

Question 1

Let $u = u(x, t)$ be an unknown scalar field on $\mathbb{R}^2$, and let a, b, c be real constants. Consider the family

$$au_{tt} + u_t - (1 + x^2)u_{xx} + buu_x + c(u_x)^2 = \sin t.$$

The order of a PDE is the highest derivative order actually present. A linear PDE has coefficients depending only on the independent variables, and the unknown and all its derivatives occur linearly. For a linear PDE, homogeneous means that its prescribed right-hand side is zero.

Tasks

1. Identify the independent and dependent variables and determine the order for every a , including $a = 0$. Explain why second order need not mean second order in time.
2. Classify exactly which pairs (b, c) make the equation linear. In that case, decide whether it has constant coefficients and whether it is homogeneous.
3. The nonlinear term uu_x vanishes on every field independent of x . Explain why testing only those fields cannot establish linearity. Give a scaling test that detects each nonzero nonlinear coefficient.
4. Replace the right-hand side by zero. Explain what changes in the linear case, and why a zero right-hand side alone cannot make a nonlinear equation linear.

Question 1 – Solution

Strategy. Classify the differential expression on arbitrary admissible fields, rather than on a special solution.

Step 1: Count the derivatives that remain. The independent variables are x, t and the dependent variable is u . Since $1 + x^2 > 0$, the term u_{xx} never disappears. The order is therefore $\boxed{2}$ for every a . When $a = 0$, the equation is first order in time but still second order as a PDE. When $a \neq 0$, it is also second order in time.

Step 2: Separate coefficients from unknowns. The expression is linear exactly when $\boxed{b = c = 0}$. Then $1 + x^2$ is an allowed variable coefficient, so the PDE is linear with variable coefficients. It is nonhomogeneous because $\sin t$ is not identically zero on the domain. A forcing term may vanish at some times without making the equation homogeneous.

Step 3: Test scaling on informative fields. Write the nonlinear part as $N[u] = buu_x + c(u_x)^2$. It satisfies $N[2u] - 2N[u] = 2N[u]$, whereas a linear operator would give zero. For the smooth test field $u = x$, this difference is

$$2(bx + c).$$

It vanishes for every x only when $b = c = 0$. In particular, if $b \neq 0$ it is not the zero function; if $b = 0$ and $c \neq 0$ it is the nonzero constant $2c$. The restricted tests $u = h(t)$ have $u_x = 0$ and conceal both nonlinear terms. Linearity must hold on all fields in the operator's domain.

Step 4: Classify the equation after changing the forcing. With zero right-hand side and $b = c = 0$, the equation is linear homogeneous. With $(b, c) \neq (0, 0)$, the same scaling obstruction remains, regardless of the right-hand side. Under the stated convention, “homogeneous linear PDE” is not a label for an arbitrary nonlinear equation written with zero on one side.

Question 2

On $\mathbb{R} \times [0, \infty)$ define $L[u] = u_t - u_{xx}$ and consider $L[u] = 2$. Set

$$u_1(x, t) = 2t, \quad u_2(x, t) = 2t + e^{-t} \cos x.$$

No initial or boundary conditions are imposed unless explicitly stated.

Tasks

1. Verify that u_1, u_2 solve the PDE and find the exact condition on real α, β for $\alpha u_1 + \beta u_2$ to solve the same PDE.
2. Determine the equations satisfied by $u_2 - u_1$ and $u_1 + u_2$. Explain why a linear equation need not have a vector space of solutions.
3. Prove that the complete solution set is $u_1 + \ker L$, where $\ker L = \{w : L[w] = 0\}$ in the chosen smooth function class. Explain the word affine using this representation.
4. Now impose $u(x, 0) = \cos x$. Decide which displayed solution satisfies the full problem. Explain why this check alone is neither a construction of all solutions nor a uniqueness proof.

Question 2 – Solution

Strategy. Apply the operator to a combination before applying a superposition rule.

Step 1: Verify the source and its coefficient. For u_1 , $u_{1,t} = 2$ and $u_{1,xx} = 0$. For u_2 , $u_{2,t} = 2 - e^{-t} \cos x$ and $u_{2,xx} = -e^{-t} \cos x$. Thus $L[u_j] = 2$. Linearity gives

$$L[\alpha u_1 + \beta u_2] = 2(\alpha + \beta).$$

Consequently the combination solves the same equation exactly when $\boxed{\alpha + \beta = 1}$, not for all coefficients.

Step 2: Distinguish differences from sums. The difference has source $2 - 2 = 0$, so $u_2 - u_1 \in \ker L$. The sum has source 4 and is not a solution of $L[u] = 2$. The zero field is not a solution either. Thus this solution set is not a vector space, although its defining operator is linear. For $L[w] = 0$, arbitrary linear combinations do remain solutions.

Step 3: Describe every solution by a translation. If $L[u] = 2$, then $L[u - u_1] = 0$, so $u = u_1 + w$ with $w \in \ker L$. Conversely $L[u_1 + w] = 2 + 0 = 2$ for every such w . Hence

$$\boxed{\{u : L[u] = 2\} = u_1 + \ker L.}$$

This is an affine space: a translate of a vector space. Combinations whose coefficients sum to one stay in it. The particular solution provides an origin for the translation, but is not the zero vector of the ambient function space.

Step 4: Check the additional data separately. At $t = 0$, $u_1 = 0$ and $u_2 = \cos x$, so only u_2 among these two fields satisfies the specified initial condition. This proves existence in the smooth class because u_2 is an explicit solution. It does not prove uniqueness among all fields; that requires a theorem or argument in a stated function class, including any necessary behavior at spatial infinity. Comparing two candidates cannot eliminate all other candidates.

Question 3

Let $u(x, t)$ solve $u_t = \kappa u_{xx}$ on $0 < x < L$, $0 < t < T$, where $\kappa, L, T > 0$. For this problem, initial data are prescribed at $t = 0$; spatial boundary data are prescribed at $x = 0, L$. The outward normal derivative is $\partial_n u = n_x u_x$, with $n_x = -1$ on the left and $n_x = 1$ on the right. Consider

$$u(x, 0) = f(x), \quad u(0, t) = A(t), \quad \kappa \partial_n u(L, t) + hu(L, t) = r(t), \quad h > 0.$$

Dirichlet data prescribe u , Neumann data prescribe $\partial_n u$, and Robin data prescribe a linear combination with nonzero coefficients of both.

Tasks

1. Classify each condition, and identify the portions of the space-time rectangle on which they are prescribed. Is $t = T$ a spatial boundary?
2. Rewrite the right condition with u_x . Write the corresponding coordinate form of $\kappa \partial_n u + hu = r$ if it were imposed at the left endpoint instead.
3. Classify the spatial boundary operator when $h = 0$, and decide exactly when the displayed spatial data are homogeneous. Distinguish this from homogeneity of the PDE and of the initial data.
4. Verify the manufactured field $u_*(x, t) = x^2 + 2\kappa t$. Find f, A, r that make it solve the full displayed problem, and illustrate where each datum acts.

Question 3 – Solution

Strategy. Identify both the operator being prescribed and the geometric location of the data.

Step 1: Locate and name the data. The bottom edge $t = 0$ carries initial displacement/temperature data $f(x)$. The left edge carries Dirichlet data $A(t)$; the right carries Robin data $r(t)$. The top $t = T$ is a final-time slice, not a spatial boundary. No data are prescribed there in this initial-boundary-value problem.

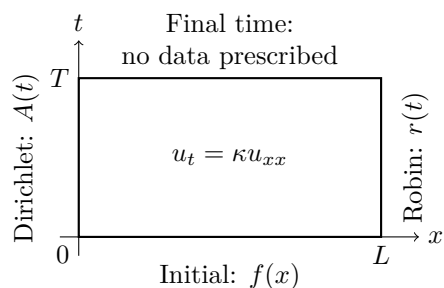
Step 2: Keep the outward signs. At the right, the condition is $\kappa u_x(L, t) + hu(L, t) = r(t)$. At the left, the analogous outward-normal condition would be $-\kappa u_x(0, t) + hu(0, t) = r(t)$. Changing the endpoint changes the sign of the normal derivative, not the sign of the prescribed value by convention.

Step 3: Classify homogeneity at each level. When $h = 0$, the right condition is Neumann, since $\kappa > 0$. The spatial data are homogeneous exactly when $A \equiv 0$ and $r \equiv 0$. The PDE is already linear homogeneous; initial data are homogeneous exactly when $f \equiv 0$. These are three separate assertions. A nonzero Robin coefficient h does not itself make the condition nonhomogeneous.

Step 4: Verify all parts of a full problem. Here $u_{*,t} = 2\kappa$ and $u_{*,xx} = 2$, so the PDE holds. Evaluation gives

$$f(x) = x^2, \quad A(t) = 2\kappa t, \quad r(t) = 2\kappa L + h(L^2 + 2\kappa t).$$

These data also agree at their shared corners. The diagram locates the data; it is a space–time domain, not a graph of the solution value.



Question 4

Consider $u_t = u_{xx}$ for $0 < x < 1$, $0 < t \leq T$, with

$$u(x, 0) = f(x) = x(1 - x), \quad u(0, t) = A(t), \quad u(1, t) = B(t).$$

For this question a solution smooth up to the corners has continuous u, u_t, u_x, u_{xx} on the closed rectangle, satisfies the PDE there by continuity, and has $u_t(x, 0)$ and $u_{xx}(x, 0)$ obtained by differentiating its traces. Assume A, B are continuously differentiable.

Tasks

1. Derive the necessary value compatibility conditions at the two initial corners.
2. Derive the next necessary compatibility conditions involving $A'(0), B'(0)$ and f'' .
3. Set $A = B = 0$. Decide whether value compatibility holds and whether a solution in the stated corner-smooth class can exist. Explain why this does not rule out solutions smooth only for positive time.
4. Keep f fixed and construct linear-in-time boundary data for which a polynomial solution exists. Verify the PDE and every datum explicitly; distinguish this sufficiency proof from checking necessary conditions alone.

Question 4 – Solution

Strategy. At a corner, two different traces must describe the same value and the same time derivative.

Step 1: Match the values. Continuity at $(0, 0)$ and $(1, 0)$ requires

$$A(0) = f(0) = 0, \quad B(0) = f(1) = 0.$$

If these equalities fail, even a continuous solution on the entire closed rectangle is impossible. Interior smoothness alone does not imply these equalities.

Step 2: Match the PDE to the time traces. The left boundary trace gives $u_t(0, 0) = A'(0)$ and the initial trace gives $u_{xx}(0, 0) = f''(0)$. The PDE therefore requires $A'(0) = f''(0)$. The right corner gives the analogous identity. Since $f'' = -2$,

$$A'(0) = B'(0) = -2.$$

These conditions use the stated corner regularity; they cannot be imposed silently on a class that has no continuous u_{xx} at the corners.

Step 3: Diagnose homogeneous endpoint data. For $A = B = 0$, both value conditions hold, but $A'(0) = B'(0) = 0 \neq -2$. Thus no solution exists in the corner-smooth class specified here. This is a regularity obstruction at $t = 0$, not a proof of nonexistence in a weaker class. A solution may be continuous up to the initial edge and smooth for $t > 0$ without its second spatial derivative extending continuously through the initial corners. No such weaker solution is needed for the contradiction.

Step 4: Construct a compatible full example. Take $A(t) = B(t) = -2t$ and

$$u(x, t) = x(1 - x) - 2t.$$

Then $u_t = u_{xx} = -2$, $u(x, 0) = f(x)$ and $u(0, t) = u(1, t) = -2t$. This polynomial has all required corner derivatives and proves existence for the modified problem. Matching a finite list of necessary compatibility conditions would not alone prove existence for arbitrary data; the explicit field supplies the sufficiency argument in this case.

Question 5

For the first-order PDE $u_t + u_x = 0$ on $\mathbb{R}^2$, consider C^1 solutions. A characteristic curve is a curve $(x(t), t)$ along which the PDE forces u to be constant. Suppose data are prescribed on the line $x = at$:

$$u(at, t) = h(t), \quad t \in \mathbb{R},$$

where $a \in \mathbb{R}$ and $h \in C^1(\mathbb{R})$.

Tasks

1. Use the chain rule to identify the characteristic lines, and prove that every C^1 solution has the form $u(x, t) = F(x - t)$.
2. For $a \neq 1$, determine the unique F and verify the resulting full solution for arbitrary h .
3. For $a = 1$, classify exactly which h permit a solution and whether that solution is unique. Give two distinct solutions when $h \equiv H$ is constant.
4. Compare $a = 0, h(t) = \sin t$ with $a = 1, h(t) = \sin t$. Explain geometrically why data on an entire line can determine a unique field, no field, or many fields.

Question 5 – Solution

Strategy. Check whether the data line intersects all characteristic lines or follows just one of them.

Step 1: Recover the characteristic form. Along $x = t + s$, the chain rule gives $\frac{d}{dt}u(t+s, t) = u_x + u_t = 0$. Thus $u(t+s, t) = u(s, 0)$. Defining $F(s) = u(s, 0)$ proves $\boxed{u(x, t) = F(x-t)}$ for every solution. Conversely this form has $u_t = -F'$ and $u_x = F'$, so it solves the PDE.

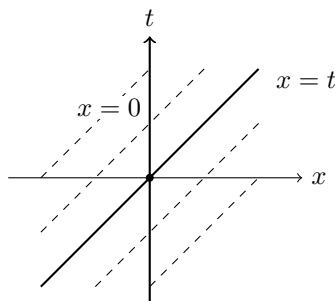
Step 2: Use a noncharacteristic data line. The condition becomes $F((a-1)t) = h(t)$. If $a \neq 1$, the argument ranges over all real numbers exactly once. Consequently

$$\boxed{F(s) = h\left(\frac{s}{a-1}\right), \quad u(x, t) = h\left(\frac{x-t}{a-1}\right)}.$$

This is C^1 , solves the PDE and takes the prescribed trace. The characteristic representation proves uniqueness in the whole stated C^1 class.

Step 3: Diagnose characteristic data. If $a = 1$, the condition reduces to $F(0) = h(t)$ for every t . A solution exists exactly when h is constant, say H . Then any C^1 function with $F(0) = H$ works. For example, $u = H$ and $u = H + x - t$ are distinct solutions. Thus compatible characteristic data leave infinitely many fields undetermined.

Step 4: Interpret the two sine traces. For $a = 0$, $u(x, t) = \sin(t-x)$ is the unique solution. For $a = 1$, the varying trace $\sin t$ contradicts constancy along that characteristic, so no solution exists. The diagram shows the unique intersection with $x = 0$ and the characteristic line $x = t$; the latter supplies information on just one member of the family.



Characteristic family: $x - t = s$

Question 6

For $u_t + u_x = 0$ on $\mathbb{R}^2$, a classical solution here means a C^1 field satisfying the PDE at every point. Compare

$$u(x, t) = |x - t|, \quad u_\varepsilon(x, t) = \sqrt{(x - t)^2 + \varepsilon^2}, \quad \varepsilon > 0.$$

For this question only, an integral solution is a locally integrable field satisfying

$$\int_{\mathbb{R}^2} u(x, t) (\phi_t + \phi_x)(x, t) dx dt = 0$$

for every smooth compactly supported test function ϕ .

Tasks

1. Check the PDE for u off $x = t$, and decide whether u is a classical solution on the whole plane.
2. Verify that each u_ε is classical and prove a uniform error bound tending to zero as $\varepsilon \downarrow 0$.
3. Explain precisely why uniform convergence of these classical solutions does not establish that u is classical. Inspect $u_{\varepsilon, x}$ near the line $x = t$.
4. Verify that u is an integral solution using $s = x - t$, $r = t$. State which conclusions about u have and have not been proved.

Question 6 – Solution

Strategy. A solution label includes a regularity requirement; checking a formula away from a singular line is not enough.

Step 1: Locate the failure of classical regularity. For $x > t$, $u_x = 1, u_t = -1$; for $x < t$, $u_x = -1, u_t = 1$. Thus the PDE holds on both open regions. At $x = t$, the two one-sided spatial derivatives disagree. The field is not differentiable there, hence is not a classical solution on $\mathbb{R}^2$.

Step 2: Verify the smooth approximations. Writing $s = x - t$, one has

$$u_{\varepsilon,x} = \frac{s}{\sqrt{s^2 + \varepsilon^2}}, \quad u_{\varepsilon,t} = -\frac{s}{\sqrt{s^2 + \varepsilon^2}}.$$

These are smooth and sum to zero. Also

$$0 \leq \sqrt{s^2 + \varepsilon^2} - |s| = \frac{\varepsilon^2}{\sqrt{s^2 + \varepsilon^2} + |s|} \leq \varepsilon.$$

Equality occurs at $s = 0$, so the uniform error on the whole plane is exactly ε .

Step 3: Separate convergence of values from derivatives. For $s \neq 0$, $u_{\varepsilon,x} \rightarrow \text{sgn}(s)$, while its value at $s = 0$ is always zero. This pointwise limit is discontinuous at zero. It cannot be a uniform limit on any closed interval about zero of these continuous derivatives. Uniform convergence of function values does not preserve differentiability, so it does not prove classical solvability.

Step 4: Check the stated integral formulation. Set $\psi(s, r) = \phi(s + r, r)$. The change of variables has Jacobian one, and $\psi_r = \phi_x + \phi_t$. Compact support and local integrability allow integration in either order. Therefore

$$\int_{\mathbb{R}^2} |x - t|(\phi_t + \phi_x) dx dt = \int_{\mathbb{R}} |s| \left(\int_{\mathbb{R}} \psi_r(s, r) dr \right) ds = 0.$$

The inner integral vanishes by compact support. Thus u is an integral solution under the supplied definition, as well as a classical solution off the line. It remains nonclassical on the whole plane; no uniqueness assertion follows from this verification alone.

Question 7

For a real second-order linear PDE in two spatial variables, use the convention

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{lower-order terms} = 0, \quad D = B^2 - AC.$$

At points where the second-order part is nonzero, call the equation hyperbolic if $D > 0$, elliptic if $D < 0$, and parabolic if $D = 0$. Consider

$$u_{xx} + 2yu_{xy} + xu_{yy} + 3u_x - u = 0.$$

Tasks

1. Identify A, B, C and classify the equation at every point of the plane, including the separating curve. Explain why the coefficient of u_{xy} is not B itself.
2. Classify the points $(0, 1), (2, 1), (1, 1)$, and sketch the type regions with correct axes and boundary curve.
3. At a fixed point, rewrite the principal quadratic form $Q(\xi, \eta) = \xi^2 + 2y\xi\eta + x\eta^2$ as a completed square. Explain its signs and rank in all three regions.
4. Replace $3u_x - u$ by $e^{x+y}u_x + 7u$ and then multiply the entire PDE by a smooth nowhere-zero function $m(x, y)$. Decide which changes can alter the type and justify your answer.

Question 7 – Solution

Strategy. Classify the principal second-order part point by point, keeping the mixed-derivative convention fixed.

Step 1: Read the discriminant correctly. Here $A = 1$, $B = y$ and $C = x$, since the mixed coefficient is $2B = 2y$. Thus $D = y^2 - x$, giving

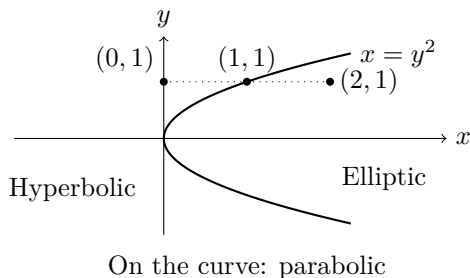
$x < y^2$: hyperbolic, $x = y^2$: parabolic, $x > y^2$: elliptic.
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Since $A = 1$, the principal part never vanishes completely. The curve is therefore parabolic under the stated convention, not an order-zero exception.

Step 2: Check the points and the geometry. The three discriminants are respectively 1, -1 , 0, so $(0, 1)$ is hyperbolic, $(2, 1)$ elliptic and $(1, 1)$ parabolic. With x horizontal and y vertical, $x = y^2$ is a right-opening parabola; ellipticity lies to its right.

Step 3: Interpret the principal form. Completing the square gives $Q = (\xi + y\eta)^2 + (x - y^2)\eta^2$. For $x > y^2$, it is positive definite, with rank two. For $x < y^2$, the choices $(\xi, \eta) = (1, 0)$ and $(-y, 1)$ give opposite signs; it is indefinite with rank two. On $x = y^2$, it is a nonnegative square of rank one, vanishing on the nonzero direction $(-y, 1)$. This explains the type boundary algebraically.

Step 4: Identify changes that preserve type. Changing lower-order terms leaves A, B, C unchanged. Multiplying by m gives $\bar{D} = (mB)^2 - (mA)(mC) = m^2D$, with the same sign because $m \neq 0$. Neither operation changes type. Allowing $m = 0$ could erase the principal part at those points, which is why the nowhere-zero restriction matters.



Question 8

Let $u_t = \kappa u_{xx}$ on $0 < x < L$, $t \geq 0$, with $\kappa, L > 0$ and

$$u(0, t) = A(t), \quad u(L, t) = B(t), \quad u(x, 0) = f(x).$$

Assume all data are smooth and $f(0) = A(0)$, $f(L) = B(0)$. Define a boundary lifting

$$\ell(x, t) = \left(1 - \frac{x}{L}\right) A(t) + \frac{x}{L} B(t), \quad v = u - \ell.$$

Tasks

1. Determine the PDE, endpoint data and initial data for v . Classify the homogeneity of its PDE and its spatial boundary conditions separately.
2. Find the exact condition on A, B for this affine lifting to leave the transformed PDE homogeneous for every x, t .
3. For $A(t) = t$, $B(t) = 2t$, $f = 0$, compute the transformed source and initial data. Does making the endpoint conditions homogeneous remove the time-dependent forcing from the full problem?
4. Show that a lifting is not unique. Replace ℓ by $\tilde{\ell} = \ell + q$, where $q(0, t) = q(L, t) = 0$, and derive the new source and initial data. Explain why this is a change of unknown, not a different physical field u .

Question 8 – Solution

Strategy. Subtract the lifting from every part of the problem, including the differential equation and the initial condition.

Step 1: Compute the transformed full problem. Since $\ell_{xx} = 0$,

$$v_t - \kappa v_{xx} = -\ell_t = -\left(1 - \frac{x}{L}\right) A'(t) - \frac{x}{L} B'(t).$$

The endpoints are $v(0, t) = v(L, t) = 0$, and $v(x, 0) = f(x) - \ell(x, 0)$. The endpoint conditions are homogeneous Dirichlet. The transformed PDE is linear, but generally nonhomogeneous. Its initial condition is homogeneous only if $f = \ell(\cdot, 0)$.

Step 2: Determine when the source vanishes. For each t , the displayed affine function of x vanishes throughout $(0, L)$ if and only if both its endpoint limits vanish. Thus $A'(t) = B'(t) = 0$ for all t , or equivalently $\boxed{A, B \text{ are constant in time}}$. Nonzero constant boundary values can therefore produce a homogeneous transformed PDE, although the original boundary data were nonhomogeneous.

Step 3: Inspect time-dependent endpoint values. For $A = t, B = 2t, \ell = t(1 + x/L)$, so

$$v_t - \kappa v_{xx} = -(1 + x/L), \quad v(0, t) = v(L, t) = 0, \quad v(x, 0) = 0.$$

The boundary forcing has moved into the interior equation; it has not been removed from the full problem. In particular, $v = 0$ would fail this PDE. The stated value compatibility holds, but no corner-smooth existence is claimed for these data; higher compatibility would require additional checks.

Step 4: Track the freedom in the lifting. Let $\tilde{v} = u - \tilde{\ell} = v - q$. Direct differentiation gives

$$\tilde{v}_t - \kappa \tilde{v}_{xx} = -\ell_t - q_t + \kappa q_{xx}, \quad \tilde{v}(x, 0) = f(x) - \ell(x, 0) - q(x, 0).$$

The endpoints still vanish. For example, $q = x(L - x)e^{-t}$ is a nonzero smooth choice with zero endpoints. Although the source and transformed initial data change, $u = v + \ell = \tilde{v} + \tilde{\ell}$ is identical. Each lifting gives an equivalent formulation when every datum is transformed consistently.

Question 9

A problem is well posed in specified function spaces if solutions exist, are unique, and depend continuously on the prescribed data. To examine the third requirement, consider the Cauchy data problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y \leq 1; \quad u(x, 0) = f(x), \quad u_y(x, 0) = g(x),$$

with $u(0, y) = u(\pi, y) = 0$. Measure the bottom data by $\|f\|_\infty + \|g\|_\infty$ on $[0, \pi]$ and the observed solution by $\|u(\cdot, 1)\|_\infty$. For integers $n \geq 1$, set

$$u_n(x, y) = \frac{e^{-n}}{n} \sin(nx) \sinh(ny).$$

Tasks

1. Verify the PDE and side conditions and find the exact bottom data f_n, g_n .
2. Compute the bottom data norm and top solution norm. Determine their limits as $n \rightarrow \infty$.
3. Prove that the amplification ratio is unbounded and explain why the calculation rules out a continuous solution map at zero in the stated norms if the problem is uniquely solvable on a linear class containing these fields.
4. Explain why convergence of $u_n(\cdot, 1)$ to zero does not rescue continuity. Rescale the fields to obtain data converging to zero while the top norm stays exactly one, and sketch the vertical amplitude profiles for three n .

Question 9 – Solution

Strategy. Small absolute outputs are not enough for stability; compare their size with the much smaller input, then normalize.

Step 1: Verify the supplied solutions. Two x derivatives multiply u_n by $-n^2$; two y derivatives multiply it by n^2 . Their sum is zero, and $\sin(nx)$ vanishes on both sides. At the bottom, $f_n = 0, \quad g_n = e^{-n} \sin(nx)$. These are smooth fields with compatible side and bottom values.

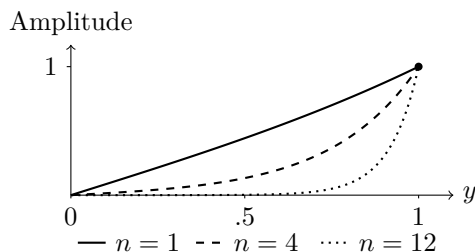
Step 2: Measure the input and output. For each integer n , $\sup_{[0,\pi]} |\sin(nx)| = 1$. Hence the bottom norm is e^{-n} and the top norm is

$$M_n = \frac{e^{-n}}{n} \sinh n = \frac{1 - e^{-2n}}{2n}.$$

Both tend to zero. Thus this sequence by itself does not contradict continuity; the relative amplification is the information needed for the next step.

Step 3: Expose unbounded amplification. The ratio is $M_n/e^{-n} = \sinh n/n \rightarrow \infty$. If uniqueness holds in a linear solution class containing these fields, solving the homogeneous linear PDE with data is a linear map. A continuous linear map between the stated normed spaces must obey a uniform bound $\|u(\cdot, 1)\|_\infty \leq C(\|f\|_\infty + \|g\|_\infty)$ on its domain. The unbounded ratios rule out that bound. No existence or uniqueness theorem for arbitrary Cauchy data has been assumed.

Step 4: Give a direct discontinuity sequence. Set $w_n = u_n/M_n = \sin(nx) \sinh(ny)/\sinh n$. Its top norm is one, whereas its bottom norm is $n/\sinh n \rightarrow 0$. The zero data have the zero solution; uniqueness would force the solution map to select these w_n , contradicting continuity at zero. The vertical envelopes are $\sinh(ny)/\sinh n$, not solution traces at one fixed x for every n .



Question 10

On the unit square $\Omega = (0, 1)^2$, consider a C^2 field smooth up to each edge, satisfying

$$\Delta u = s, \quad \partial_n u = g \text{ on } \partial\Omega,$$

where s is a constant and g is constant on each open edge. Denote these four constants by g_L, g_R, g_B, g_T . At vertices, interpret boundary data by one-sided edge traces. You may use the divergence theorem and Green's identity.

Tasks

1. Identify the boundary type and derive a necessary relation between s and the four edge values, retaining the outward-normal signs.
2. For arbitrary edge constants satisfying that relation, construct a polynomial solution of the form $u = ax^2 + by^2 + cx + dy + C$. Verify that the compatibility condition is also sufficient for this data family.
3. Prove that any two solutions for the same compatible data differ by a constant. Explain why this is uniqueness up to a constant rather than uniqueness of the original problem.
4. Take $(g_L, g_R, g_B, g_T) = (0, 2, 0, 4)$. Find s , impose the normalization $\int_{\Omega} u \, dA = 0$, and give the resulting unique solution. Would the same edge data be compatible with $s = 0$?

Question 10 – Solution

Strategy. Separate existence compatibility from the constant ambiguity left by Neumann data.

Step 1: Integrate the equation before solving it. These are Neumann conditions. Since the square has area one and each edge has length one, the divergence theorem gives

$$s = \int_{\Omega} \Delta u \, dA = \int_{\partial\Omega} \partial_n u \, ds = g_L + g_R + g_B + g_T.$$

In coordinates the derivatives are $-u_x$ on the left, u_x on the right, $-u_y$ on the bottom and u_y on the top. Replacing all by positive coordinate derivatives would give the wrong compatibility relation.

Step 2: Construct every compatible polynomial. For the proposed polynomial the four traces give

$$c = -g_L, \quad 2a + c = g_R, \quad d = -g_B, \quad 2b + d = g_T.$$

Thus

$$u = \frac{g_R + g_L}{2}x^2 + \frac{g_T + g_B}{2}y^2 - g_Lx - g_By + C.$$

Its Laplacian is the sum of the four edge values, exactly s when compatible. Every boundary derivative has the prescribed value. This explicit construction proves sufficiency for constant source and constant edge data on this square; it is not a general sufficiency theorem for arbitrary domains and functions.

Step 3: Prove the exact remaining ambiguity. The difference w of two solutions obeys $\Delta w = 0$, $\partial_n w = 0$. Green's identity yields

$$\int_{\Omega} |\nabla w|^2 \, dA = \int_{\partial\Omega} w \partial_n w \, ds - \int_{\Omega} w \Delta w \, dA = 0.$$

Continuity gives $\nabla w = 0$ everywhere, and connectedness of the square makes w constant. Conversely any added constant leaves all equations and data unchanged. A value or mean normalization is needed for actual uniqueness.

Step 4: Normalize the concrete field. Here $s = 6$ and $u = x^2 + 2y^2 + C$. Its integral is $1/3 + 2/3 + C = 1 + C$. The zero-mean condition fixes $C = -1$, so $u = x^2 + 2y^2 - 1$ is unique among solutions with that normalization. For $s = 0$, the integrated compatibility condition would require $0 = 6$, so no solution exists with these edge data.