the absolute temperature, that approaches a fixed state.

Question 6

A semi-infinite rod has $u_t = \kappa u_{xx}$ on $x > 0$, with $\kappa, \omega > 0$ and a maintained oscillating boundary $u(0, t) = A + B \cos(\omega t)$, where $A > |B| > 0$. Seek its periodic response of the form

$$u(x, t) = A + Be^{-\alpha x} \cos(\omega t - \alpha x), \quad \alpha > 0,$$

with $u(x, t) \rightarrow A$ as $x \rightarrow \infty$. The initial field is to be the trace of this periodic response, so no additional startup transient is requested. Use the rightward flux convention $j = -\kappa u_x$.

Tasks

1. Determine α by direct substitution, and state the initial field that makes the response a solution of the full initial-boundary problem.
2. Derive the amplitude attenuation and phase delay at distance x . Find the depth where the oscillation amplitude falls to $1/e$ of its boundary value, and express it in terms of the period $P = 2\pi/\omega$.
3. Compute the boundary flux and its phase relative to the imposed temperature. Explain why constant far-field temperature does not imply zero instantaneous boundary flux.
4. Prove positivity of the temperature and approach to the far-field value as $x \rightarrow \infty$, uniformly in time. For $A = 2, B = 1, \kappa = 1, \omega = 2$, sketch profiles at phases $\omega t = 0, \pi/2, \pi$.

Question 6 – Solution

Strategy. A periodic boundary signal diffuses inward with both amplitude attenuation and a spatial phase shift.

Step 1: Match the PDE and the initial trace. Put $\theta = \omega t - \alpha x$. Then $u_t = -\omega B e^{-\alpha x} \sin \theta$ and $u_{xx} = -2\alpha^2 B e^{-\alpha x} \sin \theta$. Thus $\alpha = \sqrt{\omega/(2\kappa)}$. The boundary trace is the imposed cosine, and the compatible initial field is $f(x) = A + B e^{-\alpha x} \cos(\alpha x)$. It is generally not the constant A ; a constant initial field would require an additional transient.

Step 2: Quantify attenuation and delay. The local oscillation amplitude is $|B|e^{-\alpha x}$ and its phase lag is αx . Relative to the boundary waveform, the time delay is $\alpha x/\omega$, understood modulo the period. The penetration depth is

$$d = 1/\alpha = \sqrt{2\kappa/\omega} = \sqrt{\kappa P/\pi}.$$

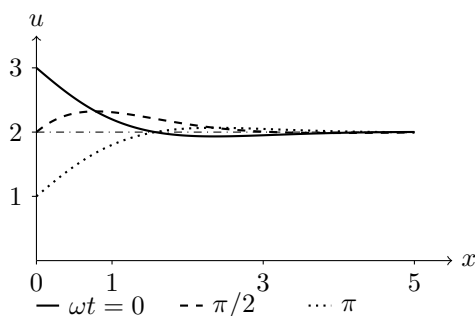
Longer periods penetrate farther, with depth proportional to the square root of the period rather than the period itself.

Step 3: Determine the phase of the boundary flux. Differentiation gives $u_x = \alpha B e^{-\alpha x} (-\cos \theta + \sin \theta)$. Hence

$$j(0, t) = B\sqrt{\kappa\omega} \cos(\omega t + \pi/4).$$

The flux leads the boundary temperature oscillation by $\pi/4$ in phase and has zero average over a full period. Instantaneous storage and release near the driven boundary are compatible with a constant far-field limit.

Step 4: Verify bounds and interpret the profiles. For all $x \geq 0, t$, $|u - A| \leq |B|e^{-\alpha x} \leq |B|$, so $u \geq A - |B| > 0$. Also $\sup_t |u(x, t) - A| \leq |B|e^{-\alpha x} \rightarrow 0$ as $x \rightarrow \infty$. This is a spatial far-field limit uniform in time, not decay of the maintained oscillation at a fixed point as time increases. For the plotted parameters, $\alpha = 1$ and the penetration depth is one.



Question 7

On $0 < x < 1$, let $u_t = u_{xx}$ and $u(x, 0) = A$. The right endpoint stays at A . The left is held at $A + B$ for $0 < t < \tau$ and returns to A for $t \geq \tau$, where $B, \tau > 0$. You may use the unit boundary-step response

$$S(x, t) = 1 - x - \frac{2}{\pi} \sum_{n \geq 1} \frac{e^{-n^2 \pi^2 t} \sin(n \pi x)}{n}, \quad t > 0,$$

which has zero initial interior trace, left boundary one and right boundary zero. Set $S(x, 0) = 0$ for $0 < x < 1$ when discussing the switching time.

Tasks

1. Construct the solution before and after the switch by superposing boundary step responses. Verify the boundary traces on both open time intervals.
2. Derive a homogeneous-boundary sine series for $t > \tau$, with coefficients at time τ . Explain continuity at the switch for interior points and why continuity at the switching boundary corner is impossible.
3. Compute the total excess heat after the switch and prove that it strictly decreases. Determine the long-time equilibrium.
4. For $A = B = 1$, $\tau = 0.1$, sketch profiles at $t = 0.03, 0.08, 0.13, 0.3$. Explain why the rod does not return instantly to its initial field when the boundary pulse ends.

Question 7 – Solution

Strategy. Switching off a boundary step is a delayed negative step; its instantaneous boundary effect does not erase the stored interior heat.

Step 1: Superpose the two boundary events. Linearity gives

$$u = A + BS(x, t) \quad (0 < t < \tau), \quad u = A + B[S(x, t) - S(x, t - \tau)] \quad (t > \tau).$$

Each term solves the heat equation in its time interval. Before the switch, the left trace is $A + B$; afterward the two unit left traces cancel and leave A . Both right traces are always A . The initial interior trace is A .

Step 2: Identify the post-switch initial data. The stationary lines cancel after the switch, leaving

$$u - A = \frac{2B}{\pi} \sum_{n \geq 1} \frac{1 - e^{-n^2 \pi^2 \tau}}{n} e^{-n^2 \pi^2 (t - \tau)} \sin(n \pi x).$$

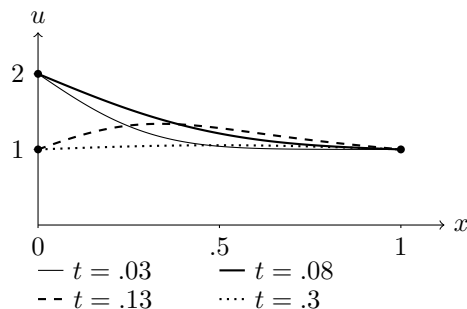
These are the sine coefficients of $BS(x, \tau)$ on the open interval. For every interior x , $S(x, t - \tau) \rightarrow 0$ as $t \downarrow \tau$, so the interior trace agrees with the pre-switch value. The left boundary itself jumps from $A + B$ to A ; no single continuous trace at that corner can match both. For times strictly after the switch, all differentiated series converge uniformly by Gaussian decay. The switching trace is also recovered in L^2 .

Step 3: Track the retained heat. For $t > \tau$, integration yields

$$H_{\text{ex}}(t) = \frac{4B}{\pi^2} \sum_{n \text{ odd}} \frac{1 - e^{-n^2 \pi^2 \tau}}{n^2} e^{-n^2 \pi^2 (t - \tau)}.$$

Its derivative is $-4B \sum_{n \text{ odd}} (1 - e^{-n^2 \pi^2 \tau}) e^{-n^2 \pi^2 (t - \tau)} < 0$. This equals $[u_x]_0^1$ after the switch. At any later fixed starting time the exponentially damped coefficients are summable, which proves uniform convergence to the equilibrium $u = A$ as $t \rightarrow \infty$.

Step 4: Interpret the finite pulse. The profiles before the switch have left value two; those after it have left value one. The right value is one throughout. At the switch the excess heat is $B \int_0^1 S(x, \tau) dx > 0$, as follows from the positive derivative of step-response heat and its zero initial value. It must subsequently diffuse out. A changed boundary value is not an instantaneous reset of the interior.



Question 8

Consider $u_t = u_{xx}$ on $0 < x < 1$, with

$$u(0, t) = e^t, \quad u(1, t) = e^t \cosh 1, \quad u(x, 0) = \cosh x.$$

Compare the affine boundary lifting $\ell(x, t) = e^t[1 + (\cosh 1 - 1)x]$ with the nonaffine lifting $P(x, t) = e^t \cosh x$. You may use the sine expansion of a smooth function vanishing at both endpoints.

Tasks

1. Verify a closed-form solution of the original problem using P , including the PDE and every datum.
2. For $v = u - \ell$, derive the homogeneous-boundary forced PDE and its initial data. Explain why setting $v = 0$ would give the wrong physical field.
3. Set $k_n = n\pi$ and $D_n = 1 - (-1)^n \cosh 1$. Compute the sine coefficients of the initial difference and the transformed source. Solve the resulting modal initial-value equations.
4. Reconstruct u from the modal solution and prove that it agrees with the closed form. Explain why different liftings can produce different source terms without producing different solutions of the original problem.

Question 8 – Solution

Strategy. A convenient lifting can solve the PDE as well as its boundary traces; a less convenient lifting is still valid if every transformed term is retained.

Step 1: Verify the exact nonaffine solution. The field $u = e^t \cosh x$ has $u_t = u_{xx} = e^t \cosh x$. Its two endpoint values and its initial trace are exactly the prescribed ones. With lifting P , the remaining field is zero, with zero source and initial data.

Step 2: Transform all parts under the affine lifting. Write $p(x) = 1 + (\cosh 1 - 1)x$, so $\ell = e^t p$. Since $\ell_{xx} = 0$,

$$v_t - v_{xx} = -e^t p(x), \quad v(0, t) = v(1, t) = 0, \quad v(x, 0) = \cosh x - p(x).$$

The zero field fails both the interior source equation and the initial trace. The affine lifting satisfies the boundary values but is not itself a heat solution.

Step 3: Solve the transformed modal equations. Direct integration gives

$$2 \int_0^1 p(x) \sin(k_n x) dx = \frac{2D_n}{k_n}, \quad 2 \int_0^1 \cosh x \sin(k_n x) dx = \frac{2k_n D_n}{1 + k_n^2}.$$

Thus the initial coefficient is $c_n = -2D_n/[k_n(1 + k_n^2)]$. The modal equation is $v'_n + k_n^2 v_n = -(2D_n/k_n)e^t$, $v_n(0) = c_n$. Since $(1 + k_n^2)c_n = -2D_n/k_n$, its solution is $v_n(t) = c_n e^t$. No additional decaying term is needed for these data.

Step 4: Reconstruct the same physical temperature. The coefficients $c_n = O(n^{-3})$ are summable. Their sine series equals the smooth zero-endpoint function $\cosh x - p(x)$ uniformly on $[0, 1]$. Consequently

$$u = \ell + e^t \sum_{n \geq 1} c_n \sin(k_n x) = e^t p + e^t (\cosh x - p) = e^t \cosh x.$$

This identity verifies the reconstructed PDE directly without claiming absolute uniform convergence of every differentiated sine series at the boundary. Different liftings change the auxiliary source and initial data together; retaining both transformations preserves the original solution.

Question 9

A rod on $0 < x < 1$ obeys $u_t = u_{xx}$ with unknown constant boundary temperatures A, B . It is known that its full field has the form

$$u(x, t) = A + (B - A)x + Ce^{-\pi^2 t} \sin(\pi x).$$

Sensors at $x = 1/3, 2/3$ record $y_L(t), y_R(t)$. Set $m(t) = (y_L(t) + y_R(t))/2$ and let $\Delta > 0$ be known.

Tasks

1. Derive which combination of the readings identifies the stationary gradient immediately, even before the transient has decayed.
2. Use readings at times $0, \Delta$ to recover A, B, C uniquely. State the consistency condition on the differences of the two pairs of readings.
3. For $\Delta = (\log 2)/\pi^2$, use readings $(8/3, 10/3)$ at zero and $(13/6, 17/6)$ at Δ to find the complete field and verify the readings and boundary values.
4. Explain why one pair of simultaneous readings cannot identify all three parameters. For small Δ , quantify amplification of errors in the two averaged readings when estimating the mean of the boundary temperatures.

Question 9 – Solution

Strategy. Equal sine values at the two sensors let their difference cancel the transient; a second observation time separates the remaining steady and decaying parts.

Step 1: Cancel the equal transient values. At both sensor positions, $\sin(\pi x) = \sqrt{3}/2$. Thus

$$d := B - A = 3(y_R - y_L), \quad m(t) = S + Ke^{-\pi^2 t}, \quad S = (A + B)/2, \quad K = \sqrt{3}C/2.$$

The difference determines the stationary gradient at any measurement time. It must be unchanged between the two observations if the assumed model holds.

Step 2: Invert the two-time averages. Put $r = e^{-\pi^2 \Delta} \in (0, 1)$. From $m_0 = S + K$, $m_\Delta = S + rK$,

$$K = \frac{m_0 - m_\Delta}{1 - r}, \quad S = \frac{m_\Delta - rm_0}{1 - r}, \quad A = S - d/2, \quad B = S + d/2, \quad C = 2K/\sqrt{3}.$$

The denominator is nonzero because $\Delta > 0$. Equality of the observed differences is the only additional consistency condition; with it these parameters reconstruct all four readings under the stated model.

Step 3: Reconstruct the numerical example. Here $r = 1/2$, $d = 2$, $m_0 = 3$ and $m_\Delta = 5/2$. Thus $S = 2$, $K = 1$ and

$$A = 1, \quad B = 3, \quad C = 2/\sqrt{3}, \quad u = 1 + 2x + \frac{2}{\sqrt{3}}e^{-\pi^2 t} \sin(\pi x).$$

The endpoint values are one and three. The sensor transients are one at zero and one half at Δ , giving exactly the four supplied readings. The linear stationary part and the decaying sine verify $u_t = u_{xx}$.

Step 4: State the identifiability and conditioning limits. A single pair gives d and $S + K$, leaving infinitely many choices of S, K . For two times, errors e_0, e_Δ in the averages cause $\delta S = (e_\Delta - re_0)/(1 - r)$. If $|e_0|, |e_\Delta| \leq \eta$,

$$|\delta S| \leq \frac{1 + r}{1 - r} \eta \sim \frac{2\eta}{\pi^2 \Delta} \quad (\Delta \downarrow 0).$$

Opposite-signed errors attain the bound. Very close observations are therefore poorly conditioned even though the noiseless parameters are unique.

Question 10

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, with fixed endpoint temperatures A, B , $\kappa, L > 0$, and stationary field $S = A + (B - A)x/L$. For smooth compatible data define the error energy $E_v = \frac{1}{2} \int_0^L (u - S)^2 dx$ and the raw quadratic quantity $E_u = \frac{1}{2} \int_0^L u^2 dx$. You may use the zero-endpoint Poincare inequality $\int v_x^2 \geq (\pi/L)^2 \int v^2$ and the parabolic maximum principle.

Tasks

1. Derive the energy identities for E_v and E_u , retaining all boundary terms. Prove the sharp bound $E_v(t) \leq e^{-2\lambda t} E_v(0)$, where $\lambda = \kappa \pi^2 / L^2$.
2. For $A = B = 1$ and $u(x, 0) = 1 - \varepsilon \sin(\pi x / L)$, $0 < \varepsilon \leq 1$, construct the solution. Prove that its raw quadratic quantity increases while its error energy decreases.
3. If $|u(x, 0) - S(x)| \leq C \sin(\pi x / L)$, $C > 0$, use comparison to prove a pointwise error bound for all later times. Give a sufficient time for uniform temperature error at most δ , where $0 < \delta < C$.
4. Give a sufficient time for reducing error energy to a fraction $\eta \in (0, 1)$ of its initial value, and explain why neither total heat nor raw quadratic energy alone measures distance from a maintained nonzero equilibrium.

Question 10 – Solution

Strategy. Dissipation applies naturally to deviations from the maintained equilibrium; the boundary reservoirs can increase the unshifted quantities.

Step 1: Identify which energy has no boundary work. For $v = u - S$, the endpoints vanish and $v_t = \kappa v_{xx}$. Therefore

$$E'_v = -\kappa \int_0^L v_x^2 dx \leq -2\lambda E_v, \quad E_v(t) \leq e^{-2\lambda t} E_v(0).$$

A first-sine error attains equality, so the rate is sharp. For the raw quantity,

$$E'_u = \kappa [uu_x]_0^L - \kappa \int_0^L u_x^2 dx.$$

The first term need not vanish with nonzero maintained boundary values.

Step 2: Exhibit simultaneous raw growth and error decay. The stated initial data give $u = 1 - \varepsilon e^{-\lambda t} \sin(\pi x/L)$. Then

$$E_u = \frac{L}{2} - \frac{2\varepsilon L}{\pi} e^{-\lambda t} + \frac{\varepsilon^2 L}{4} e^{-2\lambda t}, \quad E_v = \frac{\varepsilon^2 L}{4} e^{-2\lambda t}.$$

Writing $z = \varepsilon e^{-\lambda t} \in (0, 1]$ gives $E'_u = \lambda L z(2/\pi - z/2) > 0$, since $2/\pi > 1/2$. Meanwhile $E'_v = -2\lambda E_v < 0$. Heating by the reservoirs moves this field closer to equilibrium while increasing its raw quadratic quantity.

Step 3: Establish a pointwise stopping criterion. The barriers $\pm C e^{-\lambda t} \sin(\pi x/L)$ solve the homogeneous zero-endpoint heat equation and bound the initial error. Applying comparison to their differences with v on any finite time interval gives

$$|u - S| \leq C e^{-\lambda t} \sin(\pi x/L) \leq C e^{-\lambda t}.$$

Thus $t \geq \lambda^{-1} \log(C/\delta)$ suffices for uniform error at most δ . This conclusion uses the supplied pointwise initial bound, not merely an initial L^2 norm.

Step 4: Distinguish the two stopping criteria. The energy inequality gives $E_v(t)/E_v(0) \leq \eta$ once $t \geq (2\lambda)^{-1} \log(1/\eta)$ for nonzero initial error; zero error is already at equilibrium. The factor of two comes from squaring a decaying amplitude. Total heat can conceal spatial differences, and raw energy includes the maintained background and boundary work. Neither is a substitute for an explicit norm of $u - S$ when measuring equilibration.