

Question 1

A stationary rod has cross-sectional area $A(x) > 0$, conductivity $k(x) > 0$ and volumetric heat capacity $C(x) > 0$. Temperature $u(x, t)$ is uniform across each section; the lateral surface is insulated. A volumetric source $s(x, t)$ has units W/m^3 . Define signed heat power toward increasing x by $Q = -kAu_x$, in watts. All coefficients are independent of time.

Tasks

1. Apply energy conservation to an arbitrary interval $[a, b]$ and derive the local heat equation. State the units of every term in the balance.
2. Expand the spatial derivative and identify the diffusivity. Explain when the equation reduces to $u_t = \alpha u_{xx}$.
3. After nondimensionalization, take $0 < x < 1$, $A = e^x$, $k = C = 1$, $s = 0$ and an instantaneous profile $u = x$. Find Q , the storage rate Au_t and u_t . Sketch the first two quantities.
4. Explain why this linear temperature profile need not be steady, even though its curvature is zero. Distinguish heat flux per area from total heat power.

Question 1 – Solution

Strategy. Balance total heat power before dividing by the local cross-sectional area.

Step 1: Derive the conservation law. The stored thermal energy relative to a fixed reference has time derivative $\int_a^b CAu_t dx$. Inflow minus outflow plus generation gives

$$\int_a^b CAu_t dx = Q(a, t) - Q(b, t) + \int_a^b As dx.$$

Since this holds on every interval, with Fourier's law it yields

$$CAu_t = -Q_x + As = (kAu_x)_x + As.$$

Each local term has units W/m: C is J/(m³K), A is m² and u_t is K/s. Each integrated term, including $Q(a) - Q(b)$, is in watts.

Step 2: Identify the geometric contribution. Dividing by CA gives

$$u_t = \frac{k}{C}u_{xx} + \frac{(kA)'}{CA}u_x + \frac{s}{C}.$$

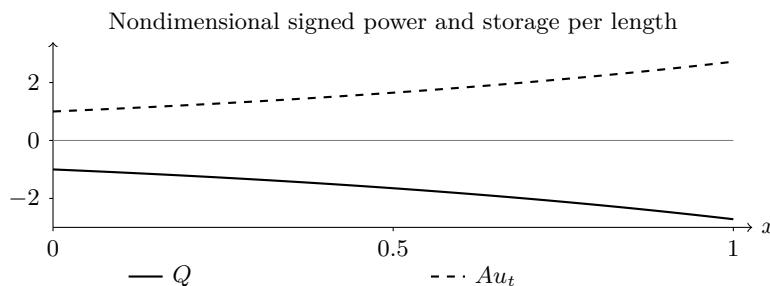
The diffusivity k/C has units m²/s. For a homogeneous constant-area rod without a source, this is $u_t = \alpha u_{xx}$ with constant $\alpha = k/C$. More generally, the first-derivative term vanishes when kA is constant; constant diffusivity additionally requires k/C constant.

Step 3: Evaluate the nondimensional example. Here $u_x = 1$ and $u_{xx} = 0$, but

$$Q(x) = -e^x, \quad Au_t = -Q_x = e^x, \quad u_t = 1.$$

The signed power is negative because heat moves toward decreasing temperature, which here is the negative x direction. Its magnitude increases toward the larger cross section.

Step 4: Interpret the nonzero storage. Flux per unit area is $q = -ku_x = -1$, constant in this example. Total power is $Q = Aq = -e^x$, which is not constant. More heat enters a small interval through its larger right face than leaves through its smaller left face, so energy accumulates. Zero curvature alone is a steady-state test only under the coefficient and geometry assumptions that remove the extra term.



Question 2

A homogeneous rod of length L and constant area A obeys $Cu_t = ku_{xx}$, with $C, k > 0$ and insulated sides. Both ends exchange heat with an ambient temperature T_a . The outward heat flux per area equals $h_j(u - T_a)$ at end j , where $h_0, h_L \geq 0$. Set $v = u - T_a$ and use the signed flux $q = -ku_x$ toward increasing x .

Tasks

1. Derive the two boundary conditions, explaining the different signs at the left and right ends.
2. Derive the rate of change of the total excess heat $E = CA \int_0^L v \, dx$. Must this signed quantity always decrease?
3. Derive a dissipation identity for $V = (CA/2) \int_0^L v^2 \, dx$.
4. Use the identity to classify every steady state, including $h_0 = h_L = 0$, and prove uniqueness for a prescribed initial temperature profile.

Question 2 – Solution

Strategy. Outward flux has opposite coordinate signs at the two ends; quadratic energy provides a sign-definite quantity.

Step 1: Orient both boundary fluxes. At the left end, outward flux is $-q(0) = ku_x(0)$; at the right it is $q(L) = -ku_x(L)$. Thus

$$kv_x(0, t) = h_0v(0, t), \quad -kv_x(L, t) = h_Lv(L, t).$$

Using the same coordinate sign at both ends would reverse one heat transfer.

Step 2: Balance the signed excess heat. Integrating the equation gives

$$E' = kA[v_x(L) - v_x(0)] = -A(h_Lv(L) + h_0v(0)).$$

It is negative when both ends are hotter than ambient and positive when both are colder. It need not always decrease, since E is a signed excess rather than a nonnegative measure of departure from ambient.

Step 3: Derive the dissipative quantity. Multiplying by Av and integrating by parts yields

$$V' = -kA \int_0^L v_x^2 dx - Ah_0v(0)^2 - Ah_Lv(L)^2 \leq 0.$$

The endpoint terms have the correct negative signs precisely because the boundary conditions were oriented outward.

Step 4: Classify steady states and prove uniqueness. For a steady state $V' = 0$, every nonnegative term on the right must vanish. Thus $v_x = 0$. If at least one $h_j > 0$, its boundary term forces this constant to be zero, so the sole equilibrium is $u = T_a$. If both coefficients vanish, every spatially constant temperature is steady. For uniqueness, the difference of two sufficiently smooth solutions with the same initial data obeys the same homogeneous equation and boundary conditions. Its quadratic energy starts at zero and cannot increase, so the difference vanishes everywhere by continuity. This also covers the insulated case: its possible equilibrium constants do not cause nonuniqueness for fixed initial data.

Question 3

A rod has constant area, insulated sides, no internal source and conductivity $k(x) = k_0(1 + x/L)$, with $k_0, L > 0$. Its end temperatures are $u(0) = T_h$, $u(L) = T_c$, with $T_h > T_c$. The steady equation is $(k(x)u_x)_x = 0$, and flux per area toward the right is $q = -k(x)u_x$.

Tasks

1. Derive the exact temperature and heat flux.
2. Verify the equation and both boundary values, and locate the largest magnitude of the temperature gradient.
3. Test the straight-line interpolant between the same endpoints in the correct variable-conductivity equation. Explain its physical failure.
4. Define k_{eff} by $q = k_{\text{eff}}(T_h - T_c)/L$. Compute it and compare it with the arithmetic spatial average of k . Sketch the normalized exact and linear temperature profiles.

Question 3 – Solution

Strategy. Constant steady heat flux requires the gradient to vary inversely with conductivity.

Step 1: Integrate the flux law. Since $(ku_x)_x = 0$, the flux q is constant. Integrating $u_x = -q/k(x)$ gives

$$T_c - T_h = -\frac{qL}{k_0} \log 2.$$

Consequently

$$q = \frac{k_0(T_h - T_c)}{L \log 2}, \quad u(x) = T_h - (T_h - T_c) \frac{\log(1 + x/L)}{\log 2}.$$

The flux is positive: heat travels from the hotter left end toward the right.

Step 2: Check the profile and gradient. The logarithm is zero at 0 and $\log 2$ at L , giving the two endpoint values. Also

$$u_x = -\frac{T_h - T_c}{L \log 2 (1 + x/L)}, \quad k(x)u_x = -\frac{k_0(T_h - T_c)}{L \log 2}.$$

Thus the product is constant as required. The gradient magnitude is largest at $x = 0$ and decreases by a factor of two from left to right.

Step 3: Diagnose the linear candidate. For $\ell = T_h - (T_h - T_c)x/L$, one has $\ell_{xx} = 0$, but

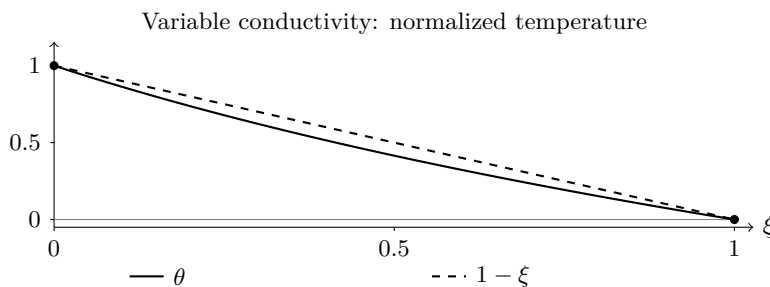
$$(k\ell_x)_x = -\frac{k_0(T_h - T_c)}{L^2} \neq 0.$$

Its rightward flux grows along the rod, so more heat leaves each small interval than enters. Without a source this profile would cool locally, not remain steady. Using $ku_{xx} = 0$ discards the essential term $k'u_x$.

Step 4: Interpret the effective conductivity. The definition gives $k_{\text{eff}} = k_0/\log 2$, while the arithmetic average is $3k_0/2$. In fact $\log 2 > 2/3$: the function $1/t$ is strictly convex and its average on $[1, 2]$ exceeds its midpoint value $2/3$. Hence

$$k_{\text{eff}} = \frac{k_0}{\log 2} < \frac{3k_0}{2}.$$

Steady conduction uses the reciprocal average $k_{\text{eff}} = L/\int_0^L k^{-1} dx$. The graph uses $\xi = x/L$ and $\theta = (u - T_c)/(T_h - T_c)$.



Question 4

Two homogeneous layers in series have thicknesses L_1, L_2 , conductivities $k_1, k_2 > 0$ and the same cross-sectional area. The outer temperatures are $T_h > T_c$. There are no sources. A contact resistance per area $R \geq 0$ satisfies $u_- - u_+ = Rq$, where u_-, u_+ are the left and right interface traces and q is the steady rightward heat flux. The interface stores no energy.

Tasks

1. Explain why flux is continuous at the interface while temperature may jump. State all equations determining the steady profiles.
2. Derive q, u_-, u_+ and the two linear temperature profiles.
3. In consistent nondimensional units, take $L_1 = L_2 = 1$, $k_1 = 1$, $k_2 = 2$, $R = 1/2$, $T_h = 100$ and $T_c = 0$. Compute and sketch the profiles, showing both interface traces.
4. Analyze the limits $R \rightarrow 0$ and $R \rightarrow \infty$. Explain why imposing temperature continuity when $R > 0$ would generally contradict the model.

Question 4 – Solution

Strategy. Add the two layer resistances and the contact resistance, retaining the interface temperature drop.

Step 1: Apply the interface balance. No interfacial storage or source means incoming flux equals outgoing flux. Thus the same q satisfies $u_x = -q/k_1$ in the first layer and $u_x = -q/k_2$ in the second. The boundary and contact relations are

$$u(0) = T_h, \quad u(L_1 + L_2) = T_c, \quad u_- - u_+ = Rq.$$

Temperature continuity is a special case of zero resistance, not the general interface law.

Step 2: Add all temperature drops. Writing $\mathcal{R} = L_1/k_1 + R + L_2/k_2$ gives

$$q = \frac{T_h - T_c}{\mathcal{R}}, \quad u_- = T_h - \frac{qL_1}{k_1}, \quad u_+ = T_c + \frac{qL_2}{k_2}.$$

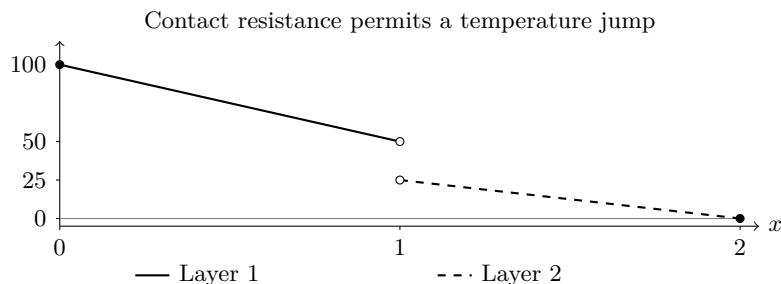
The profiles are $T_h - qx/k_1$ on the first layer and $T_c + q(L_1 + L_2 - x)/k_2$ on the second. Their traces differ by Rq .

Step 3: Verify the numerical example. Here $\mathcal{R} = 1 + 1/2 + 1/2 = 2$, so

$$q = 50, \quad u_- = 50, \quad u_+ = 25, \quad u(x) = \begin{cases} 100 - 50x, & 0 \leq x < 1, \\ 50 - 25x, & 1 < x \leq 2. \end{cases}$$

The flux is 50 on both sides, although the slopes differ. The temperature jump is $25 = Rq$. Open circles show separate traces, not a single assigned temperature at an idealized zero-thickness contact.

Step 4: Take the resistance limits. As $R \rightarrow 0$, the flux tends to $(T_h - T_c)/(L_1/k_1 + L_2/k_2)$ and the traces meet. As $R \rightarrow \infty$, $q \rightarrow 0$, each layer becomes nearly constant at its own outer temperature, and $Rq \rightarrow T_h - T_c$. A large resistance can therefore sustain a finite temperature jump while the flux vanishes. For finite $R > 0$, forcing $u_- = u_+$ would give $q = 0$, contradicting $T_h > T_c$ and the finite total resistance.



Question 5

A homogeneous rod with insulated ends is observed to have temperature

$$u(x, t) = T_a + Be^{-\lambda t} \cos(\pi x/L), \quad 0 \leq x \leq L, \quad B, \lambda, L > 0.$$

The unknown constant conductivity and volumetric heat capacity satisfy $Cu_t = ku_{xx}$ and $k, C > 0$. Assume B, L, λ are known.

Tasks

1. Verify the boundary conditions and infer the diffusivity $\alpha = k/C$ from the observed field.
2. Prove that temperature observations of this field alone do not identify k and C separately. Describe the full family of positive parameter pairs.
3. An independent measurement gives rightward heat flux per area $q_m > 0$ at $x = L/2, t = 0$. Recover k and C uniquely and check their units.
4. Explain why the midpoint is useful for this flux measurement but useless for measuring the temperature decay. Identify which temperature sensor locations can recover λ from two distinct measurement times.

Question 5 – Solution

Strategy. The temperature equation determines a ratio of material parameters; a flux measurement provides an independent scale.

Step 1: Verify the field and recover diffusivity. Differentiation gives

$$u_t = -\lambda B e^{-\lambda t} \cos(\pi x/L), \quad u_{xx} = -\frac{\pi^2}{L^2} B e^{-\lambda t} \cos(\pi x/L).$$

Also $u_x = -(\pi B/L)e^{-\lambda t} \sin(\pi x/L)$, so $u_x = 0$ at both ends. Matching the equation at points where the cosine is nonzero gives

$$\alpha = \frac{k}{C} = \frac{\lambda L^2}{\pi^2}.$$

The same equality then verifies the equation everywhere, including the node.

Step 2: Identify the nonuniqueness. For every choice $C_* > 0$, the pair

$$C = C_*, \quad k = \frac{\lambda L^2}{\pi^2} C_*$$

produces exactly the observed field. Multiplying both parameters by the same positive factor leaves the equation and insulated boundary conditions unchanged. More exact observations of this same temperature field cannot resolve that scaling ambiguity.

Step 3: Use the independent flux data. At the midpoint initially, $q = -ku_x = kB\pi/L$. Therefore

$$k = \frac{q_m L}{B\pi}, \quad C = \frac{k}{\alpha} = \frac{q_m \pi}{B\lambda L}.$$

With q_m in W/m², B in kelvins and λ in s⁻¹, these have units W/(mK) and J/(m³K), respectively. Positivity of the measured quantities ensures physically positive parameters.

Step 4: Choose informative sensor locations. At $x = L/2$, $u = T_a$ for all time: both u_t and u_{xx} vanish, so a ratio of these derivatives would be 0/0. But $|u_x|$ is largest there, giving a strong flux signal. At any fixed $x \neq L/2$ in $[0, L]$, the excess temperature is nonzero and

$$\lambda = \frac{1}{t_2 - t_1} \log \left| \frac{u(x, t_1) - T_a}{u(x, t_2) - T_a} \right|, \quad t_2 > t_1.$$

The absolute values allow sensors on either side of the node; the ratio eliminates the unknown local amplitude.

Question 6

On an infinite homogeneous rod, consider

$$u(x, t) = T_a + \frac{B}{\sqrt{t + \tau}} \exp\left(-\frac{x^2}{4\alpha(t + \tau)}\right), \quad B, \tau, \alpha > 0, \quad t \geq 0.$$

Use $C > 0$ for volumetric heat capacity and interpret integrated heat per unit cross-sectional area. You may use $\int_{\mathbb{R}} e^{-z^2} dz = \sqrt{\pi}$ and $\int_{\mathbb{R}} z^2 e^{-z^2} dz = \sqrt{\pi}/2$.

Tasks

1. Verify $u_t = \alpha u_{xx}$ directly, retaining the derivative of the time-dependent prefactor.
2. Compute the total excess heat per area and show that it is constant.
3. Compute the variance of the excess-temperature profile about zero and describe its peak and width as time increases.
4. Determine where the temperature is increasing at a given time, and the time of maximum temperature at a fixed point x . Sketch the profiles for $\alpha = B = \tau = 1$ at $t = 0, 3$, marking the inflection points.

Question 6 – Solution

Strategy. Differentiate the amplitude and width together; local warming can coexist with a falling global peak and conserved heat.

Step 1: Verify the differential equation. Put $v = u - T_a$ and $\sigma = t + \tau > 0$. Logarithmic differentiation gives

$$v_t = v \left(-\frac{1}{2\sigma} + \frac{x^2}{4\alpha\sigma^2} \right), \quad v_{xx} = v \left(-\frac{1}{2\alpha\sigma} + \frac{x^2}{4\alpha^2\sigma^2} \right).$$

Thus $v_t = \alpha v_{xx}$. Omitting the derivative of $\sigma^{-1/2}$ would remove the first term and invalidate the equation.

Step 2: Integrate the excess heat. With $z = x/\sqrt{4\alpha\sigma}$,

$$\mathcal{H} = C \int_{\mathbb{R}} v \, dx = 2CB\sqrt{\pi\alpha}.$$

The factor from the increasing width cancels the decreasing prefactor. The Gaussian flux tends to zero at both infinities, consistent with no net heat loss from the infinite rod.

Step 3: Measure spreading. Symmetry gives center zero. The provided Gaussian integrals yield

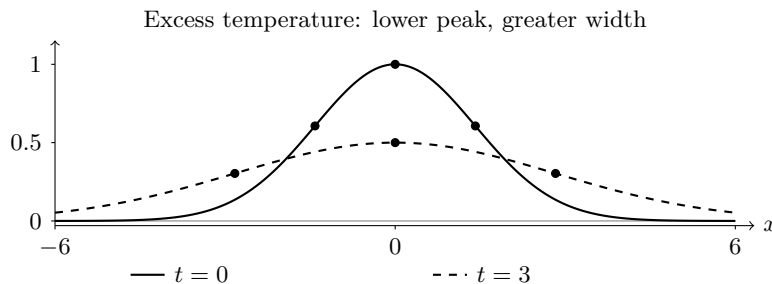
$$\frac{\int_{\mathbb{R}} x^2 v \, dx}{\int_{\mathbb{R}} v \, dx} = 2\alpha\sigma, \quad \max_x v = \frac{B}{\sqrt{\sigma}}.$$

Thus the root-mean-square width grows as $\sqrt{2\alpha(t + \tau)}$, while the peak decreases as $(t + \tau)^{-1/2}$. The inflection points occur at $x = \pm\sqrt{2\alpha\sigma}$.

Step 4: Locate local warming. Since $v > 0$, $u_t > 0$ precisely when $x^2 > 2\alpha(t + \tau)$. A fixed point reaches its maximum on $t \geq 0$ at

$$t_* = \max\left(0, \frac{x^2}{2\alpha} - \tau\right).$$

Outside the initial inflection points it first warms and then cools. Inside them it cools from the start; equality gives an initially zero derivative. Therefore diffusion does not force temperature to decrease at every location. The graph uses excess temperature v , so the ambient offset is removed.



Question 7

A constant-area rod with constant $k, C > 0$ is prescribed the candidate field

$$u(x, t) = T_a + at + bx(L - x), \quad a, b, L > 0, \quad 0 \leq x \leq L.$$

The model permits a volumetric source s and obeys $Cu_t = ku_{xx} + s$. Its lateral surface is insulated; end conditions must be determined consistently.

Tasks

1. Determine the unique source field that makes the candidate satisfy the equation.
2. State the initial and Dirichlet boundary data realized by the candidate. Decide whether it also satisfies insulated-end conditions.
3. Compute both outward end heat fluxes and verify the integrated energy balance, using cross-sectional area A .
4. Suppose the source is removed while the same formula is retained but a is allowed to be real. Find the required a and explain why changing only the boundary values cannot fix a nonzero interior residual.

Question 7 – Solution

Strategy. A manufactured temperature field determines its source and boundary data; those data cannot be prescribed independently afterward.

Step 1: Compute the interior residual. The derivatives are $u_t = a$, $u_x = b(L - 2x)$ and $u_{xx} = -2b$. Substitution requires

$$s = Ca + 2kb.$$

This is a positive spatially and temporally constant source, in W/m^3 . It supplies both the rising stored energy and heat lost at the ends.

Step 2: State the compatible data. At $t = 0$, $u(x, 0) = T_a + bx(L - x)$. Both endpoint temperatures equal $T_a + at$. The endpoint derivatives are bL and $-bL$, neither zero, so this field does not satisfy insulated-end conditions when $b > 0$. Prescribing insulation in addition to the realized end temperatures would contradict the candidate rather than add useful information.

Step 3: Verify total energy accounting. The outward fluxes are $ku_x(0) = kbL$ and $-ku_x(L) = kbL$. The stored heat relative to T_a is

$$E = CA \int_0^L (u - T_a) dx = CA \left(atL + \frac{bL^3}{6} \right), \quad E' = CAaL.$$

Total source power minus total outward power is

$$AL(Ca + 2kb) - 2AkbL = CAaL = E'.$$

This independently checks the source sign and both boundary signs.

Step 4: Remove the source consistently. For $s = 0$, the formula can satisfy the PDE only if

$$a = -\frac{2kb}{C}.$$

Its initially warm curved profile then decreases at a constant rate while both endpoint temperatures follow the same decreasing law. If a retains a different value, the residual $Ca + 2kb$ is nonzero at every interior point. Boundary values alone cannot repair that local equation while keeping the same candidate field. Changing boundary data generally requires changing the field itself.

Question 8

Let u be continuous on $[0, L] \times [0, T]$, sufficiently differentiable in the interior, and satisfy $u_t = \alpha u_{xx}$ with $\alpha > 0$. Assume $0 \leq u(x, 0) \leq M$ and $0 \leq u(0, t), u(L, t) \leq M$, where $M > 0$.

Tasks

1. Prove $u \leq M$ by applying a maximum argument to $w = u - M - \varepsilon t$ and then letting $\varepsilon \downarrow 0$. Address a maximum occurring at the final observation time.
2. Prove $u \geq 0$ and explain why these bounds concern the global range rather than monotonic cooling at every fixed position.
3. Test $v(x, t) = Me^{\alpha(\pi/L)^2 t} \sin(\pi x/L)$ against the source-free heat equation and the range bound.
4. Find the source needed to realize this growing candidate in $Cv_t = kv_{xx} + s$, with $k/C = \alpha$. Give and verify a source-free decaying candidate with the same initial and endpoint data.

Question 8 – Solution

Strategy. A small time-dependent perturbation makes the maximum argument strict, while a growing test field exposes a missing source.

Step 1: Prove the upper bound. For $\varepsilon > 0$, w is nonpositive initially and at the spatial endpoints. If it has a positive maximum on the space-time rectangle, that maximum occurs at an interior spatial point and a time $t_* > 0$. There $w_{xx} \leq 0$ and $w_t \geq 0$: at the latest time, use the derivative from earlier times; at an interior time it is zero. Yet

$$w_t - \alpha w_{xx} = -\varepsilon < 0,$$

a contradiction. One can first apply the argument on any shorter time rectangle and pass to T by continuity if needed. Thus $u \leq M + \varepsilon t$, and letting $\varepsilon \downarrow 0$ gives $u \leq M$.

Step 2: Prove the lower bound. Apply the same argument to $-u$ with upper bound zero to obtain $-u \leq 0$. Hence $0 \leq u \leq M$ throughout the rectangle. The theorem does not say $u_t \leq 0$ at every point. At a local temperature minimum with positive curvature, the equation gives $u_t = \alpha u_{xx} > 0$. Redistribution can warm a cooler location without exceeding the allowed global range.

Step 3: Diagnose the growing sine. Write $\mu = (\pi/L)^2$. Then $v_t = \alpha \mu v$ and $v_{xx} = -\mu v$, so

$$v_t - \alpha v_{xx} = 2\alpha \mu v \neq 0$$

in the spatial interior. It satisfies zero endpoint values and the allowed initial profile, but for $t > 0$ its midpoint exceeds M . This violates the conclusion because it fails the source-free PDE hypothesis.

Step 4: Restore the missing physics or correct the time factor. The required source is $s = C(v_t - \alpha v_{xx}) = 2k\mu v > 0$ in the interior. With that source, the earlier source-free maximum argument no longer applies. Alternatively,

$$\tilde{v}(x, t) = M e^{-\alpha \mu t} \sin(\pi x/L)$$

has $\tilde{v}_t = -\alpha \mu \tilde{v} = \alpha \tilde{v}_{xx}$, the same initial profile and zero endpoints. It remains between zero and M .

Question 9

Two homogeneous rods have lengths L and $2L$, the same $k, C > 0$, insulated sides and no sources. At both ends each rod exchanges heat with the same ambient T_a by an outward convection law. The first rod has coefficient $h > 0$. The two initial profiles have the same shape as functions of relative position. Set $\alpha = k/C$ and choose a nonzero temperature scale ΔT .

Tasks

1. Nondimensionalize the first problem using $\xi = x/L$, $\eta = \alpha t/L^2$ and $\theta = (u - T_a)/\Delta T$. Identify the dimensionless boundary parameter.
2. Verify the PDE and initial shape for the proposed rescaling $u_2(x, t) = u_1(x/2, t/4)$ on the longer rod.
3. Derive exactly which convection coefficient h_2 makes this rescaling satisfy both boundary laws.
4. Assess the claim that doubling length always quadruples every thermal response time when the same h is used. State when the proposed scaling is valid, including the perfectly insulated case.

Question 9 – Solution

Strategy. Match the dimensionless boundary conditions as well as the interior diffusion time scale.

Step 1: Identify the dimensionless problem. The chain rule gives $\theta_\eta = \theta_{\xi\xi}$. The outward convection laws become

$$\theta_\xi(0, \eta) = \text{Bi } \theta(0, \eta), \quad -\theta_\xi(1, \eta) = \text{Bi } \theta(1, \eta), \quad \text{Bi} = \frac{hL}{k}.$$

This dimensionless coefficient is the Biot number for the chosen length scale. The interior time scale is L^2/α , but the boundary parameter also affects the dimensionless response.

Step 2: Verify the rescaled interior field. For $u_2(x, t) = u_1(x/2, t/4)$,

$$(u_2)_t = \frac{1}{4}(u_1)_t, \quad (u_2)_{xx} = \frac{1}{4}(u_1)_{xx}.$$

Thus the same diffusivity equation holds. At $t = 0$, relative positions agree: $x/(2L) = (x/2)/L$, so the stated initial-shape requirement is met.

Step 3: Match both convection conditions. At the left endpoint,

$$k(u_2)_x(0, t) = \frac{1}{2}k(u_1)_x(0, t/4) = \frac{h}{2}(u_2(0, t) - T_a).$$

The right outward derivative scales by the same factor. Therefore the coefficient making this a rescaling of the full problem is

$$h_2 = h/2, \quad h_2(2L)/k = hL/k.$$

Both rods then have the same dimensionless boundary problem.

Step 4: Qualify the length-squared claim. If $h_2 = h$, the longer rod instead has Biot number $2hL/k$. The proposed field generally fails the boundary law; the full response is not obtained merely by multiplying time by four. A special zero-excess equilibrium cannot establish a general scaling rule. The factor-four rescaling is valid when convection is also adjusted to $h/2$ and the initial shapes match. If both rods are perfectly insulated ($h = 0$), the zero-flux conditions are preserved automatically. Interior dimensional analysis alone does not justify ignoring changed boundary parameters.

Question 10

Divide an insulated homogeneous rod into three equal cells of width Δx . Let U_i^n be cell temperatures and $r = \alpha \Delta t / (\Delta x)^2 \geq 0$. Approximate each internal rightward flux by $-k(U_{i+1}^n - U_i^n) / \Delta x$, set outer fluxes to zero, and use a forward Euler time step for the cell energy balances.

Tasks

1. Derive the three-cell update matrix and prove that the sum of temperatures is conserved.
2. Find the exact range of r for which every update preserves the range of every input vector. Prove necessity as well as sufficiency.
3. Using the orthogonal vectors $(1, 1, 1)$, $(1, 0, -1)$ and $(1, -2, 1)$, determine exactly when $\sum_i (U_i^n - \bar{U})^2$ cannot increase in one step.
4. Apply one step with $r = 3/5$ to $(0, 1, 0)$. Check heat conservation and squared norm, and sketch the old and new cell values. Explain why energy stability need not imply preservation of nonnegative temperatures.

Question 10 – Solution

Strategy. Conservation, preservation of the temperature range and decay of a quadratic norm impose different conditions on a numerical scheme.

Step 1: Balance each cell. For the first cell, inflow minus outflow gives $U_1^{n+1} = U_1^n + r(U_2^n - U_1^n)$. Applying the same balance to the others yields

$$U^{n+1} = \begin{pmatrix} 1-r & r & 0 \\ r & 1-2r & r \\ 0 & r & 1-r \end{pmatrix} U^n.$$

Every column sums to one, so $\sum_i U_i$ is conserved. Equal cell volumes and constant capacity make this proportional to total heat. Every row also sums to one, so constant profiles are fixed.

Step 2: Characterize range preservation. All entries are nonnegative exactly when $0 \leq r \leq 1/2$. Each new value is then a convex combination of old values, preserving their range. For any $r > 1/2$, input $(0, 1, 0)$ gives a middle value $1 - 2r < 0$, outside $[0, 1]$. Thus the exact range is $\boxed{0 \leq r \leq 1/2}$.

Step 3: Test the quadratic energy on each mode. The stated mutually orthogonal vectors have update factors 1 , $1 - r$ and $1 - 3r$, respectively. The mean is fixed; the remaining squared mode components do not grow exactly when $|1 - r| \leq 1$ and $|1 - 3r| \leq 1$. For $r \geq 0$, this is

$$\boxed{0 \leq r \leq 2/3}.$$

Necessity follows by choosing the offending eigenvector as input. This threshold belongs to this particular three-cell insulated matrix.

Step 4: Exhibit the distinction numerically. At $r = 3/5$,

$$\boxed{(0, 1, 0) \mapsto (3/5, -1/5, 3/5)}.$$

Both sums equal one. The uncentered squared norm drops from 1 to $9/25 + 1/25 + 9/25 = 19/25$; since the mean is unchanged, the centered squared norm also drops. Nevertheless the middle temperature becomes negative. The stem plot shows individual cell values, without suggesting a smooth interpolated temperature profile between cell centers.

