

Question 1

An ideal string has constant tension $\mathcal{T} > 0$ and mass per unit length $\mu > 0$. Its small transverse displacement is $u(x, t)$. Assume $|u_x| \ll 1$, negligible longitudinal motion and bending stiffness, and a transverse applied force $p(x, t)$ per unit length. A linear drag force per unit length is $-bu_t$, with $b \geq 0$.

Tasks

1. Apply transverse momentum balance on an arbitrary interval and derive the linear wave model, explaining the small-slope approximation.
2. State the units of $\mu, \mathcal{T}, p, b$ and check each term. Identify the undamped wave speed and the effect of quadrupling μ at fixed tension.
3. Explain why both initial displacement and initial velocity are needed. Give two different undamped, source-free solutions on the whole line with the same zero initial displacement.
4. For $p = b = 0$, test $u = F(x - vt)$ with $F \in C^2$. Determine when this represents a traveling wave of the string and identify the exceptional profiles that do not determine v .

Question 1 – Solution

Strategy. Transverse acceleration is driven by the difference of vertical tension components, not by a first time derivative.

Step 1: Derive the momentum equation. The vertical tension component is $\mathcal{T} \sin \theta \approx \mathcal{T} u_x$ under the small-slope approximation. For an interval $[a, d]$,

$$\int_a^d \mu u_{tt} dx = \mathcal{T} u_x(d, t) - \mathcal{T} u_x(a, t) + \int_a^d (p - bu_t) dx.$$

Localization gives

$$\mu u_{tt} + bu_t = \mathcal{T} u_{xx} + p.$$

The linearization neglects changes of length and tension of higher order in the slope.

Step 2: Check dimensions and speed. The units are $\mu : \text{kg/m}$, $\mathcal{T} : \text{N}$, $p : \text{N/m}$ and $b : \text{Ns/m}^2$. Since u is a length, each term in the local equation has units N/m. Without drag or forcing,

$$u_{tt} = c^2 u_{xx}, \quad c = \sqrt{\mathcal{T}/\mu}.$$

The speed is in m/s. Quadrupling μ at fixed tension halves c .

Step 3: Explain the two initial data. Acceleration determines how velocity changes, so displacement alone cannot specify the motion. For example, on the whole line, both $u = 0$ and $u = Vt$, with constant $V \neq 0$, solve the source-free undamped equation and have $u(x, 0) = 0$. Their initial velocities differ. A prescribed pair $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ removes this particular ambiguity.

Step 4: Verify a moving profile. For $u = F(x - vt)$, $u_{tt} = v^2 F''$ and $u_{xx} = F''$. Hence

$$(v^2 - c^2)F'' = 0.$$

If F'' is not identically zero, the admissible speeds are $v = \pm c$, corresponding to opposite directions. If F is affine, both second derivatives vanish for every v ; a translating straight line cannot by itself identify a propagation speed. A non-affine smooth profile is needed to draw that inference from the PDE.

Question 2

For the whole-line wave equation $u_{tt} = c^2 u_{xx}$, $c > 0$, you may use

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Take $f = 0$ and $g(s) = (1 - s^2)^2$ for $|s| < 1$, $g(s) = 0$ for $|s| \geq 1$, in consistent nondimensional units. Here g is continuously differentiable.

Tasks

1. Differentiate the integral formula to verify the equation and both initial conditions for these data.
2. Find the exact center displacement $u(0, t)$ for all $t \geq 0$.
3. Determine where the solution is zero and when a fixed point reaches the constant plateau behind the departing waves. Explain why compact initial velocity can leave nonzero displacement behind.
4. Sketch the profiles for $c = 1$ at $t = 1/2, 3/2, 3$. Check that the plateau height does not grow without bound and distinguish zero displacement from zero velocity.

Question 2 – Solution

Strategy. Initial velocity contributes an integral over a growing interval; it is not simply two displaced copies of the initial displacement.

Step 1: Verify the integral solution. Leibniz differentiation gives

$$u_t = \frac{g(x+ct) + g(x-ct)}{2}, \quad u_{tt} = \frac{c}{2}(g'(x+ct) - g'(x-ct)), \quad u_{xx} = \frac{1}{2c}(g'(x+ct) - g'(x-ct)).$$

Thus $u_{tt} = c^2 u_{xx}$. At $t = 0$, the integral is zero and $u_t(x, 0) = g(x)$. The C^1 regularity of g makes these second derivatives continuous.

Step 2: Integrate at the center. Symmetry gives $u(0, t) = c^{-1} \int_0^{\min(ct, 1)} (1-s)^2 ds$. Consequently

$$u(0, t) = \begin{cases} t - \frac{2c^2 t^3}{3} + \frac{c^4 t^5}{5}, & 0 \leq ct \leq 1, \\ \frac{8}{15c}, & ct \geq 1. \end{cases}$$

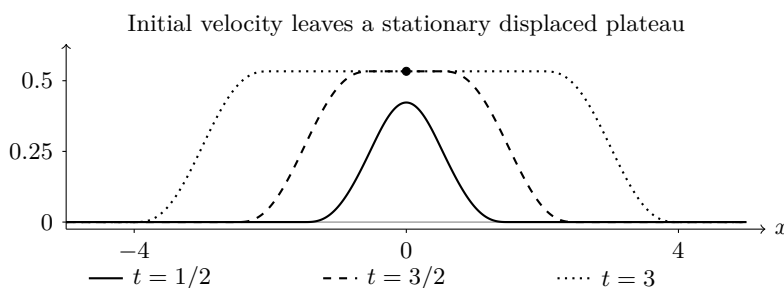
The two expressions agree at $ct = 1$.

Step 3: Locate the moving fronts and plateau. For $t > 0$, the integral is positive exactly when $[x-ct, x+ct]$ overlaps $(-1, 1)$ in positive length. It is therefore zero for $|x| \geq 1+ct$. When $ct \geq |x|+1$, the integration interval contains the entire support of g , so

$$u(x, t) = \frac{1}{2c} \int_{-1}^1 g(s) ds = \frac{8}{15c}.$$

The same value holds at equality because the endpoints of g contribute nothing. Behind the fronts, the string is displaced but locally stationary and flat. Displacement itself need not return to zero after a velocity pulse.

Step 4: Interpret the plotted profiles. Since $g \geq 0$, every interval integral lies between zero and the full integral. Thus $0 \leq u \leq 8/(15c)$ for every x, t . For $c = 1$, the fronts are at $x = \pm(1+t)$; a plateau appears once $t \geq 1$ and occupies $|x| \leq t-1$. Its height stays $8/15$ as its width grows. Zero local velocity on that plateau does not mean zero local displacement.



Question 3

Use the whole-line wave formula

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds, \quad c > 0.$$

At an observation time $t_0 > 0$, compare two solutions with initial data (f, g) and $(\tilde{f}, \tilde{g})$.

Tasks

1. Identify exactly which initial positions can influence an observation (x_0, t_0) in this formula.
2. If $|f - \tilde{f}| \leq \varepsilon$ and $|g - \tilde{g}| \leq \eta$ on that interval, prove $|u(x_0, t_0) - \tilde{u}(x_0, t_0)| \leq \varepsilon + t_0 \eta$. Show the two constants are sharp.
3. Take $c = 1$, $g = 0$ and $f(x) = (x - 2)^3(3 - x)^3$ on $2 < x < 3$, zero elsewhere. Find the entire time trace at $x = 0$, including its first and last nonzero times and maximum.
4. Sketch the normalized trace $128u(0, t)$ for $0 \leq t \leq 4$. Explain why changes to initial data outside the backward interval cannot affect a given observation, however large those changes are.

Question 3 – Solution

Strategy. Read causality directly from the interval sampled by the solution formula.

Step 1: Identify the backward interval. Only the initial positions in

$$[x_0 - ct_0, x_0 + ct_0]$$

enter: f is sampled at its two endpoints, while g is integrated across it. This interval is a sufficient domain of dependence for both types of data; interior values of f alone do not enter this one-dimensional formula.

Step 2: Bound and attain the data error. The endpoint contribution is at most $(\varepsilon + \varepsilon)/2$. The integral contribution is at most $(2ct_0)\eta/(2c)$. Hence

$$|u(x_0, t_0) - \tilde{u}(x_0, t_0)| \leq \varepsilon + t_0\eta.$$

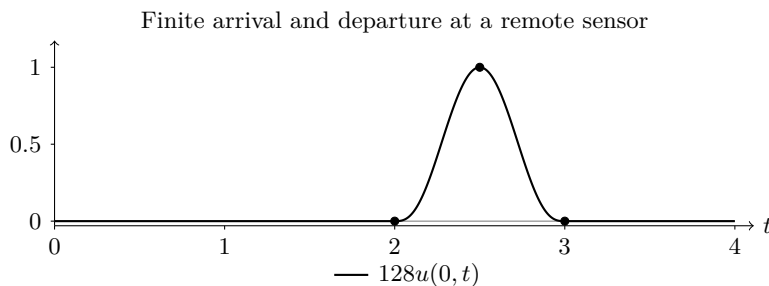
Constant differences $f - \tilde{f} = \varepsilon$ and $g - \tilde{g} = \eta$ attain equality. Taking either difference zero proves the corresponding coefficient cannot be improved.

Step 3: Compute the sensor trace. At $x = 0$, $u(0, t) = [f(-t) + f(t)]/2$. For $t \geq 0$, the first term is always zero. Thus

$$u(0, t) = \begin{cases} \frac{1}{2}(t-2)^3(3-t)^3, & 2 < t < 3, \\ 0, & 0 \leq t \leq 2 \text{ or } t \geq 3. \end{cases}$$

The trace starts to become nonzero immediately after $t = 2$ and returns to zero at $t = 3$. Since $(t-2)(3-t) = 1/4 - (t-5/2)^2$, its unique maximum is $1/128$ at $t = 5/2$. The initial profile is C^2 even at its support edges.

Step 4: Interpret finite propagation. At times before 2, the backward interval does not reach the initial pulse, so the sensor cannot respond. No alteration outside the sampled interval changes either endpoint value or the integral in the formula. This conclusion places no bound on the magnitude of the remote alteration. It is a statement about finite propagation and dependence on initial data, not about small remote effects that have merely been neglected.



Question 4

Two semi-infinite strings meet at $x = 0$, share tension $\mathcal{T} > 0$ and have densities $\mu_1, \mu_2 > 0$. Set $c_j = \sqrt{\mathcal{T}/\mu_j}$ and $Z_j = \mathcal{T}/c_j$. The massless junction has continuous displacement and continuous transverse force $\mathcal{T}u_x$. For a smooth incident pulse with $0 < \int_{\mathbb{R}} (F')^2 < \infty$, write

$$u_1 = F(t - x/c_1) + RF(t + x/c_1) \quad (x < 0), \quad u_2 = DF(t - x/c_2) \quad (x > 0).$$

Tasks

1. Verify each traveling term solves its local wave equation and derive the two junction equations for R, D .
2. Solve for R, D and classify the sign of the reflected displacement when the second string is denser or lighter.
3. Using rightward power $\mathcal{P} = -\mathcal{T}u_t u_x$, derive the reflected and transmitted energy fractions and prove their sum is one.
4. Evaluate the case $\mu_2 = 9\mu_1$. Analyze the limits $Z_2/Z_1 \rightarrow 0$ and $\rightarrow \infty$, and explain why a transmitted displacement amplitude above one need not violate energy conservation.

Question 4 – Solution

Strategy. Displacement amplitudes and energy fractions differ because the strings have different wave impedances.

Step 1: Apply the interface conditions. Every term of the form $F(t \mp x/c_j)$ has $u_{tt} = c_j^2 u_{xx}$. At the junction, displacement continuity gives $1 + R = D$. The slope condition gives

$$\frac{\mathcal{T}}{c_1}(R - 1)F'(t) = -\frac{\mathcal{T}}{c_2}DF'(t), \quad \boxed{Z_1(1 - R) = Z_2D.}$$

Since F is nonconstant, these amplitude equations follow.

Step 2: Solve and interpret the amplitudes. Solving the two equations gives

$$\boxed{R = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \quad D = \frac{2Z_1}{Z_1 + Z_2}.}$$

Here $Z_j = \sqrt{\mathcal{T}\mu_j}$. A denser second string gives $R < 0$, an inverted reflected displacement. A lighter string gives $R > 0$. Equal impedances give no reflection and $D = 1$.

Step 3: Account for energy flux. The incident rightward power is $Z_1(F')^2$, the reflected leftward power has magnitude $Z_1R^2(F')^2$, and transmitted power is $Z_2D^2(F')^2$. The cross terms in $-\mathcal{T}u_tu_x$ on the incident side cancel. For a pulse with finite nonzero $\int(F')^2$, integration in time therefore gives

$$\boxed{\mathcal{R} = R^2, \quad \mathcal{D} = \frac{Z_2}{Z_1}D^2, \quad \mathcal{R} + \mathcal{D} = \frac{(Z_1 - Z_2)^2 + 4Z_1Z_2}{(Z_1 + Z_2)^2} = 1.}$$

Step 4: Test examples and extreme impedances. If $\mu_2 = 9\mu_1$, then $Z_2 = 3Z_1$, so $R = -1/2$, $D = 1/2$, $\mathcal{R} = 1/4$ and $\mathcal{D} = 3/4$. As $Z_2/Z_1 \rightarrow 0$, $R \rightarrow 1$ and $D \rightarrow 2$, but $\mathcal{D} \rightarrow 0$. As $Z_2/Z_1 \rightarrow \infty$, $R \rightarrow -1$, $D \rightarrow 0$ and again $\mathcal{D} \rightarrow 0$. These are the free-end and fixed-end reflection limits, respectively. Energy depends on impedance as well as squared displacement amplitude.

Question 5

On $x \geq 0$, use wave speed one and the even profile $F(z) = e^{-z^2}$. For $a > 0$, compare

$$u_D(x, t) = F(x + t - a) - F(x - t + a), \quad u_N(x, t) = F(x + t - a) + F(x - t + a).$$

These formulas include the reflected field from the start; the Gaussian has noncompact tails.

Tasks

1. Verify the wave equation for both fields and show which one satisfies a fixed end and which one satisfies a free end at $x = 0$.
2. Compute their initial displacements and velocities explicitly. Explain why keeping only the incident Gaussian would violate the boundary condition.
3. At $t = a$, compute each field and its velocity. Decide whether zero displacement everywhere implies the string is at rest.
4. For $a = 3$, sketch u_D at $t = 1, 3, 5$ on $0 \leq x \leq 6$. Interpret the sign of the reflected pulse and compare it with the free-end reflection.

Question 5 – Solution

Strategy. An odd or even spatial reflection enforces the corresponding boundary condition.

Step 1: Verify the two boundary models. Each translated profile solves $u_{tt} = u_{xx}$. Since F is even and F' is odd,

$$u_D(0, t) = F(t - a) - F(a - t) = 0, \quad (u_N)_x(0, t) = F'(t - a) + F'(a - t) = 0.$$

Thus u_D satisfies the fixed-end condition and u_N the free-end condition.

Step 2: Recover the actual initial data. Differentiating the time arguments carefully gives

$$\begin{aligned} u_D(x, 0) &= F(x - a) - F(x + a), & (u_D)_t(x, 0) &= F'(x - a) + F'(x + a), \\ u_N(x, 0) &= F(x - a) + F(x + a), & (u_N)_t(x, 0) &= F'(x - a) - F'(x + a), \end{aligned}$$

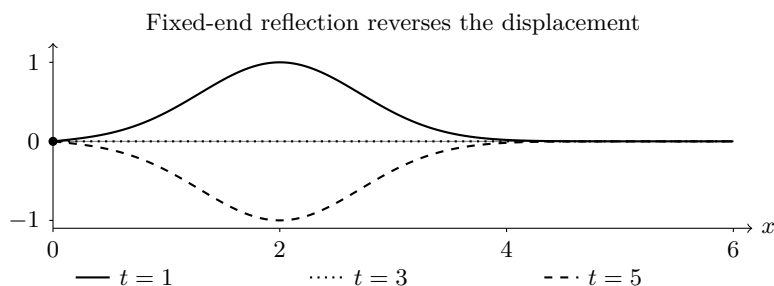
where $F'(z) = -2ze^{-z^2}$. The incident term alone generally has neither zero boundary displacement nor zero boundary slope. Its reflected partner supplies the exact cancellation; small Gaussian tails are not identically zero.

Step 3: Inspect the collision instant. At $t = a$, the two profile arguments both equal x . Therefore

$$u_D(x, a) = 0, \quad (u_D)_t(x, a) = 2F'(x), \quad u_N(x, a) = 2F(x), \quad (u_N)_t(x, a) = 0.$$

The fixed-end string has zero displacement but nonzero velocity for $x > 0$: it is not at rest. The free-end field has a doubled displacement profile at that instant and zero instantaneous velocity. Displacement and velocity must both be known to specify the state.

Step 4: Interpret the reflected sign. For $a = 3$, the incoming peak at $t = 1$ is near $x = 2$. At $t = 3$ the fixed-end displacement is exactly zero everywhere. At $t = 5$ the reflected pulse is near $x = 2$ with opposite sign. Indeed $u_D(x, 5) = -u_D(x, 1)$. A fixed end reverses displacement upon reflection; a free end preserves its sign. The plotted zero snapshot is an exact cancellation, not disappearance of the motion or its energy.



Question 6

A string on $0 \leq x \leq L$ has fixed ends, constant $\mu, \mathcal{T} > 0$ and equation $\mu u_{tt} = \mathcal{T} u_{xx} + p(x, t)$. Assume sufficient smoothness and define

$$E(t) = \frac{1}{2} \int_0^L (\mu u_t^2 + \mathcal{T} u_x^2) dx.$$

Tasks

1. Derive the energy identity, first retaining boundary power and then using the fixed-end conditions.
2. Prove uniqueness for prescribed initial displacement and velocity when the applied force is prescribed.
3. For $p = 0$ and $A \neq 0$, verify $u = A \sin(\pi x/L) \cos(\omega t)$ with $\omega = (\pi/L)\sqrt{\mathcal{T}/\mu}$. Compute its kinetic, elastic and total energies.
4. Sketch the two energy fractions over $0 \leq \omega t \leq 2\pi$. Explain why a zero-displacement snapshot can have positive energy, and why the energy identity does not claim that displacement at each point is monotone.

Question 6 – Solution

Strategy. Integrate the PDE against velocity so that acceleration and curvature become time derivatives of kinetic and elastic energies.

Step 1: Derive the work balance. Multiplying by u_t and integrating by parts gives

$$E'(t) = \mathcal{T}[u_t u_x]_0^L + \int_0^L p u_t dx.$$

At a fixed endpoint, $u = 0$ for all time, hence $u_t = 0$ there. Thus $E' = \int p u_t$. With no applied force, total energy is constant. The sign of the work rate depends on the force and velocity together.

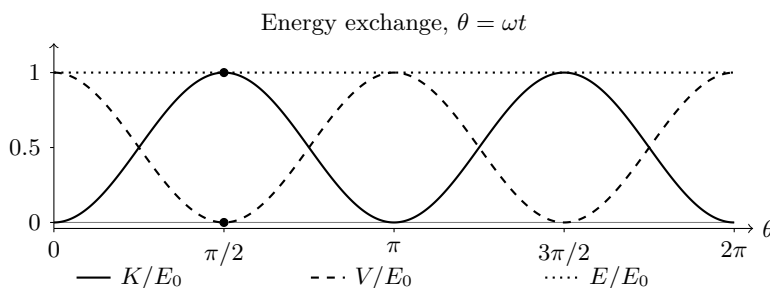
Step 2: Prove uniqueness for the full initial state. The difference w of two solutions with the same force, endpoints and initial data obeys the homogeneous equation and has zero initial energy. Its energy stays zero, so $w_t = w_x = 0$ everywhere by continuity. The fixed endpoints then force the spatial constant to be zero. Initial displacement alone would not make the kinetic part of the initial difference energy vanish; both initial data are essential.

Step 3: Compute the standing-wave energy. Put $k = \pi/L$, so $\omega^2 = (\mathcal{T}/\mu)k^2$. Then $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$, and both endpoints are zero. Using $\int_0^L \sin^2 kx = \int_0^L \cos^2 kx = L/2$ gives

$$K(t) = E_0 \sin^2(\omega t), \quad V(t) = E_0 \cos^2(\omega t), \quad E_0 = \frac{\mathcal{T} A^2 k^2 L}{4}.$$

Their sum is the constant E_0 .

Step 4: Interpret the exchange of energy. For $A \neq 0$, at $\omega t = \pi/2$ the displacement and elastic energy are zero, but the velocity is nonzero and all energy is kinetic. At $\omega t = 0$ the velocity is zero and all energy is elastic. The field exchanges these forms while oscillating; total-energy conservation does not require pointwise monotonic motion. The graph shows K/E_0 , V/E_0 and their constant sum.



Question 7

A fixed-end string is known to have displacement

$$u(x, t) = A \sin(\pi x/L) \cos(c\pi t/L), \quad A, L > 0,$$

with unknown speed $c > 0$. A snapshot at a known time $\Delta > 0$ shows $u(x, \Delta) = 0$ for every x . Initially the shape and zero velocity are known.

Tasks

1. Find every positive speed consistent with the zero snapshot and prove the list is complete.
2. Determine the speed if the independent prior bound $0 < c < L/\Delta$ is available. Explain why the snapshot alone is insufficient without it.
3. Derive the midpoint velocity at time Δ for each admissible speed. Give a consistency test for an exact signed velocity measurement and show that it identifies the speed uniquely when consistent.
4. If the measured velocity is $-5\pi A/(2\Delta)$, find c and predict the midpoint displacement at time $\Delta/2$. Verify the signs rather than using only the velocity magnitude.

Question 7 – Solution

Strategy. A snapshot can hide how many oscillations have occurred; a velocity measurement retains both rate and direction.

Step 1: List all compatible speeds. Since the spatial sine is not identically zero, the snapshot requires $\cos(c\pi\Delta/L) = 0$. For positive c ,

$$c_n = \frac{L}{\Delta} \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots$$

Every listed speed gives the same zero snapshot, as well as the specified initial displacement and zero velocity. The positive zeros of cosine prove completeness.

Step 2: Use an actual prior restriction. The bound $c < L/\Delta$ allows only $n = 0$, so $c = L/(2\Delta)$. Without this independent information, arbitrarily many half-oscillations may have occurred before the snapshot. A zero snapshot does not imply zero speed or determine which crossing of equilibrium was observed.

Step 3: Recover speed from signed velocity. At the midpoint,

$$v_n = u_t(L/2, \Delta) = (-1)^{n+1} \frac{\pi A}{\Delta} \left(n + \frac{1}{2} \right).$$

A measured v is consistent exactly when $n = |v|\Delta/(\pi A) - 1/2$ is a nonnegative integer and its sign is $(-1)^{n+1}$. The strictly increasing magnitudes then select one n uniquely. In particular, a zero velocity measurement is incompatible with the stated nonzero-amplitude zero snapshot.

Step 4: Verify the numerical multiple. For $v = -5\pi A/(2\Delta)$, the magnitude gives $n = 2$ and the predicted sign $(-1)^3$ is negative, as measured. Thus

$$c = \frac{5L}{2\Delta}, \quad u(L/2, \Delta/2) = A \cos(5\pi/4) = -\frac{A}{\sqrt{2}}.$$

The signed result distinguishes this phase from other times at which the displacement vanishes. Inferring the lowest possible speed without a prior bound would produce the wrong prediction for the intermediate snapshot.

Question 8

Compare the ideal wave equation with the given dispersive extension

$$u_{tt} = c^2 u_{xx} - \beta u_{xxxx}, \quad c > 0, \quad \beta \geq 0.$$

For a sinusoidal wave $u = \cos(kx - \omega t)$ with $k > 0, \omega > 0$, define phase speed $v_p = \omega/k$ and group speed $v_g = d\omega/dk$. No general theory of wave packets is required.

Tasks

1. Substitute the sinusoid and derive the frequency relation. Determine the units of β .
2. Compute v_p, v_g and compare them for $\beta > 0$. Find their small- k limits and their values when $\beta = 0$.
3. For $\beta > 0$, consider a sum of two nonzero sinusoidal components with distinct positive wave numbers. Prove that the sum cannot translate rigidly with one fixed speed while both components follow the derived positive-frequency relation.
4. Sketch v_p, v_g for $c = 1, \beta = 1/4, 0 < k \leq 3$, showing the limiting value at zero separately. Explain which feature of the ordinary wave equation is lost.

Question 8 – Solution

Strategy. Test sinusoidal components before claiming that an arbitrary composite profile moves without changing shape.

Step 1: Derive the frequency relation. The derivatives give $u_{tt} = -\omega^2 u$, $u_{xx} = -k^2 u$ and $u_{xxxx} = k^4 u$. Therefore

$$\omega^2 = c^2 k^2 + \beta k^4, \quad \omega = k\sqrt{c^2 + \beta k^2}.$$

Since k has units m^{-1} and ω units s^{-1} , β has units m^4/s^2 .

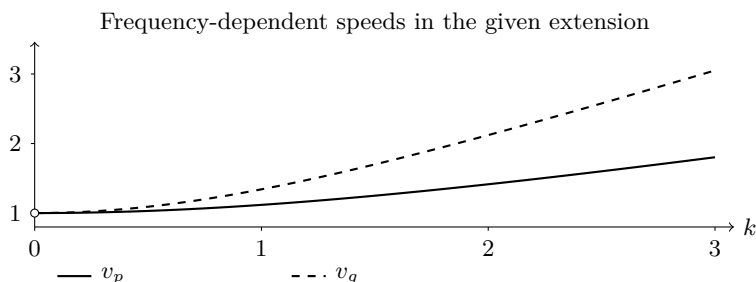
Step 2: Compare the two speeds. Differentiating the positive-frequency branch gives

$$v_p = \sqrt{c^2 + \beta k^2}, \quad v_g = \frac{c^2 + 2\beta k^2}{\sqrt{c^2 + \beta k^2}}.$$

Their difference is $\beta k^2 / \sqrt{c^2 + \beta k^2}$, strictly positive if $\beta > 0$. Both tend to c as $k \downarrow 0$ and both equal c for every k when $\beta = 0$.

Step 3: Test rigid translation of two frequencies. A fixed profile $\cos(k_1 x + \phi_1) + a \cos(k_2 x + \phi_2)$ with $a \neq 0$ translated by speed v gives temporal frequencies vk_1, vk_2 . Independence of the two spatial frequencies requires $v = \omega(k_1)/k_1 = \omega(k_2)/k_2$ for agreement at every x, t . For $\beta > 0$, $v_p(k)$ is strictly increasing for $k > 0$, so distinct wave numbers cannot share this value. Relative phases change in a way that cannot be represented by one common translation.

Step 4: Interpret the comparison. In the ideal equation, all right-moving sinusoidal components travel with the same speed c , so a smooth profile $F(x - ct)$ retains its shape. The given extension makes phase speed depend on frequency; a multi-frequency profile generally changes shape. The graph shows this dependence and the strict separation of the two defined speeds for positive k . The open point at $k = 0$ records their common limit, since the definitions above were made for $k > 0$.



Question 9

Let a sufficiently smooth displacement be periodic in x with period 2π and satisfy $u_{tt} = c^2 u_{xx}$, $c > 0$. Define its mean $m(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} u(x, t) dx$ and momentum $P(t) = \mu \int_{-\pi}^{\pi} u_t dx$, with $\mathcal{T} = \mu c^2$.

Tasks

1. Derive the evolution of the mean and prove momentum conservation.
2. Derive conservation of $E = \frac{1}{2} \int_{-\pi}^{\pi} (\mu u_t^2 + \mathcal{T} u_x^2) dx$. Does bounded energy force bounded displacement for all time?
3. Test the explicit family $u(x, t) = Vt + A \cos x \cos(ct)$. Compute its mean, momentum and energy exactly and use it to answer the preceding question.
4. If the mean initial velocity is zero, prove that a uniform displacement bound does follow from energy and the initial mean. You may use continuity, the intermediate value theorem and Cauchy–Schwarz.

Question 9 – Solution

Strategy. Wave energy controls velocity and slope, but a spatially uniform motion can carry the mean displacement indefinitely.

Step 1: Track the constant spatial mode. Integrating the PDE and using matching periodic slopes gives

$$m''(t) = \frac{c^2}{2\pi} [u_x]_{-\pi}^{\pi} = 0.$$

Hence

$$m(t) = m(0) + m'(0)t, \quad P(t) = 2\pi\mu m'(0).$$

The mean velocity, rather than the mean displacement, is conserved.

Step 2: Derive the periodic energy balance. Multiplication by u_t and integration by parts gives $E' = \mathcal{T}[u_t u_x]_{-\pi}^{\pi} = 0$ by periodicity. The energy contains u_t and u_x , but not u itself. A nonzero constant mean velocity is therefore compatible with a linearly growing mean displacement and a fixed total energy.

Step 3: Verify a drifting example. For $u = Vt + A \cos x \cos(ct)$, the term Vt and the oscillatory term each solve the equation. Orthogonality gives

$$m(t) = Vt, \quad P = 2\pi\mu V, \quad E = \pi\mu V^2 + \frac{\pi\mathcal{T}A^2}{2}.$$

The cosine's kinetic and elastic contributions sum to the second constant. For $V \neq 0$, displacement is unbounded in time even though the energy is finite and constant.

Step 4: Bound displacement when the mean cannot drift. If $m'(0) = 0$, then $m(t) = m(0)$. At each time the continuous function $w = u - m(0)$ has zero spatial mean, so it has a zero at some x_* (or is identically zero). Integrate its derivative from that point over a path of length at most 2π . Cauchy-Schwarz gives

$$|w(x, t)| \leq \sqrt{2\pi} \|u_x(\cdot, t)\|_2 \leq \sqrt{\frac{4\pi E(0)}{\mathcal{T}}}.$$

Thus $|u(x, t)| \leq |m(0)| + \sqrt{4\pi E(0)/\mathcal{T}}$ for all time. The estimate need not be sharp; it identifies the missing mean-velocity condition.

Question 10

A fixed-end string is driven by

$$\mu u_{tt} = \mathcal{T} u_{xx} + F \sin(kx) \cos(\omega t), \quad k = \pi/L, \quad \omega = k\sqrt{\mathcal{T}/\mu}, \quad F > 0.$$

Both initial displacement and velocity are zero. Test solutions of the supplied form $u(x, t) = q(t) \sin(kx)$; no general separation procedure is required.

Tasks

1. Derive the scalar equation for q and verify $q(t) = Ft \sin(\omega t)/(2\mu\omega)$.
2. Check both initial conditions and both fixed-end conditions. Explain why the factor t does not alter the propagation-speed parameter of the PDE.
3. Compute the energy and verify its rate equals the applied work rate. Evaluate the energy at $t_j = j\pi/\omega$ and describe its growth.
4. Replace the driving frequency by $\Omega > 0$, $\Omega \neq \omega$. Verify the zero-initial-data response $q_\Omega = F(\cos \Omega t - \cos \omega t)/(\mu(\omega^2 - \Omega^2))$ and recover the resonant formula as $\Omega \rightarrow \omega$ for fixed t .

Question 10 – Solution

Strategy. A force at the natural frequency produces growing amplitude while the same local wave equation retains its original speed.

Step 1: Verify the resonant amplitude equation. Substitution gives $q'' + \omega^2 q = (F/\mu) \cos \omega t$. If $D = F/(2\mu\omega)$ and $q = Dt \sin \omega t$, then

$$q' = D(\sin \omega t + \omega t \cos \omega t), \quad q'' + \omega^2 q = 2D\omega \cos \omega t.$$

Thus the proposed response satisfies the forced equation.

Step 2: Check the full initial-boundary data. One has $q(0) = q'(0) = 0$. The spatial sine vanishes at $0, L$, giving both fixed ends for all time. The growing factor is a resonant response to sustained external forcing. It changes the amplitude history, not the coefficient $c^2 = \mathcal{T}/\mu$ that sets the source-free propagation speed.

Step 3: Measure energy and work. Spatial integration gives

$$E(t) = \frac{\mu L}{4}(q'^2 + \omega^2 q^2) = \frac{F^2 L}{16\mu\omega^2} [(\sin \omega t + \omega t \cos \omega t)^2 + (\omega t \sin \omega t)^2].$$

Differentiating the first expression and using the amplitude equation yields $E' = (FL/2)q' \cos \omega t$, exactly $\int_0^L F \sin(kx) \cos(\omega t) u_t dx$. At $t_j = j\pi/\omega$,

$$E(t_j) = \frac{F^2 L}{16\mu} t_j^2.$$

Energy is unbounded along these times. Its instantaneous rate can have either sign; sustained resonant input produces the growing envelope.

Step 4: Take the frequency limit carefully. The supplied q_Ω has zero value and derivative at zero. Direct differentiation gives $q''_\Omega + \omega^2 q_\Omega = (F/\mu) \cos \Omega t$. For fixed t , differentiating numerator and denominator with respect to Ω at ω gives

$$\lim_{\Omega \rightarrow \omega} q_\Omega(t) = \frac{F}{\mu} \frac{-t \sin \omega t}{-2\omega} = \frac{Ft \sin \omega t}{2\mu\omega}.$$

Each fixed nonresonant response is bounded in time, but those bounds are not uniform as its frequency approaches resonance.