

Question 1

Solve $u_t = u_{xx}$ for $0 < x < \pi$, $t > 0$, with

$$u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = \sin x + \frac{1}{2} \sin(2x).$$

The temperature is measured relative to the zero endpoint temperature.

Tasks

1. Construct the solution from the initial modes and verify the PDE, both endpoint conditions and the initial trace.
2. Prove directly from the formula that the initial temperature and every positive-time temperature are nonnegative. Does this force the temperature at every fixed interior point to decrease?
3. At $x = 5\pi/6$, determine the initial rate of change and the exact time of the unique temperature maximum. Justify the sign change of the time derivative.
4. Compute the total heat $H(t) = \int_0^\pi u(x, t) dx$ and its boundary-flux balance. Sketch profiles at $t = 0, 1/5, 1$ and explain how local warming is consistent with net heat loss.

Question 1 – Solution

Strategy. Different spatial modes decay at different rates; local temperature and total heat need not have the same monotonicity.

Step 1: Evolve each initial mode. Since $(\sin nx)'' = -n^2 \sin nx$, the solution is

$$u = e^{-t} \sin x + \frac{1}{2} e^{-4t} \sin 2x.$$

Termwise differentiation of this finite sum gives $u_t = u_{xx}$. Both sines vanish at the endpoints, and setting $t = 0$ gives the prescribed field. The formula is smooth on the closed space interval for all $t \geq 0$.

Step 2: Prove nonnegativity without a plot. Factor the solution as $u = e^{-t} \sin x [1 + e^{-3t} \cos x]$. For $0 \leq x \leq \pi$ and $t \geq 0$, $\sin x \geq 0$ and the bracket is nonnegative. At positive time the bracket is strictly positive; hence the interior temperature is positive. This sign statement concerns values, not their time derivatives. Diffusion can transport heat into a cooler interior region.

Step 3: Find the local warming interval. At $x = 5\pi/6$,

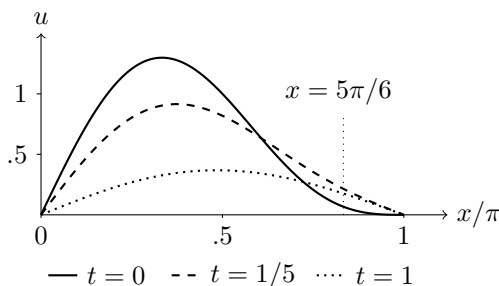
$$u = \frac{1}{2} e^{-t} - \frac{\sqrt{3}}{4} e^{-4t}, \quad u_t = -\frac{1}{2} e^{-t} + \sqrt{3} e^{-4t}.$$

Thus $u_t(x, 0) = \sqrt{3} - 1/2 > 0$. Factoring out e^{-4t} shows that the derivative changes sign exactly when $e^{3t} = 2\sqrt{3}$. The unique maximum occurs at

$$t_* = \frac{1}{3} \log(2\sqrt{3}).$$

Before t_* the derivative is positive; afterward it is negative. This gives a concrete counterexample to pointwise monotone cooling.

Step 4: Compare with the global heat balance. Integration gives $H(t) = 2e^{-t}$, since the second sine has zero integral. Also $u_x = e^{-t} \cos x + e^{-4t} \cos 2x$, so $H'(t) = [u_x]_0^\pi = -2e^{-t} < 0$. The right endpoint derivative minus the left is the signed flux balance obtained by integrating the PDE. Total heat falls even while a neighborhood of $5\pi/6$ initially warms.



Question 2

Let $\kappa, L > 0$ and solve $u_t = \kappa u_{xx}$ on $0 < x < L, t > 0$, with zero endpoint values and initial temperature $f(x) = x(L - x)$. You may use Fourier sine convergence for piecewise smooth odd extensions.

Tasks

1. Compute every coefficient $b_n = (2/L) \int_0^L f(x) \sin(n\pi x/L) dx$ and construct the solution series.
2. Justify uniform convergence to the initial trace as $t \downarrow 0$ and termwise verification of the PDE for $t > 0$. Do not infer corner smoothness from uniform convergence of the values.
3. Explain the absence of the even modes from reflection symmetry. Determine the total heat $H(t)$ as a convergent series and evaluate $H(0)$ directly.
4. Differentiate H for positive time and verify $H' = \kappa[u_x]_0^L$. Explain why H strictly decreases and why the solution is not a single separated product.

Question 2 – Solution

Strategy. Exact coefficients control the initial convergence, while Gaussian mode decay justifies later differentiation.

Step 1: Compute the sine coefficients. Put $k = n\pi/L$. Twice integrating by parts, using $f(0) = f(L) = 0$ and $f'' = -2$, gives $\int_0^L f \sin(kx) dx = 2(1 - (-1)^n)/k^3$. Thus $b_n = 4L^2(1 - (-1)^n)/(n^3\pi^3)$, and

$$u(x, t) = \frac{8L^2}{\pi^3} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sin(n\pi x/L)}{n^3} e^{-\kappa(n\pi/L)^2 t}.$$

Step 2: Verify the full initial-value solution. The coefficients are absolutely summable, so the series at $t = 0$ converges uniformly. Fourier convergence identifies its sum as f on $[0, L]$, including the zero endpoints. Dominated convergence of the coefficient sum then gives $\|u(\cdot, t) - f\|_\infty \rightarrow 0$. For $t \geq \tau > 0$, any fixed number of spatial or time derivatives only adds powers of n ; the factor $e^{-\kappa(n\pi/L)^2 \tau}$ makes these series uniformly summable. Hence $u_t = \kappa u_{xx}$ termwise for positive time. At an initial corner, $f'' = -2$ conflicts with the zero boundary time derivative; continuity of all second spatial derivatives through those corners is not claimed.

Step 3: Use symmetry and integrate the solution. Since $f(L - x) = f(x)$ and $\sin(n\pi(L - x)/L) = (-1)^{n+1} \sin(n\pi x/L)$, even coefficients vanish. Integrating the uniformly convergent series gives

$$H(t) = \frac{16L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^4}, \quad H(0) = \int_0^L x(L - x) dx = \frac{L^3}{6}.$$

Step 4: Check heat loss and the mode content. For $t > 0$, differentiation gives

$$H'(t) = -\frac{16\kappa L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t}}{n^2} = \kappa[u_x]_0^L.$$

The equality on the right follows by differentiating the spatial sine series and using the odd-mode cosine endpoint values. Every summand is positive, so $H' < 0$. Infinitely many nonzero coefficients decay at distinct rates; in particular the ratio of the third coefficient to the first changes with time. A single product would have fixed ratios wherever its time factor is nonzero.

Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$ with $\kappa, L > 0$, zero endpoint values, and $0 < a < L$, take the continuous tent-shaped initial temperature

$$f(x) = \begin{cases} x/a, & 0 \leq x \leq a, \\ (L-x)/(L-a), & a \leq x \leq L. \end{cases}$$

Its height is one. You may use Fourier convergence for continuous piecewise smooth functions with zero endpoint values.

Tasks

1. Derive all sine coefficients by splitting the integral at a , retaining the change in slope, and write the heat solution.
2. Justify the initial trace and positive-time smoothing even though f' jumps at a .
3. Show how the location a can be recovered from the ratio b_2/b_1 . State its allowed range and explain why the recovery is unique when $0 < a < L$.
4. For $L = \kappa = 1$, $a = 1/3$, simplify the coefficients, identify the missing modes, and sketch the initial tent and the profiles at $t = 0.01, 0.05, 0.15$.

Question 3 – Solution

Strategy. The corner in the initial profile appears as a slope jump in the coefficient calculation, not as a failure of the positive-time heat solution.

Step 1: Compute the contribution of the slope jump. With $k = n\pi/L$, integration by parts once gives $\int f \sin(kx) = k^{-1} \int f' \cos(kx)$ since f vanishes at both endpoints. Using $f' = 1/a$ to the left and $-1/(L-a)$ to the right gives

$$\int_0^L f \sin(kx) dx = \frac{L \sin(ka)}{a(L-a)k^2}.$$

Therefore

$$b_n = \frac{2L^2 \sin(n\pi a/L)}{\pi^2 n^2 a(L-a)}, \quad u = \sum_{n=1}^{\infty} b_n e^{-\kappa(n\pi/L)^2 t} \sin(n\pi x/L).$$

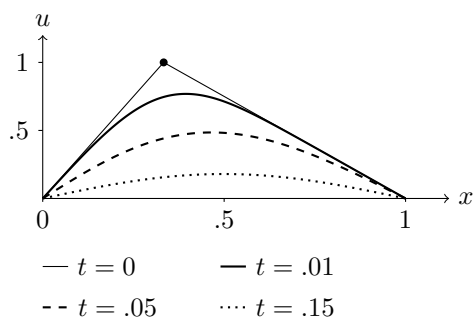
Step 2: Separate initial regularity from later regularity. Since $|b_n| \leq C/n^2$, the sine series converges absolutely and uniformly, and Fourier convergence identifies its initial sum as the continuous tent. Summable domination gives uniform convergence to f as $t \downarrow 0$. For every $\tau > 0$, polynomial factors from any finite number of derivatives are dominated by Gaussian decay for $t \geq \tau$. Thus the series is smooth for positive time, satisfies the PDE termwise and retains zero endpoints. Its initial derivative jump does not persist as a corner in a positive-time profile.

Step 3: Recover the peak location from two modes. Put $\theta = \pi a/L \in (0, \pi)$. Then $b_1 > 0$ and

$$\frac{b_2}{b_1} = \frac{\sin 2\theta}{4 \sin \theta} = \frac{1}{2} \cos \theta, \quad a = \frac{L}{\pi} \arccos(2b_2/b_1).$$

The ratio must lie strictly between $-1/2$ and $1/2$. Cosine is strictly decreasing on $(0, \pi)$, so this reconstruction is unique in the stated family.

Step 4: Specialize and interpret the profiles. For $L = \kappa = 1$, $a = 1/3$, $b_n = 9 \sin(n\pi/3)/(\pi^2 n^2)$. Exactly the multiples of three vanish, and $b_2/b_1 = 1/4$ recovers $a = 1/3$. The figure uses the exact piecewise initial trace and rapidly convergent positive-time series. The peak rounds as the profile diffuses toward zero.



Question 4

Solve $u_t = u_{xx}$ on $0 < x < \pi$, $t > 0$, with $u(0, t) = u(\pi, t) = 0$ and initial data in the L^2 sense

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2, \\ 1/2, & x = \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$$

The L^2 norm is $\|v\|_2^2 = \int_0^\pi |v|^2 dx$. You may use sine-series convergence at continuity points and to the average at a jump, and Parseval's identity.

Tasks

1. Compute the sine coefficients and construct the heat solution.
2. Prove that the solution is smooth and solves the PDE for positive time, and that it approaches f in L^2 as $t \downarrow 0$.
3. State the pointwise limit at the interior jump and at each boundary when boundary values are retained as zero. Explain why convergence cannot be uniform on $[0, \pi]$, even if the endpoints of f are defined to be zero.
4. Prove uniqueness among solutions smooth up to the spatial boundaries for $t > 0$ and continuous into L^2 at $t = 0$, with these zero endpoint values. Use an energy estimate starting at $t = \delta > 0$ and then let $\delta \downarrow 0$.

Question 4 – Solution

Strategy. Interpret the initial condition in its stated norm; incompatible corners and jumps do not invalidate a positive-time series solution.

Step 1: Compute all modal amplitudes. Integration over the nonzero half of the interval gives

$$b_n = \frac{2}{\pi n} (1 - \cos(n\pi/2)), \quad u(x, t) = \sum_{n \geq 1} b_n e^{-n^2 t} \sin(nx).$$

The point value assigned at $\pi/2$ does not affect the integral. Modes divisible by four vanish; the other coefficients must not all be replaced by the odd-only coefficients for a constant temperature on the entire interval.

Step 2: Verify positive-time smoothness and the norm trace. For $t \geq \tau > 0$, $|b_n| \leq 4/(\pi n)$ and the exponential makes all finitely differentiated series uniformly convergent. Thus the PDE and endpoints hold. Parseval and the sine expansion of f give

$$\|u(\cdot, t) - f\|_2^2 = \frac{\pi}{2} \sum_{n \geq 1} b_n^2 (1 - e^{-n^2 t})^2 \rightarrow 0,$$

by summable domination, since $\sum b_n^2 < \infty$. This is precisely the specified initial condition.

Step 3: State the pointwise limits correctly. The limits are 1 on $(0, \pi/2)$, 0 on $(\pi/2, \pi)$ and $1/2$ at the jump. To justify Gaussian damping of a pointwise convergent Fourier series, apply summation by parts: the decreasing weights $e^{-n^2 t}$ average its partial sums, have total weight tending to one, and move past each fixed index as $t \downarrow 0$. Thus they preserve the Fourier limit. At both endpoints u is identically zero for $t > 0$, so the limit is zero. A uniform limit of continuous functions on $[0, \pi]$ would be continuous; the jump and the left corner mismatch preclude uniform convergence to these data.

Step 4: Prove uniqueness in the stated class. Let w be the difference of two such solutions. For $t \geq \delta > 0$,

$$\frac{d}{dt} \|w\|_2^2 = 2 \int_0^\pi w w_{xx} dx = -2 \int_0^\pi w_x^2 dx \leq 0.$$

The boundary term vanishes. Hence $\|w(\cdot, t)\|_2 \leq \|w(\cdot, \delta)\|_2$. Both solutions have the same L^2 initial trace, so the right side tends to zero as $\delta \downarrow 0$. Positive-time continuity then makes $w = 0$ pointwise. No smoothness at the discontinuous initial trace was assumed.

Question 5

Let $\kappa, L > 0$. Solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $t > 0$, with insulated ends $u_x(0, t) = u_x(L, t) = 0$ and initial temperature $f(x) = x$. Use the convention $f = a_0 + \sum_{n \geq 1} a_n \cos(n\pi x/L)$, where $a_0 = L^{-1} \int_0^L f$ and $a_n = (2/L) \int_0^L f \cos(n\pi x/L)$. You may use cosine-series convergence and Parseval's identity.

Tasks

1. Compute the coefficients and construct the solution, retaining the zero mode.
2. Prove conservation of the mean and the reflection identity $u(L - x, t) = L - u(x, t)$. Determine the limiting equilibrium.
3. Justify uniform recovery of the initial values, and explain why the spatial derivative cannot extend continuously through the initial boundary corners.
4. Compute $V(t) = \int_0^L (u - L/2)^2 dx$ as a series, find $V(0)$ and establish $V(t) \leq e^{-2\kappa(\pi/L)^2 t} V(0)$. Sketch profiles for $L = \kappa = 1$ at $t = 0, 0.03, 0.15$.

Question 5 – Solution

Strategy. Insulation preserves the constant component while all nonconstant cosine modes decay.

Step 1: Compute and evolve the cosine coefficients. Integration gives $a_0 = L/2$ and $a_n = 2L((-1)^n - 1)/(\pi^2 n^2)$. Therefore

$$u(x, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{e^{-\kappa(n\pi/L)^2 t} \cos(n\pi x/L)}{n^2}.$$

Gaussian decay justifies the PDE and derivative boundary conditions termwise for $t > 0$. The constant mode remains stationary.

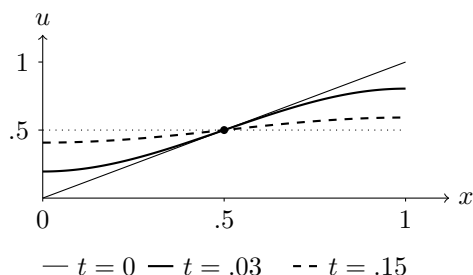
Step 2: Identify the conserved and symmetric parts. Integrating the PDE yields $(\int u)' = \kappa[u_x]_0^L = 0$, so the mean is $L/2$. For odd n , $\cos(n\pi(L-x)/L) = -\cos(n\pi x/L)$. The series therefore gives $u(L-x, t) = L - u(x, t)$, in particular $u(L/2, t) = L/2$ at all times. The absolutely summable coefficients and positive modal decay imply uniform convergence to the equilibrium $L/2$ as $t \rightarrow \infty$.

Step 3: Check values and derivatives separately. The n^{-2} coefficients give absolute uniform convergence at $t = 0$; the cosine convergence theorem identifies the sum as x , including both endpoints. Summable domination proves uniform recovery of those values as $t \downarrow 0$. But the initial trace has derivative one, whereas $u_x = 0$ at the endpoints for every positive time. A continuously extending spatial derivative would have to take both values at each initial corner, which is impossible. Uniform convergence of temperatures does not imply convergence of derivatives.

Step 4: Quantify equilibration. Orthogonality gives

$$V(t) = \frac{8L^3}{\pi^4} \sum_{n \text{ odd}} \frac{e^{-2\kappa(n\pi/L)^2 t}}{n^4}, \quad V(0) = \int_0^L (x - L/2)^2 dx = \frac{L^3}{12}.$$

Each exponential is at most $e^{-2\kappa(\pi/L)^2 t}$, proving the required bound. The profiles flatten toward their conserved mean, rather than toward zero. Only the initial profile has the incompatible endpoint slopes.



Question 6

Solve $u_t = \kappa u_{xx}$ for $0 < x < L$, $t > 0$, with $\kappa, L > 0$,

$$u(0, t) = 0, \quad u_x(L, t) = 0, \quad u(x, 0) = x(2L - x).$$

You may use the complete orthogonal family $\sin(k_n x)$, $k_n = (n + 1/2)\pi/L$, $n = 0, 1, \dots$, with squared norm $L/2$, and its Fourier convergence for the odd-at-zero, even-at- L extension of these data.

Tasks

1. Compute the expansion coefficients using the actual mixed conditions and construct the solution.
2. Verify the initial trace, PDE and two different endpoint conditions, justifying the series operations.
3. Compute total heat and its rate of loss. Explain why only the left endpoint contributes to the heat flux.
4. Determine the limiting shape of $e^{\kappa k_0^2 t} u$ uniformly on $[0, L]$. Compare its slowest exponential decay rate with that of a rod of the same length and diffusivity with both endpoints fixed at zero.

Question 6 – Solution

Strategy. Use half-integer modes for the mixed boundary problem; integer sine modes would solve a different problem.

Step 1: Evaluate the coefficient integral. For $f = x(2L - x)$, $f(0) = 0$, $f'(L) = 0$ and $f'' = -2$. Two integrations by parts, using $\cos(k_n L) = 0$, give

$$\int_0^L f \sin(k_n x) dx = \frac{2}{k_n^3}, \quad u(x, t) = \frac{4}{L} \sum_{n=0}^{\infty} \frac{e^{-\kappa k_n^2 t} \sin(k_n x)}{k_n^3}.$$

The vanishing boundary terms use the derivative condition at L , not a zero value of $f(L)$, which is actually L^2 .

Step 2: Verify the complete solution. The coefficients are summable, so the initial series is uniformly convergent; the supplied Fourier theorem identifies it as f . Summable domination gives uniform convergence to f as $t \downarrow 0$. For $t \geq \tau > 0$, differentiated series converge uniformly by exponential decay and verify the PDE. At zero, all sine factors vanish; at L , all derivative cosine factors vanish. The initial function also satisfies these value/first-derivative data, although higher corner compatibility is a separate issue.

Step 3: Compute heat loss through the open thermal boundary. Since $\int_0^L \sin(k_n x) dx = 1/k_n$,

$$H(t) = \frac{4}{L} \sum_{n \geq 0} \frac{e^{-\kappa k_n^2 t}}{k_n^4}, \quad H(0) = \frac{2L^3}{3}.$$

For $t > 0$, $H' = -(4\kappa/L) \sum e^{-\kappa k_n^2 t}/k_n^2 = -\kappa u_x(0, t)$. This is $\kappa[u_x]_0^L$ because the right derivative is zero. Every term in the loss sum is positive; insulation blocks flux at the right endpoint only.

Step 4: Extract the slowest mode rigorously. Multiplying the series by $e^{\kappa k_0^2 t}$ leaves the first term and a remainder bounded by $e^{-\kappa(k_1^2 - k_0^2)t} \sum_{n \geq 1} 4/(Lk_n^3)$, which tends to zero. Hence uniformly,

$$e^{\kappa k_0^2 t} u \longrightarrow \frac{32L^2}{\pi^3} \sin(\pi x/(2L)).$$

The slowest rate is $\kappa\pi^2/(4L^2)$, one quarter of the Dirichlet–Dirichlet rate $\kappa\pi^2/L^2$. Equal material properties and length do not imply equal decay rates when the boundary conditions differ.

Question 7

For the zero-endpoint heat problem $u_t = u_{xx}$ on $(0, \pi)$, let $f \in L^2(0, \pi)$ with $\|f\|_2 \leq M$, where $M > 0$. Its solution for $t > 0$ is

$$u = \sum_{n \geq 1} b_n e^{-n^2 t} \sin(nx), \quad u_N = \sum_{n=1}^N b_n e^{-n^2 t} \sin(nx), \quad N \geq 1.$$

You may use Parseval: $\sum b_n^2 = 2\|f\|_2^2/\pi$, and Cauchy-Schwarz.

Tasks

1. Prove $\|u - u_N\|_2 \leq M e^{-(N+1)^2 t}$. Show that the bound is sharp over the stated data class for each N, t .
2. Derive a uniform-in- x error bound using Cauchy-Schwarz and the tail sum $\sum_{n > N} e^{-2n^2 t}$.
3. Bound that sum by a geometric series using $n = N + 1 + j$, $j \geq 0$, and obtain an explicit bound requiring no infinite summation.
4. For $M = 1, t \geq 0.1$, certify that six modes give a uniform error below 0.01. Explain why the available bound does not certify five modes, and why this does not prove that five modes fail for every particular initial field.

Question 7 – Solution

Strategy. A truncation certificate needs a bound on the omitted coefficients, not just a graph of the retained modes.

Step 1: Prove the sharp norm estimate. Orthogonality gives

$$\|u - u_N\|_2^2 = \frac{\pi}{2} \sum_{n>N} b_n^2 e^{-2n^2 t} \leq e^{-2(N+1)^2 t} \|f\|_2^2.$$

Taking square roots proves the bound. Equality is attained by $f = M\sqrt{2/\pi} \sin((N+1)x)$: all its energy lies in the first omitted mode. Thus the rate and constant are sharp for the whole L^2 ball.

Step 2: Control pointwise errors uniformly. Using $|\sin nx| \leq 1$ and Cauchy–Schwarz,

$$|u - u_N| \leq \left(\sum_{n>N} b_n^2 \right)^{1/2} \left(\sum_{n>N} e^{-2n^2 t} \right)^{1/2} \leq \sqrt{\frac{2}{\pi}} M \left(\sum_{n>N} e^{-2n^2 t} \right)^{1/2}.$$

This bound is independent of x and finite for $t > 0$. It also justifies uniform convergence of the positive-time solution for arbitrary L^2 data.

Step 3: Replace the tail by a geometric bound. For the integer $j \geq 0$, $(N+1+j)^2 \geq (N+1)^2 + (2N+3)j$, since $j^2 \geq j$. Therefore

$$\sum_{n>N} e^{-2n^2 t} \leq \frac{e^{-2(N+1)^2 t}}{1 - e^{-2(2N+3)t}}, \quad \|u - u_N\|_\infty \leq \sqrt{\frac{2}{\pi}} \frac{M e^{-(N+1)^2 t}}{\sqrt{1 - e^{-2(2N+3)t}}}.$$

The denominator is positive because $t > 0$. Each factor makes this certificate decrease as time increases.

Step 4: Make a numerical guarantee with the correct scope. At $t = 0.1$, the bound is less than 0.00611 for $N = 6$ and greater than 0.0226 for $N = 5$. Hence six modes certify the requested 0.01 uniform error for every $t \geq 0.1$ and every datum with $\|f\|_2 \leq 1$. The five-mode certificate is too large; it does not prove failure for each field. For example, a datum supported entirely in the first five modes has zero truncation error with five modes. The bound is a sufficient test, not an exact error formula for unknown data.

Question 8

For $\alpha > 0$, solve the forced heat problem

$$u_t = u_{xx} + e^{-\alpha t} \sin x, \quad 0 < x < \pi, \quad t > 0, \quad u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = 0.$$

Use a field of the form $u(x, t) = q_\alpha(t) \sin x$. This is interior forcing; the boundary temperatures remain zero.

Tasks

1. Derive and solve the scalar initial-value problem for q_α , treating $\alpha = 1$ separately.
2. Verify the PDE and all data, and prove that the constructed amplitude is positive for $t > 0$. Explain why uniqueness holds in the class of smooth solutions continuous into L^2 initially.
3. Determine the unique time of maximum amplitude for $\alpha \neq 1$ and its limit as $\alpha \rightarrow 1$. Check the maximum directly for $\alpha = 1$.
4. Derive the total-heat balance and compare matching temporal decay rates here with unbounded oscillatory resonance. Does the factor t in the exceptional solution cause unbounded temperature?

Question 8 – Solution

Strategy. The source has one spatial eigenfunction, so the full problem reduces to a forced scalar decay equation.

Step 1: Solve the amplitude equation including the exceptional rate. Substitution gives $q'_\alpha + q_\alpha = e^{-\alpha t}$, $q_\alpha(0) = 0$. Multiplying by e^t and integrating yields

$$q_\alpha(t) = \begin{cases} (e^{-\alpha t} - e^{-t})/(1 - \alpha), & \alpha \neq 1, \\ te^{-t}, & \alpha = 1. \end{cases}$$

The exceptional formula follows from the same integrating-factor integral; it is not obtained by substituting into a zero denominator.

Step 2: Verify existence, sign and uniqueness. The scalar ODE verifies the PDE, while the sine factor and $q_\alpha(0) = 0$ verify endpoints and initial data. The integral representation

$$q_\alpha(t) = \int_0^t e^{-(t-s)} e^{-\alpha s} ds$$

is strictly positive for $t > 0$. The difference of two solutions with the same source solves the homogeneous zero-data problem. Its squared L^2 norm is nonincreasing by integration by parts; starting at a positive time and using the zero initial norm proves uniqueness in the stated class.

Step 3: Locate the unique maximum. For $\alpha \neq 1$, $q'_\alpha = (e^{-t} - \alpha e^{-\alpha t})/(1 - \alpha)$. The equation $q'_\alpha = 0$ is $e^{(\alpha-1)t} = \alpha$, giving

$$t_\alpha = \frac{\log \alpha}{\alpha - 1} > 0.$$

There is exactly one root; $q'_\alpha(0) = 1 > 0$ and the derivative is negative for sufficiently large time, proving it is the unique maximum. As $\alpha \rightarrow 1$, $t_\alpha \rightarrow 1$. Directly $(te^{-t})' = e^{-t}(1 - t)$, with maximum $1/e$ at $t = 1$.

Step 4: Interpret the forced heat balance. The total heat is $H = 2q_\alpha$. Since $[u_x]_0^\pi = -2q_\alpha$ and the source integral is $2e^{-\alpha t}$,

$$H' = -2q_\alpha + 2e^{-\alpha t},$$

which agrees with the scalar ODE. At $\alpha = 1$, matching the source decay rate to the homogeneous decay produces te^{-t} . This field is bounded and tends to zero; its time factor does not imply unbounded oscillatory resonance. For every fixed $\alpha > 0$, the source and the solution ultimately decay.

Question 9

For $u_t = u_{xx}$ on $(0, \pi)$ with zero endpoint values, let $\phi_n(x) = \sqrt{2/\pi} \sin(nx)$ be the orthonormal spatial modes. Write $u(x, 0) = \sum c_n \phi_n(x)$ and $u(x, T) = \sum d_n \phi_n(x)$, where $T > 0$. All norms in this question are $L^2(0, \pi)$ norms.

Tasks

1. Relate c_n and d_n . Prove that there is at most one L^2 initial field for given exact final data, and state the precise coefficient condition for such an initial field to exist.
2. Use initial fields ϕ_n to show that uniqueness of backward reconstruction does not imply continuous dependence on final data in these norms.
3. Suppose measurement noise in the final field has norm at most δ . Reconstruct only modes $1, \dots, N$ and prove that the reconstructed noise has norm at most $e^{N^2 T} \delta$. Show that this bound is sharp.
4. For $T = 0.1$, $\delta = 10^{-6}$, choose the largest integer N certified to keep reconstructed noise below 0.01. Plot $\log_{10}$ of modal amplification against integer mode number, and explain why controlling noise alone does not bound the omitted true initial modes.

Question 9 – Solution

Strategy. Each forward-decaying mode becomes an exponentially amplified mode when time is reversed.

Step 1: Characterize exact inversion. Forward evolution gives $d_n = c_n e^{-n^2 T}$, so the only possible coefficients are $c_n = e^{n^2 T} d_n$. Completeness implies at most one initial field. It exists in L^2 exactly when

$$\sum_{n \geq 1} e^{2n^2 T} |d_n|^2 < \infty.$$

Indeed this condition constructs an L^2 sine expansion whose forward solution has the given coefficients. Arbitrary L^2 final data need not satisfy this stronger condition.

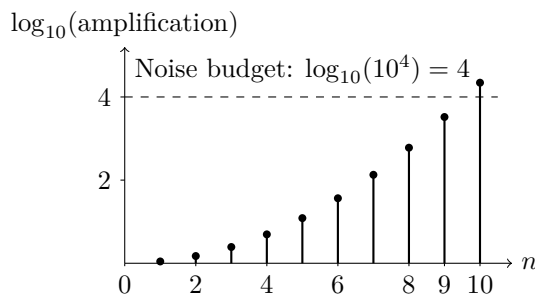
Step 2: Disprove continuity despite uniqueness. Take $f_n = \phi_n$, whose initial norm is one. Its final field has norm $e^{-n^2 T} \rightarrow 0$. Thus final data tend to zero while their uniquely determined initial fields do not. The backward map on its attainable-data domain is not continuous at zero in the stated norms. No contradiction arises with the stable forward problem, which damps these same modes.

Step 3: Bound noise after truncating the inverse. Let η_n be the noise coefficients, so $\sum |\eta_n|^2 \leq \delta^2$. The retained reconstructed noise satisfies

$$\left\| \sum_{n=1}^N e^{n^2 T} \eta_n \phi_n \right\|_2^2 = \sum_{n=1}^N e^{2n^2 T} |\eta_n|^2 \leq e^{2N^2 T} \delta^2.$$

Taking square roots gives the bound, attained by noise $\delta \phi_N$.

Step 4: Choose a noise cutoff and state its limitation. The requirement is $e^{0.1 N^2} 10^{-6} < 10^{-2}$, or $N^2 < 40 \log 10$. Thus $\lfloor N = 9 \rfloor$ is the largest certified integer: the bounds for 9, 10 are approximately 0.003295 and 0.022027. The plot uses discrete stems, since n is an integer; their heights are $0.1 n^2 / \log 10$. The omitted initial tail has norm $(\sum_{n > N} |c_n|^2)^{1/2}$, for which no numerical bound was supplied. A noise cutoff controls amplification but requires additional prior information to control that approximation error.



Question 10

Let $u_t = \kappa u_{xx}$ on $(0, L)$ with $\kappa, L > 0$, zero endpoint values, and initial field $f \in L^2(0, L)$. Let b_n be its sine coefficients and $E(t) = \frac{1}{2} \int_0^L u(x, t)^2 dx$ its quadratic energy. You may use Parseval's identity, positive-time termwise differentiation and sine completeness.

Tasks

1. Derive the energy identity and prove $E(t) \leq e^{-2\kappa(\pi/L)^2 t} E(0)$. Characterize every nonzero datum attaining equality at a fixed positive time.
2. Suppose $f(L - x) = -f(x)$ almost everywhere. Prove that only even modes remain and obtain the improved bound with rate $8\kappa\pi^2/L^2$. Identify when that bound is attained.
3. Show that a nonnegative, nonzero datum cannot have $b_1 = 0$. Explain why the enhanced asymptotic decay caused by removing the first mode requires sign changes in nonzero real initial data.
4. Prove the long-time L^2 asymptotic formula $e^{\kappa(\pi/L)^2 t} u \rightarrow b_1 \sin(\pi x/L)$, with a quantitative remainder bound. State what the formula says when $b_1 = 0$.

Question 10 – Solution

Strategy. Orthogonality turns energy decay and symmetry restrictions into precise statements about which coefficients are present.

Step 1: Prove the sharp basic energy decay. For $t > 0$, integration by parts gives $E' = \kappa \int u u_{xx} = -\kappa \int u_x^2$ because $u = 0$ at both ends. With $\lambda_1 = (\pi/L)^2$, Parseval yields

$$E(t) = \frac{L}{4} \sum_{n \geq 1} b_n^2 e^{-2\kappa n^2 \lambda_1 t} \leq e^{-2\kappa \lambda_1 t} E(0).$$

For fixed $t > 0$, every term with $n \geq 2$ has strictly smaller decay factor. Equality for nonzero data holds exactly when $f = b_1 \sin(\pi x/L)$ with $b_1 \neq 0$. This also proves sharpness of the uniform energy rate.

Step 2: Use antisymmetry to improve the rate. Changing variable $x \mapsto L - x$ in the coefficient integral and using $\sin(n\pi(L - x)/L) = (-1)^{n+1} \sin(n\pi x/L)$ gives $b_n = (-1)^n b_n$. Thus odd coefficients vanish. The smallest remaining index is two, giving

$$E(t) \leq e^{-8\kappa \pi^2 t/L^2} E(0).$$

Nonzero equality data are precisely scalar multiples of $\sin(2\pi x/L)$. The symmetry is preserved by the evolved even-mode series.

Step 3: Identify the sign constraint. If $f \geq 0$ almost everywhere and is nonzero in L^2 , then it is positive on a set of positive measure. The first sine is strictly positive on $(0, L)$; therefore $b_1 = (2/L) \int f \sin(\pi x/L) > 0$. Similarly a nonpositive nonzero datum has $b_1 < 0$. Thus any nonzero real datum with $b_1 = 0$ must have both positive and negative parts of positive measure. Removing the first mode is possible, but not for a nontrivial nonnegative initial temperature relative to the zero boundaries.

Step 4: Prove the normalized asymptotic statement. After removing the first term, Parseval gives

$$\|e^{\kappa \lambda_1 t} u - b_1 \sin(\pi x/L)\|_2^2 = \frac{L}{2} \sum_{n \geq 2} b_n^2 e^{-2\kappa(n^2-1)\lambda_1 t} \leq e^{-6\kappa \lambda_1 t} \|f\|_2^2.$$

The remainder norm is at most $e^{-3\kappa \lambda_1 t} \|f\|_2$ and tends to zero. If $b_1 = 0$, the normalized limit is zero and the actual field decays at least at the second-mode norm rate $e^{-4\kappa \lambda_1 t}$. The next nonzero coefficient, if present, can give an even faster decay rate.