

Question 1

Let I, J be open intervals, $\kappa > 0$, $r \in \mathbb{R}$, and consider

$$u_t = \kappa u_{xx} + ru \quad ((x, t) \in I \times J).$$

A separated field is $u(x, t) = X(x)T(t)$, with $X \in C^2(I)$ and $T \in C^1(J)$. Assume neither factor is identically zero. Do not assume either factor is nonzero at every point.

Tasks

1. Substitute the product into the PDE. Choose one time where T is nonzero and use it to derive a spatial ODE on all of I , without dividing by X .
2. Show that a single real constant λ gives $X'' + \lambda X = 0$ and $T' = (r - \kappa\lambda)T$ everywhere on their respective intervals. Prove the converse as well.
3. Determine whether a nontrivial T can vanish at a time in J . Explain why zeros of X do not invalidate the separated field and what happens if a factor is identically zero.
4. Verify $X(x) = \sin(2x)$ when $I = \mathbb{R}$. Find all corresponding time factors, determine the growth threshold in r , and explain the harmless freedom to rescale the two factors reciprocally.

Question 1 – Solution

Strategy. Evaluate at a nonzero anchor value of one factor before deriving equations valid even at zeros of the other.

Step 1: Derive a global spatial identity. Substitution yields $XT' = \kappa X''T + rXT$. Choose $t_0 \in J$ with $T(t_0) \neq 0$. Evaluating there gives, for every $x \in I$,

$$\kappa X''(x) = \left(\frac{T'(t_0)}{T(t_0)} - r \right) X(x).$$

Set $\lambda = (r - T'(t_0)/T(t_0))/\kappa$. Then $\boxed{X'' + \lambda X = 0}$ on all of I , including any zeros of X .

Step 2: Recover the time equation and the converse. Substituting $X'' = -\lambda X$ into the product identity gives $X[T' - (r - \kappa\lambda)T] = 0$. Choose x_0 with $X(x_0) \neq 0$ to obtain

$$\boxed{T' = (r - \kappa\lambda)T \quad \text{on } J.}$$

The constant is independent of both variables; a different anchor must give the same value because X is not identically zero. Conversely, these two ODEs imply the original product identity directly, without any divisions. Thus every such pair of factors produces a PDE solution.

Step 3: Treat zeros and the excluded trivial field. The time equation gives $T(t) = Ce^{(r-\kappa\lambda)(t-t_0)}$. For a nontrivial factor $C \neq 0$, it never vanishes. Spatial zeros are allowed; the identities above remain valid there. A factor identically zero produces $u \equiv 0$, which solves the PDE but determines no separation constant from its chosen factors. The trivial field must be retained separately, not mistakenly assigned restrictions obtained by division.

Step 4: Verify a concrete mode and its normalization. For $X = \sin(2x)$, $X'' = -4X$, so $\lambda = 4$ and

$$u(x, t) = C \sin(2x) e^{(r-4\kappa)t}.$$

Its nonzero amplitude grows if $r > 4\kappa$, is constant if $r = 4\kappa$, and decays if $r < 4\kappa$. Direct differentiation verifies the PDE, including at $x = n\pi/2$. Replacing (X, T) by $(aX, T/a)$ for $a \neq 0$ leaves u unchanged; the two factors are not individually unique until a normalization is chosen.

Question 2

Seek nonzero separated solutions of $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and mixed endpoint conditions

$$u(0, t) = 0, \quad u_x(L, t) = 0.$$

Use the convention $X'' + \lambda X = 0$, $T' = -\kappa \lambda T$.

Tasks

1. Derive the endpoint conditions on X . Use integration by parts to prove that every nonzero spatial factor has $\lambda > 0$; address equality explicitly.
2. Solve the spatial problem and list all eigenvalues and eigenfunctions, using an index $n = 0, 1, 2, \dots$. Explain why checking only a sine condition at both ends would give the wrong spectrum.
3. Find the time factor and compare the amplitude half-lives of the first two modes. Distinguish amplitude half-life from the half-life of squared amplitude.
4. Sketch the first three spatial factors normalized to have sine coefficient one, using $\xi = x/L$. Locate their interior zeros and verify their right-end slopes.

Question 2 – Solution

Strategy. Let the actual boundary operators select the admissible separation constants.

Step 1: Prove positivity before solving the ODE. At a time where $T \neq 0$, the endpoint conditions give $X(0) = 0$, $X'(L) = 0$. Multiplication by X and integration yield

$$\lambda \int_0^L X^2 dx = \int_0^L (X')^2 dx - [XX']_0^L = \int_0^L (X')^2 dx.$$

If $\lambda = 0$, then X' vanishes identically; $X(0) = 0$ forces $X = 0$. Thus a nonzero factor has $\boxed{\lambda > 0}$; negative values are impossible.

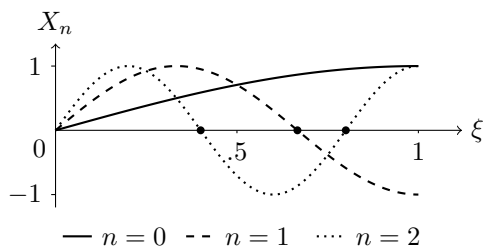
Step 2: Apply the two different endpoint conditions. Put $k = \sqrt{\lambda}$. The general solution is $A \cos(kx) + B \sin(kx)$. The left condition gives $A = 0$, and the right gives $Bk \cos(kL) = 0$. For $B \neq 0$,

$$\boxed{k_n = \frac{(n + 1/2)\pi}{L}, \quad \lambda_n = k_n^2, \quad X_n(x) = \sin(k_n x), \quad n = 0, 1, \dots}$$

The right condition is on the derivative, so it selects zeros of cosine. Using $\sin(kL) = 0$ would impose an unrequested Dirichlet condition there.

Step 3: Compare decay measures. The modes are $CX_n(x)e^{-\kappa\lambda_n t}$. Their amplitude half-lives are $\tau_n = \log 2/(\kappa\lambda_n)$, so $\tau_1 = \tau_0/9$. Squared amplitude decays as $e^{-2\kappa\lambda_n t}$ and has half-life $\tau_n/2$. Confusing those two quantities introduces a factor of two.

Step 4: Check the plotted nodes and slopes. For $n = 0, 1, 2$, the normalized factors are $\sin((n + 1/2)\pi\xi)$. There are respectively zero, one and two interior zeros: none; $2/3$; and $2/5, 4/5$. Every right slope is zero because $\cos((n + 1/2)\pi) = 0$. The right values alternate $1, -1, 1$ and need not vanish.



Question 3

For $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, take $\kappa, L > 0$ and insulated endpoints $u_x(0, t) = u_x(L, t) = 0$. Use $X'' + \lambda X = 0$ and $T' = -\kappa \lambda T$. All fields are smooth enough for the stated derivatives and integrals.

Tasks

1. Solve the spatial eigenvalue problem, including $\lambda = 0$, and exclude $\lambda < 0$ by an integral identity.
2. Show directly from the PDE that the spatial mean is constant in time. Prove that each positive-eigenvalue spatial factor has zero mean, without evaluating its trigonometric integral.
3. Construct a finite separated-mode sum satisfying $u(x, 0) = 2 + 3 \cos(\pi x/L)$. Verify the PDE, both endpoints and the initial data, and identify the limiting field as $t \rightarrow \infty$.
4. Explain precisely what is lost if one divides the eigenvalue calculation by λ and discards the zero case. Can any finite sum of positive-eigenvalue modes reproduce these initial data?

Question 3 – Solution

Strategy. The zero eigenvalue carries a conserved quantity and must be analyzed before any division by the separation constant.

Step 1: Retain the constant spatial mode. Integration by parts gives $\lambda \int X^2 = \int (X')^2 \geq 0$ because both endpoint derivatives vanish. Thus negative eigenvalues are excluded. For $\lambda = 0$, $X = ax + b$ and the endpoints force $a = 0$; nonzero constants are allowed. For $\lambda = k^2 > 0$, the left derivative condition removes the sine term and the right requires $\sin(kL) = 0$. Consequently

$$\lambda_0 = 0, X_0 = 1; \quad \lambda_n = (n\pi/L)^2, X_n = \cos(n\pi x/L), n \geq 1.$$

Step 2: Track the conserved mean. For $m(t) = L^{-1} \int_0^L u(x, t) dx$, $m'(t) = \kappa[u_x]_0^L/L = 0$. Integrating $X'' + \lambda X = 0$ gives $\lambda \int_0^L X dx = -[X']_0^L = 0$. Thus every $\lambda > 0$ factor has zero mean. The zero mode has a constant time factor and can carry a nonzero mean.

Step 3: Verify the two-mode construction. Let $\omega = \kappa(\pi/L)^2$. Then

$$u(x, t) = 2 + 3e^{-\omega t} \cos(\pi x/L).$$

Its time derivative is $-3\omega e^{-\omega t} \cos(\pi x/L)$, equal to κu_{xx} . Its endpoint derivatives vanish, and its initial trace is the prescribed function. Its mean is two for every time. Since the decaying term has absolute value at most $3e^{-\omega t}$, it tends uniformly to $u = 2$. No infinite-series argument is required for this finite construction.

Step 4: Diagnose the missing mode. Dividing by λ loses the legitimate constant eigenfunction and the corresponding stationary PDE solutions. Every finite linear combination of positive-eigenvalue modes has zero spatial mean at each time, while the specified initial data have mean two. Such a combination cannot reproduce these data. This obstruction follows from the integral, not from an inability to find enough trigonometric coefficients.

Question 4

A product field $u(x, t) = X(x)T(t)$ has proportional spatial snapshots. Consider the two-mode heat solution on $0 \leq x \leq \pi$, $t \geq 0$,

$$u(x, t) = e^{-t} \sin x + e^{-4t} \sin(2x).$$

For any two positions and times define

$$D = u(x_1, t_1)u(x_2, t_2) - u(x_1, t_2)u(x_2, t_1).$$

Tasks

1. Prove that a product field has $D = 0$. Conversely, if every such determinant vanishes and $u(x_0, t_0) \neq 0$ somewhere, construct factors X, T representing u everywhere.
2. Verify the PDE $u_t = u_{xx}$ and homogeneous Dirichlet endpoints for the displayed sum. Compute D at $x_1 = \pi/2, x_2 = \pi/6, t_1 = 0, t_2 = \log 2$ and decide whether the sum is a single product.
3. Find every interior spatial zero of $u(\cdot, t)$, the time at which it leaves the interior, and the sign of the field after that time.
4. Sketch the normalized snapshots $v(x, t) = e^t u(x, t)$ at $t = 0$, $(\log 2)/6$ and $(\log 2)/3$. Explain why division by a nonzero time factor cannot make a nonseparable field separable.

Question 4 – Solution

Strategy. Test separation with an invariant determinant, then interpret the changing spatial shape.

Step 1: Establish an exact product test. For a product, both terms of D equal $X(x_1)X(x_2)T(t_1)T(t_2)$. Conversely, use the nonzero anchor to write $u(x, t)u(x_0, t_0) = u(x, t_0)u(x_0, t)$. Hence

$$\boxed{X(x) = u(x, t_0), \quad T(t) = u(x_0, t)/u(x_0, t_0)}$$

gives a product everywhere, including zeros. The nonzero-anchor restriction excludes only the identically zero field, which is already a trivial product.

Step 2: Verify the solution and reject one-product form. Each term has time derivative equal to its second spatial derivative; the sum solves the PDE and vanishes at $x = 0, \pi$. At the stated points,

$$D = \frac{\sqrt{3}}{2} \left(\frac{1}{16} - \frac{1}{2} \right) = \boxed{-\frac{7\sqrt{3}}{32} \neq 0}.$$

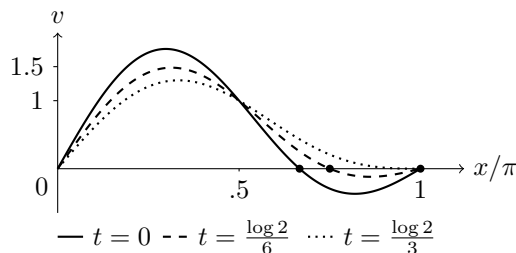
Thus superposition preserves the linear PDE but does not preserve the class of single products. The two spatial modes decay at different rates.

Step 3: Track the moving interior node. Factoring gives $u = e^{-t} \sin x [1 + 2e^{-3t} \cos x]$. On $0 < x < \pi$, $\sin x > 0$, so the only possible interior zero is

$$x_*(t) = \arccos(-e^{3t}/2), \quad 0 \leq t < t_* := \frac{\log 2}{3}.$$

It moves from $2\pi/3$ toward π . At $t = t_*$ it is at the endpoint, not an interior zero. For $t \geq t_*$ the field is strictly positive in the interior. Before then it is positive to the left of the node and negative to its right.

Step 4: Interpret the normalized plots. The three normalized fields are $\sin x + \sin 2x$, $\sin x + 2^{-1/2} \sin 2x$, and $\sin x + \frac{1}{2} \sin 2x$. Their changing zeros reveal changing shape. If $e^t u$ were a product, multiplying its time factor by e^{-t} would make u a product too, contradicting the determinant. Normalization changes amplitudes but cannot repair this failure of separation.



Question 5

For $u_{tt} = c^2 u_{xx}$ on $0 < x < \pi$, $t \in \mathbb{R}$, with $c > 0$ and $u(0, t) = u(\pi, t) = 0$, seek a nonzero product $u = X(x)T(t)$ with both factors C^2 . Use the convention $X'' + \lambda X = 0$.

Tasks

1. Derive the two factor equations using nonzero anchor values instead of assuming T has no zeros. Determine the allowed spatial factors and time factors.
2. Prove the precise restriction a single product places on its two initial functions $f(x) = u(x, 0)$ and $g(x) = u_t(x, 0)$. Include the cases where one of these functions vanishes identically.
3. Determine whether $f = \sin(2x), g = \sin(3x)$ can come from one separated mode. Construct a sum of two modes that satisfies both initial functions and verify it directly.
4. Use $u(x, t) = \sin(2x) \sin(2ct)$ to explain why a zero-displacement snapshot need not mean zero solution or zero velocity. Explain the error in dividing by T at that time.

Question 5 – Solution

Strategy. The time equation is second order and may cross zero; derive it globally and match both initial traces.

Step 1: Separate without excluding oscillation zeros. The product identity is $XT'' = c^2X''T$. A time t_0 with $T(t_0) \neq 0$ gives $X'' = -\lambda X$ everywhere, with $\lambda = -T''(t_0)/(c^2T(t_0))$. A position with $X \neq 0$ then gives $T'' + c^2\lambda T = 0$ for all times. Dirichlet endpoints and $\lambda \int X^2 = \int (X')^2$ exclude $\lambda \leq 0$. Solving the endpoint ODE gives

$$X = \sin(nx), \quad \lambda = n^2, \quad T = A \cos(cnt) + B \sin(cnt), \quad n = 1, 2, \dots,$$

with overall spatial scale absorbed into A, B .

Step 2: Match both initial functions to one spatial shape. A single mode necessarily has $f = A \sin(nx)$ and $g = cnB \sin(nx)$ for the same integer n . Conversely, any such pair gives the displayed mode. Either coefficient may vanish: zero initial displacement permits a pure sine time factor, and zero initial velocity permits a pure cosine time factor. If both vanish, uniqueness for the scalar time ODE gives only the zero field. Thus proportionality alone is insufficient unless the nonzero shape is also an admissible spatial eigenfunction.

Step 3: Construct a field beyond one product. The functions $\sin 2x$ and $\sin 3x$ are not proportional, so one mode cannot provide the prescribed pair. Instead use

$$u = \sin(2x) \cos(2ct) + \frac{1}{3c} \sin(3x) \sin(3ct).$$

Each term satisfies the wave equation and both endpoint values. At $t = 0$, the displacement is $\sin 2x$; differentiating first gives initial velocity $\sin 3x$. These checks establish this finite construction directly.

Step 4: Preserve the solution at a zero time factor. For $u = \sin 2x \sin 2ct$, the initial displacement vanishes everywhere, while $u_t(x, 0) = 2c \sin 2x$ is not identically zero. The time factor crosses zero at $t = 0$ and other times, but its ODE and the PDE remain valid. Dividing by $T(0)$ is undefined; inferring that this mode is forbidden would discard a legitimate oscillatory solution. The anchor argument requires a nonzero value somewhere, not at every time.

Question 6

Consider a single product $u(x, y) = X(x)Y(y)$ satisfying $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < \pi$, $0 < y < H$, where $H > 0$, with

$$u(0, y) = u(\pi, y) = 0, \quad u(x, 0) = 0, \quad u(x, H) = \sin(2x).$$

Restrict attention to nonzero smooth separated fields; a general Fourier solution for arbitrary top data is not requested.

Tasks

1. Derive $X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$ and determine the allowable λ from the two vertical sides.
2. Use the top data to identify the spatial mode and find Y . Verify all four boundary conditions and the PDE directly.
3. Explain why replacing the hyperbolic sine in Y by an ordinary sine generally fails. Compute the PDE residual for the proposed replacement $\sin(2x) \sin(2y)$.
4. Prove that the solution's vertical amplitude increases from zero to one, yet its absolute value never exceeds one in the rectangle. Compare the signs of the remaining factor equations for heat, wave and Laplace separation with the same positive λ .

Question 6 – Solution

Strategy. The boundary conditions select the spatial eigenvalue, while the PDE determines the sign in the other factor equation.

Step 1: Separate with the correct opposite signs. Substitution yields $X''Y + XY'' = 0$. Nonzero anchors, as in the previous separation arguments, give $X'' = -\lambda X$, $Y'' = \lambda Y$ globally. The vertical sides impose $X(0) = X(\pi) = 0$. Integration by parts excludes $\lambda \leq 0$ for nonzero X ; solving the positive case gives $\lambda = n^2$ and $X = \sin(nx)$, with the spatial constant absorbed into Y .

Step 2: Match the one-mode top trace. The top data are nonzero, so $Y(H) \neq 0$ and X must be proportional to $\sin 2x$. Thus $n = 2$. The remaining conditions are $Y(0) = 0$, $Y(H) = 1$, which determine

$$u(x, y) = \sin(2x) \frac{\sinh(2y)}{\sinh(2H)}.$$

Its second x derivative is $-4u$ and its second y derivative is $4u$. The sum is zero. The sine factor vanishes at the sides, $\sinh 0 = 0$ gives the bottom, and the ratio equals one at the top. The denominator is nonzero since $H > 0$. This uniquely determines the field within the stipulated product class.

Step 3: Reject a plausible sign error. For $\tilde{u} = \sin 2x \sin 2y$, both second derivatives equal $-4\tilde{u}$, so $\Delta \tilde{u} = -8\tilde{u}$, not identically zero. Rescaling a nonzero field to repair top values would not remove this interior residual. The required equation $Y'' - 4Y = 0$ has hyperbolic, rather than sinusoidal, solutions because the Laplace equation sums the second derivatives.

Step 4: Interpret growth in a bounded spatial direction. For $a(y) = \sinh(2y)/\sinh(2H)$, one has $a(0) = 0$, $a(H) = 1$ and $a'(y) = 2 \cosh(2y)/\sinh(2H) > 0$. Hence $0 \leq a \leq 1$ and $|u| \leq 1$. For the same $X'' = -\lambda X$ with $\lambda > 0$, the remaining equations are

$$\begin{aligned} \text{heat:} \quad & T' + \kappa \lambda T = 0, \\ \text{wave:} \quad & T'' + c^2 \lambda T = 0, \\ \text{Laplace:} \quad & Y'' - \lambda Y = 0. \end{aligned}$$

Spatial hyperbolic growth in the last equation is compatible with bounded boundary data on a finite rectangle; it is not a statement about time instability.

Question 7

On $1 < x < e$, $t \geq 0$, consider the variable-coefficient equation

$$\frac{1}{x}u_t = (xu_x)_x, \quad u(1, t) = u(e, t) = 0.$$

For a nonzero product $u = X(x)T(t)$, use the separation convention $(xX')' + \lambda X/x = 0$, $T' = -\lambda T$. Introduce $s = \log x$ and $Z(s) = X(e^s)$.

Tasks

1. Derive the factor equations from the PDE and verify, by the chain rule, the transformed spatial equation and its endpoint conditions in s .
2. Determine every eigenvalue and spatial factor and verify the resulting separated fields in the original x equation.
3. Derive an integral identity that proves eigenvalue positivity. Identify the weight in the squared spatial norm; explain why an unweighted denominator would not follow from this equation.
4. Prove weighted orthogonality of distinct spatial modes by subtracting their ODE identities. Evaluate their squared weighted norms and locate the interior zeros of the third mode in the original coordinate.

Question 7 – Solution

Strategy. A change of spatial coordinate can simplify separation, but the original equation determines the correct integration weight.

Step 1: Transform the spatial equation carefully. Substitution gives $XT'/x = (xX')'T$. Nonzero anchors yield $(xX')' = -\lambda X/x$ and $T' = -\lambda T$. Since $X' = Z_s/x$, one has $xX' = Z_s$ and $(xX')' = Z_{ss}/x$. Multiplying the spatial equation by x gives

$$Z_{ss} + \lambda Z = 0, \quad Z(0) = Z(1) = 0.$$

The physical endpoints $1, e$ become $0, 1$, not an interval of length $e - 1$.

Step 2: Recover the modes in the original variable. The transformed Dirichlet problem has $\lambda_n = n^2\pi^2$ and $Z_n(s) = \sin(n\pi s)$ for $n \geq 1$. Thus

$$u_n(x, t) = C \sin(n\pi \log x) e^{-n^2\pi^2 t}.$$

Indeed $xX'_n = n\pi \cos(n\pi \log x)$ and $(xX'_n)' = -n^2\pi^2 X_n/x$. Multiplication by the time factor gives exactly u_t/x . At $x = 1, e$ the sine values vanish. Zero and negative eigenvalues produce only the trivial Dirichlet factor.

Step 3: Derive the weighted energy quotient. Multiplication of $-(xX')' = \lambda X/x$ by X and integration give

$$\lambda \int_1^e \frac{X^2}{x} dx = \int_1^e x(X')^2 dx.$$

The endpoint term vanishes. For a nonzero factor the denominator is positive; zero numerator would force a constant and then the zero field. Hence $\lambda > 0$. The weight $1/x$ comes from the coefficient of u_t ; dropping it would change the identity and the associated eigenvalue quotient.

Step 4: Verify orthogonality, norms and nodes. For distinct modes m, n , subtracting the two integrated spatial identities cancels the derivative terms and yields $(\lambda_n - \lambda_m) \int_1^e X_n X_m dx/x = 0$. Thus their weighted inner product vanishes. With $s = \log x$,

$$\int_1^e \frac{X_n^2}{x} dx = \int_0^1 \sin^2(n\pi s) ds = \frac{1}{2}.$$

For $n = 3$, the interior zeros occur at $s = 1/3, 2/3$, hence at $x = e^{1/3}, e^{2/3}$. Equal spacing in s is not equal spacing in x .

Question 8

For $u_t = u_{xx}$ on $0 < x < L$, $t \geq 0$, with $L > 0$, impose

$$u(0, t) = 0, \quad u_x(L, t) + hu(L, t) = 0, \quad h \in \mathbb{R}.$$

A separated mode obeys $X'' + \lambda X = 0$, $T' = -\lambda T$. Define the dimensionless parameter $q = hL$. No sign restriction is imposed on h ; a negative value need not model passive heat loss.

Tasks

1. Derive the identity $\lambda \int_0^L X^2 = \int_0^L (X')^2 + hX(L)^2$. Show that $h \geq 0$ guarantees positive eigenvalues, and determine exactly when $\lambda = 0$ has a nonzero factor.
2. For $\lambda > 0$, derive an eigenvalue equation using $z = L\sqrt{\lambda}$. For $\lambda < 0$, use $\rho = L\sqrt{-\lambda}$ to derive $\rho \coth \rho = -q$.
3. Prove that the negative-eigenvalue equation has exactly one root if $q < -1$ and none if $q \geq -1$. Determine the time behavior of this mode and of the zero mode.
4. For $q = -2$, bracket the positive root ρ to within 0.001, state the corresponding eigenvalue in terms of L , and sketch the intersection defining it. Mark the limiting value at $\rho = 0$ as a limit, not a negative eigenvalue.

Question 8 – Solution

Strategy. A Robin boundary term can change the sign of the eigenvalue; analyze zero before treating the oscillatory and hyperbolic cases.

Step 1: Retain the boundary contribution. Integration by parts gives $\lambda \int X^2 = \int (X')^2 - [XX']_0^L = \int (X')^2 + hX(L)^2$. For $h \geq 0$, equality at zero would force $X' = 0$ and then $X = 0$; nonzero factors therefore have $\lambda > 0$. At $\lambda = 0$, $X = Ax$ and the right condition is $A(1 + hL) = 0$. Thus a nonzero zero mode exists exactly when $\boxed{q = -1}$.

Step 2: Separate the positive and negative cases. For $\lambda > 0$, $X = A \sin(\sqrt{\lambda}x)$ and

$$\boxed{z \cos z + q \sin z = 0, \quad z > 0.}$$

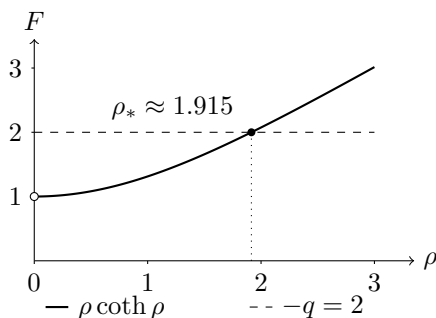
For $\lambda = -\mu^2 < 0$, $X = A \sinh(\mu x)$ and the right condition becomes $\mu \cosh(\mu L) + h \sinh(\mu L) = 0$. Since $\rho = \mu L > 0$,

$$\boxed{\rho \coth \rho = -q, \quad \lambda = -\rho^2/L^2.}$$

Dividing by $\sinh \rho$ here is valid only in this strictly negative case.

Step 3: Prove the threshold and uniqueness. Let $F(\rho) = \rho \coth \rho$. It has limit 1 as $\rho \downarrow 0$ and $F(\rho) \rightarrow \infty$. Its derivative is $F' = (\sinh \rho \cosh \rho - \rho) / \sinh^2 \rho > 0$: the numerator is zero at zero and has derivative $2 \sinh^2 \rho > 0$ for $\rho > 0$. Therefore its range is $(1, \infty)$, giving exactly one negative eigenvalue iff $q < -1$. That mode grows as $e^{\rho^2 t/L^2}$. At $q = -1$, the zero mode $X = x$ is stationary; it is not a negative eigenmode.

Step 4: Quantify the concrete crossing. For $q = -2$, numerical substitution gives $F(1.915) < 2 < F(1.916)$. Strict monotonicity certifies one root in this interval, approximately $\rho = 1.915008$. Thus $\lambda \approx -3.667256/L^2$. The open point at $(0, 1)$ records the excluded endpoint limit of F .



Question 9

Consider smooth product candidates $u(x, t) = X(x)T(t)$, neither factor identically zero, on an open rectangle $I \times J$, where each interval has positive length. Compare

$$(A) \quad u_t = u_{xx} + xtu, \quad (B) \quad u_t = u_{xx} + [a(x) + b(t)]u.$$

In (B), the given coefficient functions are smooth. No boundary data are imposed.

Tasks

1. Show that a nonzero product solving (A) would imply $T'/T - X''/X = xt$ on some smaller open rectangle where both factors are nonzero.
2. Prove that this identity is impossible by comparing two positions and two times. Explain why zeros of the factors cannot avoid the contradiction.
3. Derive separated ODEs for (B), using $X'' + a(x)X = -\lambda X$. Explain how the additive coefficient structure permits separation even though the PDE has variable coefficients.
4. Take $a(x) = -x^2$, $b(t) = t$. Construct a nonzero product with $X = e^{-x^2/2}$, determine λ and T , and verify the resulting field directly in (B).

Question 9 – Solution

Strategy. Test the coefficient's dependence on both variables before assuming that the product ansatz can succeed.

Step 1: Divide only on a legitimate local rectangle. A nonzero value of each continuous factor has a neighborhood where that factor is nonzero. On the product of those neighborhoods, substitution into (A) gives

$$\frac{T'(t)}{T(t)} - \frac{X''(x)}{X(x)} = xt.$$

The left side is a function of time minus a function of position. This local identity is already enough to test whether any nonzero product can exist.

Step 2: Use a mixed difference, not extra derivatives. Choose distinct x_1, x_2 and distinct t_1, t_2 in that rectangle. Subtract the four identities in the pattern $(x_1, t_1) - (x_1, t_2) - (x_2, t_1) + (x_2, t_2)$. All terms on the left cancel, whereas the right gives

$$0 = (x_1 - x_2)(t_1 - t_2),$$

which is false. Thus (A) has no nonzero product solution on an open rectangle. Factor zeros elsewhere cannot undo this contradiction in a nonzero neighborhood. The zero solution still exists; the conclusion does not assert that all solutions of (A) vanish.

Step 3: Separate an additive coefficient instead. For (B), substitution rearranges to $X[T' - b(t)T] = [X'' + a(x)X]T$. Nonzero anchors give, with one constant λ ,

$$X'' + [a(x) + \lambda]X = 0, \quad T' = [b(t) - \lambda]T.$$

The remaining coefficients each depend on only the appropriate variable. Conversely these ODEs imply the product PDE everywhere. Variable coefficients do not alone prevent separation; their dependence on the variables matters.

Step 4: Verify a nonconstant-coefficient example. For $X = e^{-x^2/2}$, $X'' = (x^2 - 1)X$, so $X'' - x^2X = -X$ and $\lambda = 1$. The time equation $T' = (t - 1)T$ gives $T = Ce^{t^2/2 - t}$. Hence

$$u = Ce^{-x^2/2 + t^2/2 - t}, \quad C \neq 0.$$

Directly $u_t = (t - 1)u$ and $u_{xx} = (x^2 - 1)u$; adding $(-x^2 + t)u$ to the second expression gives the first. This is an explicit product solution of (B), not a solution of (A).

Question 10

For $u_t = u_{xx} + ru$ on $0 < x < \pi$, $t \geq 0$, impose homogeneous Dirichlet endpoints. The separated spatial modes are $\sin(nx)$ with eigenvalues n^2 , $n = 1, 2, \dots$. Consider $1 < r < 4$ and the finite-mode initial field

$$u(x, 0) = \sin(2x) + \varepsilon \sin x, \quad 0 < \varepsilon < 1.$$

Only this finite-mode construction is requested; no arbitrary-data expansion or infinite-series convergence theorem is needed.

Tasks

1. Derive the time equation for each mode and construct a solution with the given initial field. Verify its PDE and endpoints.
2. Compare the behavior when $\varepsilon = 0$ with every fixed $\varepsilon > 0$. Prove that in the latter case $e^{-(r-1)t}u$ converges uniformly to $\varepsilon \sin x$.
3. Using $\|v\|_2^2 = \int_0^\pi v^2 dx$, compute the squared norm of each modal component and determine the time when the growing component overtakes the decaying component in this norm.
4. For $r = 2$, $\varepsilon = 1/8$, sketch the two positive modal amplitudes and mark their crossing. Explain why finding a decaying separated solution alone does not show that all solutions decay, and state the first-mode threshold in r .

Question 10 – Solution

Strategy. Separation reveals individual growth rates; superposition shows how a small unstable component can dominate a decaying one.

Step 1: Construct and verify the finite sum. For $X_n = \sin nx$, $X_n'' = -n^2 X_n$, so the factor equation is $T_n' = (r - n^2)T_n$. Consequently

$$u = e^{(r-4)t} \sin 2x + \varepsilon e^{(r-1)t} \sin x.$$

Each time derivative has coefficient $r - n^2$, matching $u_{xx} + ru$. Both sine factors vanish at $0, \pi$, and their initial coefficients are $1, \varepsilon$. Linearity justifies adding these two verified modes.

Step 2: Compare exact cancellation with small contamination. For $\varepsilon = 0$, the solution decays uniformly because $r - 4 < 0$. For fixed $\varepsilon > 0$, the first mode grows because $r - 1 > 0$. Moreover

$$e^{-(r-1)t} u = \varepsilon \sin x + e^{-3t} \sin 2x, \quad \|e^{-(r-1)t} u - \varepsilon \sin x\|_\infty = e^{-3t}.$$

This proves uniform convergence of the normalized shape. The unnormalized solution grows without bound in sup norm, as is also seen at $x = \pi/2$.

Step 3: Compare modal norms exactly. Since $\int_0^\pi \sin^2(nx) dx = \pi/2$, the squared component norms are

$$N_2^2 = \frac{\pi}{2} e^{2(r-4)t}, \quad N_1^2 = \frac{\pi}{2} \varepsilon^2 e^{2(r-1)t}.$$

Their equality is $\varepsilon^2 e^{6t} = 1$, giving $t_c = \frac{1}{3} \log(1/\varepsilon)$. The growing component is larger for $t > t_c$. The crossing time is independent of r in the stated range because the difference of modal rates is always three. Orthogonality also gives $\|u\|_2^2 = N_1^2 + N_2^2$.

Step 4: Interpret the crossing and the threshold. For $r = 2$, $\varepsilon = 1/8$, the amplitudes are $a_1 = e^t/8$ and $a_2 = e^{-2t}$. They cross at $t_c = \log 2$, with common value $1/4$. A decaying mode only describes its own initial shape. A nonzero first-mode component decays for $r < 1$, is stationary for $r = 1$, and grows for $r > 1$. Thus one decaying example cannot establish decay for every admissible field.

