

Question 1

Define a 2π -periodic function by $f = -1$ on $(-\pi, 0)$, $f = 2$ on $(0, \pi)$, $f(0) = 7$ and $f(\pi) = -4$. Let S_N be its real Fourier partial sum with mean $a_0/2$. Use the piecewise smooth Fourier theorem (the sum is the average of the one-sided limits) and Parseval. Here $\|u\|_2^2 = \int_{-\pi}^{\pi} |u|^2$.

Tasks

1. Derive the mean and all coefficients. Determine the pointwise series sum everywhere, including both types of assigned exceptional values.
2. Compute $\|f - S_N\|_2^2$ as a coefficient tail and prove it tends to zero.
3. Prove that S_N does not converge uniformly to f . Also prove a lower bound of $3/2$ for the essential supremum error, which ignores sets of measure zero.
4. Decide whether redefining only the jump values can restore uniform convergence. Sketch f and S_9 , displaying the assigned values separately from the open one-sided limits.

Question 1 – Solution

Strategy. Distinguish the assigned function, its Fourier pointwise limit and the equivalence class measured by an integral norm.

Step 1: Compute coefficients and limits. Apart from isolated values, $f = 1/2 + (3/2) \operatorname{sgn}(\sin x)$. Thus $a_0 = 1$, every positive cosine coefficient is zero, and

$$b_n = \frac{3(1 - (-1)^n)}{\pi n} = \begin{cases} 6/(\pi n), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The series sum equals -1 or 2 on the open intervals and $1/2$ at every jump. It does not equal either assigned value 7 or -4 there.

Step 2: Verify mean-square convergence. Isolated values do not affect integrals. Parseval and orthogonality give

$$\|f - S_N\|_2^2 = \frac{36}{\pi} \sum_{\substack{n > N \\ n \text{ odd}}} \frac{1}{n^2} \leq \frac{36}{\pi N} \rightarrow 0.$$

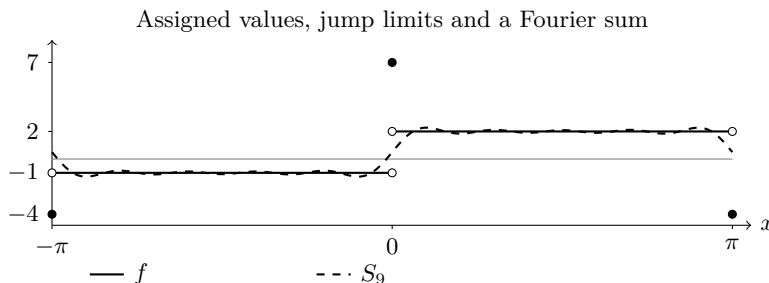
The coefficient tail is square summable despite failure at the assigned jump values.

Step 3: Separate point and essential errors. Every $S_N(0) = 1/2$, so $\|f - S_N\|_\infty \geq 13/2$ for every N . Moreover, by continuity of S_N , as $x \downarrow 0$ its error against the right-hand value 2 tends to $3/2$. For any $\varepsilon > 0$, that error exceeds $3/2 - \varepsilon$ on an interval of positive length. Hence

$$\operatorname{ess\,sup} |f - S_N| \geq 3/2.$$

Removing isolated exceptional values cannot remove this obstruction.

Step 4: Test possible redefinitions. The one-sided limits remain -1 and 2 after any change only at the jumps. The resulting target is still discontinuous. A uniform limit of the continuous trigonometric polynomials S_N would be continuous, so no such redefinition restores uniform convergence. Mean-square convergence and the open-interval pointwise limits are unchanged. Filled points in the graph show the assigned values.



Question 2

For integers $m \geq 1$, let

$$h_m(x) = \max(1 - m|x|, 0), \quad -\pi \leq x \leq \pi,$$

and extend periodically. Write $S_N^{(m)}$ for the degree- N Fourier partial sum of this particular function. The target changes with m ; these are not partial sums of one fixed function.

Tasks

1. Compute $\|h_m\|_\infty$, $\|h_m\|_2^2$ and the pointwise limit as $m \rightarrow \infty$. Distinguish mean-square convergence to zero from pointwise convergence to zero.
2. Derive the mean and all Fourier coefficients of h_m .
3. For each fixed m , prove $S_N^{(m)} \rightarrow h_m$ uniformly as $N \rightarrow \infty$. State a bound whose dependence on m is explicit.
4. Evaluate both iterated limits at $x = 0$, first letting N tend to infinity and then reversing the order. Explain why they differ, and sketch h_1, h_3, h_9 near the origin.

Question 2 – Solution

Strategy. A shrinking support can make integral errors small while retaining a fixed peak.

Step 1: Compare the modes of convergence. The peak is always one and direct integration gives

$$\|h_m\|_\infty = 1, \quad \|h_m\|_2^2 = 2 \int_0^{1/m} (1 - mx)^2 dx = \frac{2}{3m}.$$

The pointwise limit is one at 0 modulo 2π and zero elsewhere. Thus $h_m \rightarrow 0$ in mean square and almost everywhere, but not at every point and not uniformly. The pointwise limit has a single-point spike in each period.

Step 2: Compute the spectrum. The function is even, so $b_n = 0$. Its area is $1/m$, giving mean $1/(2\pi m)$. Integration over its support gives

$$a_n = \frac{2}{\pi} \int_0^{1/m} (1 - mx) \cos nx dx = \frac{2m}{\pi n^2} (1 - \cos(n/m)).$$

For each fixed n , $0 \leq a_n \leq 1/(\pi m)$ by $1 - \cos u \leq u^2/2$. Therefore every fixed coefficient tends to zero as $m \rightarrow \infty$.

Step 3: Keep the target fixed when taking a Fourier limit. For fixed m , $|a_n| \leq 4m/(\pi n^2)$, so the series converges absolutely and uniformly. The piecewise smooth Fourier theorem identifies its sum with the continuous target h_m . In particular,

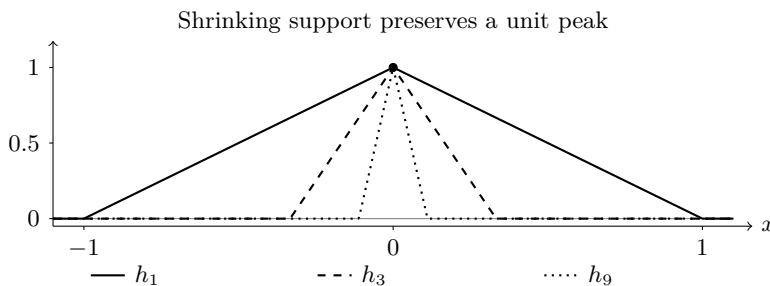
$$\|h_m - S_N^{(m)}\|_\infty \leq \frac{4m}{\pi N}.$$

The factor m prevents using this bound uniformly over all targets.

Step 4: Compare the iterated limits. For fixed m , the inner N -limit at zero is $h_m(0) = 1$. For fixed N , the mean and finitely many coefficients all tend to zero. Thus

$$\lim_{m \rightarrow \infty} \lim_{N \rightarrow \infty} S_N^{(m)}(0) = 1, \quad \lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} S_N^{(m)}(0) = 0.$$

Fixed-mode convergence does not control the collective contribution of modes whose number grows with m . There is no common uniform estimate permitting this interchange.



Question 3

You may use the Fourier identity

$$F(x) = \frac{\pi^2}{6} - \frac{\pi|x|}{2} + \frac{x^2}{4} = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \quad -\pi \leq x \leq \pi,$$

extended periodically. Let $P_N = \sum_{n=1}^N \cos(nx)/n^2$, $N \geq 1$.

Tasks

1. Prove uniform convergence and find the exact value of $\|F - P_N\|_{\infty}$ as a positive numerical series.
2. Determine all points where this maximum absolute error occurs, modulo 2π .
3. Prove strict two-sided bounds using integrals and determine the smallest integer N for which the uniform error is less than $1/100$.
4. Find the exact squared mean-square error as a series and compare its decay rate with that of the uniform error. Sketch $F - P_5$ and its upper and lower uniform-error levels.

Question 3 – Solution

Strategy. Positive coefficients make the usual triangle bound attain its value at a common phase.

Step 1: Turn the tail estimate into an equality. The Weierstrass test applies because $\sum n^{-2} < \infty$. Put $T_N = \sum_{n>N} n^{-2}$. For every x , $|F(x) - P_N(x)| \leq T_N$, while at $x = 0$ every cosine equals one. Hence

$$\|F - P_N\|_{\infty} = T_N.$$

This is an exact error, not merely a sufficient upper bound.

Step 2: Identify every equality point. For the tail to equal T_N , every positive summand must have $\cos nx = 1$. Using two consecutive integers $n > N$ forces $x = 0$ modulo 2π . For the tail to equal $-T_N$, every $\cos nx$ would have to be -1 . But then $\cos(2nx) = 1$ for a frequency also in the tail, a contradiction. Thus the maximum absolute error occurs exactly at $x = 0$ modulo 2π .

Step 3: Certify the minimal truncation. Strict monotonicity of t^{-2} gives

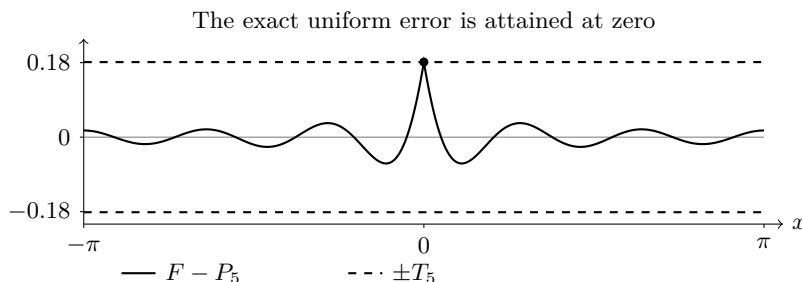
$$\frac{1}{N+1} < T_N < \frac{1}{N}.$$

At $N = 100$, $T_N < 1/100$. At $N = 99$, $T_N > 1/100$; monotonicity excludes every smaller N . Therefore the exact smallest choice is $N = 100$. The two integral bounds straddle the threshold tightly enough to decide it.

Step 4: Compare the two norms. Orthogonality gives

$$\|F - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^4}, \quad \frac{\pi}{3(N+1)^3} < \|F - P_N\|_2^2 < \frac{\pi}{3N^3}.$$

The uniform error is of order N^{-1} , whereas the L^2 error is of order $N^{-3/2}$ (its square is of order N^{-3}). Distinguish the norm from its square.



Question 4

Let $G(x) = (\pi - x)/2$ on $0 < x < 2\pi$, extended periodically with value zero at the joins. Its Fourier coefficients give

$$G(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (0 < x < 2\pi).$$

This identity follows from the piecewise smooth Fourier theorem. Set $P_N = \sum_{n=1}^N \sin(nx)/n$ and fix $0 < \delta < \pi$.

Tasks

1. Use a finite geometric sum to bound $\left| \sum_{n=p}^q \sin nx \right|$ uniformly in p, q for $x \in [\delta, 2\pi - \delta]$.
2. Apply summation by parts to prove $\|G - P_N\|_{\infty, [\delta, 2\pi - \delta]} \leq 1/((N+1)\sin(\delta/2))$.
3. Prove that convergence is not uniform even on the full open interval $(0, 2\pi)$. Explain why the preceding bound cannot be used with $\delta = 0$.
4. Compute the squared $L^2(0, 2\pi)$ error and bound it. Sketch the error for $N = 8$, marking the endpoint limits separately from the assigned endpoint errors.

Question 4 – Solution

Strategy. Cancellation controls a tail away from a jump, but the cancellation bound degenerates near the join.

Step 1: Bound finite oscillatory blocks. For $x \notin 2\pi\mathbb{Z}$,

$$\left| \sum_{n=p}^q e^{inx} \right| = \left| \frac{e^{ipx}(1 - e^{i(q-p+1)x})}{1 - e^{ix}} \right| \leq \frac{1}{|\sin(x/2)|}.$$

The same bound holds for the imaginary part. On the specified closed interval, it is at most $B = 1/\sin(\delta/2)$.

Step 2: Sum by parts with a decreasing weight. Put $A_k = \sum_{n=N+1}^k \sin nx$, so $|A_k| \leq B$. Then

$$\sum_{n=N+1}^M \frac{\sin nx}{n} = \frac{A_M}{M} + \sum_{k=N+1}^{M-1} A_k \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

The absolute value is bounded by $B/(N+1)$. Passing to the convergent tail gives

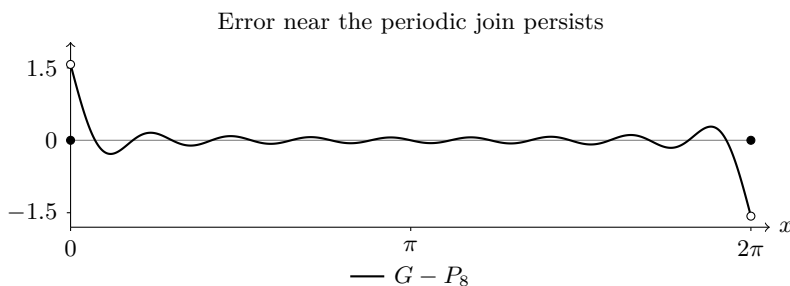
$$\sup_{\delta \leq x \leq 2\pi - \delta} |G - P_N| \leq \frac{1}{(N+1)\sin(\delta/2)} \rightarrow 0.$$

Step 3: Locate the global obstruction. For every fixed N , $P_N(x) \rightarrow 0$ as $x \downarrow 0$, while $G(x) \rightarrow \pi/2$. Therefore $\sup_{0 < x < 2\pi} |G - P_N| \geq \pi/2$ for every N . Excluding the endpoint itself does not exclude points arbitrarily close to it. At $\delta = 0$ the geometric-sum bound has a zero denominator and gives no finite uniform control.

Step 4: Verify convergence in the integral norm. The sine functions have squared norm π over this period. Hence

$$\|G - P_N\|_2^2 = \pi \sum_{n>N} \frac{1}{n^2} < \frac{\pi}{N}.$$

The error tends to zero in mean square despite persistent near-jump suprema. Its assigned endpoint errors are zero, whereas its interior limits are $\pi/2$ at 0 and $-\pi/2$ at 2π .



Question 5

Consider

$$H(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$

You may use the theorem: if a series of C^1 functions converges at one point and its derivative series converges uniformly, its sum is C^1 and may be differentiated term by term. Also use

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4}, \quad 0 \leq x \leq 2\pi.$$

Tasks

1. Verify every hypothesis needed to differentiate the series once on $[0, 2\pi]$.
2. Integrate the given derivative formula to obtain a closed form for H . Check the endpoint values and slopes of its periodic extension.
3. Determine whether the periodic extension is C^2 . Identify precisely where the formal second derivative series agrees with the classical second derivative.
4. Prove that the second derivative partial sums cannot converge uniformly to the interior second derivative on $(0, 2\pi)$. Explain why the first differentiation remains valid.

Question 5 – Solution

Strategy. Check a separate convergence condition for each differentiation; one valid differentiation does not authorize the next.

Step 1: Justify the first differentiation. Every summand is C^1 . The original series converges at 0 (all terms are zero), and the derivative terms satisfy $|\cos nx/n^2| \leq 1/n^2$. The Weierstrass test gives uniform convergence of the derivative series. The stated theorem therefore gives

$$H'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

This includes one-sided endpoint derivatives on the closed interval.

Step 2: Verify the explicit function. Integrating the provided formula and using $H(0) = 0$ gives

$$H(x) = \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12}, \quad 0 \leq x \leq 2\pi.$$

Both endpoint values are zero. Both endpoint slopes equal $\pi^2/6$, so the periodic extension is C^1 across the join.

Step 3: Identify the second derivative obstruction. Inside the period, $H''(x) = -\pi/2 + x/2$. Its right limit at zero is $-\pi/2$, whereas its left periodic limit is $\pi/2$. Thus the periodic function is not twice differentiable at the join and is not C^2 . The formal second derivative series is $-\sum \sin(nx)/n$. The piecewise smooth Fourier theorem, or direct coefficients of the displayed linear function, gives H'' on $0 < x < 2\pi$. At a join the series equals zero, the average of the limits, where the classical periodic second derivative does not exist.

Step 4: Rule out uniform second differentiation. Every finite sum $-\sum_{n=1}^N \sin(nx)/n$ tends to zero as $x \downarrow 0$, but the interior target tends to $-\pi/2$. Thus its error supremum on the full open period is at least $\pi/2$ for every N . The derivative series for the first differentiation had summable bounds n^{-2} ; the next series has no such global uniform convergence. There is no contradiction between a valid first differentiation and failure of the stronger second-derivative conclusion.

Question 6

For $N \geq 1$, consider real trigonometric polynomials $p(x) = A_0 + \sum_{n=1}^N (A_n \cos nx + B_n \sin nx)$ on $[-\pi, \pi]$. Use $\|p\|_2^2 = \int_{-\pi}^{\pi} p^2$ and define $D_N(x) = 1 + 2 \sum_{n=1}^N \cos nx$.

Tasks

1. Prove $\|p\|_{\infty} \leq \sqrt{(2N+1)/(2\pi)} \|p\|_2$ by an orthonormal-basis argument.
2. Prove the constant is sharp and characterize all nonzero equality cases at a prescribed point x_0 .
3. For $p_N = D_N/(2N+1)$, compute both norms exactly and determine its pointwise limit. Explain what this says about point evaluation under mean-square convergence.
4. Derive the sine-quotient form of D_N and sketch p_2, p_8 over a period. Explain why this sequence is not the sequence of Fourier partial sums of one fixed function.

Question 6 – Solution

Strategy. Point evaluation on a finite-dimensional space has a norm that grows with the number of available modes.

Step 1: Apply Cauchy–Schwarz in coefficient space. An orthonormal basis is $1/\sqrt{2\pi}$, $\cos nx/\sqrt{\pi}$, $\sin nx/\sqrt{\pi}$. The sum of their squared values at any point is

$$\frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^N (\cos^2 nx + \sin^2 nx) = \frac{2N+1}{2\pi}.$$

Cauchy–Schwarz against the orthonormal coefficient vector proves

$$\|p\|_{\infty} \leq \sqrt{\frac{2N+1}{2\pi}} \|p\|_2.$$

Step 2: Identify equality cases. Equality at x_0 holds exactly when the coefficient vector is a real nonzero multiple of the basis-value vector at x_0 . Equivalently,

$$p(x) = CD_N(x - x_0), \quad C \in \mathbb{R} \setminus \{0\}.$$

This exhibits the sharp constant, not merely an upper estimate.

Step 3: Build a vanishing-energy sequence with a fixed peak. The triangle inequality and the value at zero give $\|D_N\|_{\infty} = 2N+1$. Orthogonality gives $\|D_N\|_2^2 = 2\pi(2N+1)$. Consequently

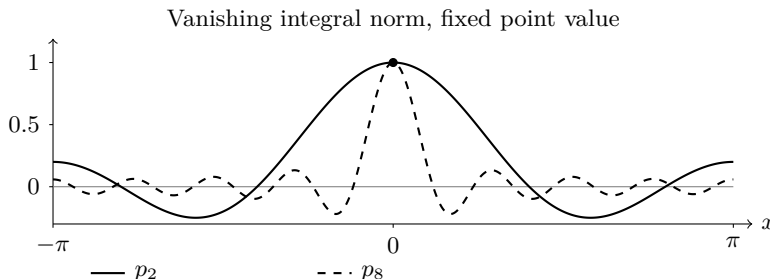
$$\|p_N\|_{\infty} = 1, \quad \|p_N\|_2^2 = \frac{2\pi}{2N+1} \rightarrow 0.$$

At 0 modulo 2π , $p_N = 1$. At every other fixed point it tends to zero by the quotient below. Point evaluation is therefore not continuous in the L^2 norm on the union of these polynomial spaces.

Step 4: Explain the oscillatory peak. Summing $\sum_{n=-N}^N e^{inx}$ yields

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)} \quad (x \notin 2\pi\mathbb{Z}).$$

The removable value at zero is $2N+1$. The normalized peak narrows with N . These p_N cannot be successive partial sums of one fixed function: their constant coefficients $1/(2N+1)$ change with N , whereas truncation preserves every already-included Fourier coefficient.



Question 7

For real bounded piecewise continuous 2π -periodic f , its degree-one Fourier projection is

$$(S_1 f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)(1+2\cos t) dt.$$

Use the essential supremum norm, which ignores changes on sets of measure zero.

Tasks

1. Prove $\|S_1 f\|_{\infty} \leq \Lambda \|f\|_{\infty}$, where $\Lambda = (2\pi)^{-1} \int_{-\pi}^{\pi} |1+2\cos t| dt$.
2. Compute Λ exactly, locating every sign change of the kernel.
3. Construct an admissible f with $\|f\|_{\infty} = 1$ for which equality is attained at $x = 0$. Verify its mean and first cosine coefficient directly.
4. Use the result to give a sharp worst-case amplification factor for a bounded input error under S_1 . Explain why orthogonal projection being a contraction in L^2 does not imply contraction in the supremum norm.

Question 7 – Solution

Strategy. The signed Fourier kernel controls integral projection, while its absolute integral controls worst-case pointwise error.

Step 1: Bound the projection in the supremum norm. Taking absolute values in the given integral gives

$$|S_1 f(x)| \leq \frac{\|f\|_\infty}{2\pi} \int_{-\pi}^{\pi} |1 + 2 \cos t| dt.$$

Null-set changes do not affect this estimate or the projected polynomial. Taking the supremum over x proves the claimed inequality.

Step 2: Integrate the absolute kernel. The kernel is nonnegative for $|t| \leq 2\pi/3$ and negative outside. Its full signed integral is 2π . Its integral on the two negative intervals is

$$2 \int_{2\pi/3}^{\pi} (1 + 2 \cos t) dt = \frac{2\pi}{3} - 2\sqrt{3}.$$

Changing the sign of that contribution gives

$$\Lambda = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} > 1.$$

Step 3: Attain the bound with an explicit input. Choose $f(t) = \operatorname{sgn}(1 + 2 \cos t)$, with value zero at its two zeros. It is even and piecewise constant with essential supremum one. Its positive region has length $4\pi/3$, so its mean is $1/3$. Direct integration gives

$$a_1 = \frac{2}{\pi} \left(\int_0^{2\pi/3} \cos t dt - \int_{2\pi/3}^{\pi} \cos t dt \right) = \frac{2\sqrt{3}}{\pi}, \quad b_1 = 0.$$

Hence $S_1 f(0) = 1/3 + 2\sqrt{3}/\pi = \Lambda$, proving sharpness.

Step 4: Interpret the stability statement. For input error e with $\|e\|_\infty \leq \varepsilon$, linearity gives $\|S_1(f + e) - S_1 f\|_\infty \leq \Lambda \varepsilon$. The choice $e = \varepsilon \operatorname{sgn}(1 + 2 \cos t)$ attains this bound. Orthogonal projection still satisfies $\|S_1 e\|_2 \leq \|e\|_2$ by the Pythagorean identity. That statement uses an integral norm, and does not prevent a larger pointwise peak. Contraction is a property of a specified norm.

Question 8

Let real coefficient sequences satisfy

$$\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) \leq M, \quad M > 0.$$

Define the formal series $f = m + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx)$ and let S_N be its degree- N truncation, $N \geq 1$. Write $T_N = \sum_{n > N} n^{-2}$.

Tasks

1. Prove that the series defines a continuous periodic function by establishing uniform convergence.
2. Prove $\|f - S_N\|_{\infty} \leq \sqrt{MT_N} < \sqrt{M/N}$.
3. For a fixed N and prescribed point x_0 , construct coefficients satisfying the constraint for which the bound $\sqrt{MT_N}$ is attained.
4. Prove $\|f - S_N\|_2^2 \leq \pi M/(N+1)^2$ and construct an equality case for this bound. Explain why the two optimizing coefficient patterns differ.

Question 8 – Solution

Strategy. Use the weighted coefficient budget with two different dual estimates: one for a point value and one for energy.

Step 1: Prove uniform convergence from the budget. For any finite tail from p to q , Cauchy–Schwarz gives

$$\left| \sum_{n=p}^q (a_n \cos nx + b_n \sin nx) \right| \leq \left(\sum_{n=p}^q n^2 (a_n^2 + b_n^2) \right)^{1/2} \left(\sum_{n=p}^q \frac{\cos^2 nx + \sin^2 nx}{n^2} \right)^{1/2}.$$

It is at most $\sqrt{M}(\sum_{n \geq p} n^{-2})^{1/2}$, independently of x , and tends to zero as $p \rightarrow \infty$. The uniform Cauchy criterion applies. The limit of these continuous periodic partial sums is continuous and periodic.

Step 2: Bound the infinite tail. Passing to the uniform limit of the finite-tail inequality gives

$$\|f - S_N\|_\infty \leq \sqrt{MT_N} < \sqrt{M/N} \quad (N \geq 1).$$

The weighted hypothesis controls the full collection of omitted modes, not just a finite list of observed coefficients.

Step 3: Attain the exact pointwise bound. Set every coefficient with $n \leq N$ to zero and, for $n > N$, choose

$$a_n = \sqrt{\frac{M}{T_N}} \frac{\cos(nx_0)}{n^2}, \quad b_n = \sqrt{\frac{M}{T_N}} \frac{\sin(nx_0)}{n^2}.$$

The weighted sum equals M , and at x_0 the tail equals $\sqrt{MT_N}$. This proves sharpness of the first bound over all admissible sequences. The mean m may be arbitrary because it cancels from the error.

Step 4: Optimize the integral error. Orthogonality, justified by the uniform convergence, gives

$$\|f - S_N\|_2^2 = \pi \sum_{n > N} (a_n^2 + b_n^2) \leq \frac{\pi}{(N+1)^2} \sum_{n > N} n^2 (a_n^2 + b_n^2) \leq \boxed{\frac{\pi M}{(N+1)^2}}.$$

Equality is attained by $a_{N+1} = \sqrt{M}/(N+1)$ and all other coefficients zero. The integral optimizer concentrates its budget in the cheapest omitted mode; the pointwise optimizer distributes it across all omitted modes with phases aligned at the chosen point.

Question 9

For $p > 0$, study $F_{p,N}(x) = \sum_{n=1}^N \cos(nx)/n^p$ on $[-\pi, \pi]$. You may use completeness of L^2 : a Cauchy sequence in its norm has an L^2 limit; taking an inner product with any fixed L^2 function is continuous. Also use Dirichlet's test for a scalar series with bounded partial sums multiplied by decreasing positive weights tending to zero.

Tasks

1. Classify exactly those p for which the sequence converges in L^2 , proving both directions by orthogonality.
2. Determine convergence at $x = 0$ modulo 2π , and at every other fixed point.
3. Classify uniform convergence on the full period. Prove uniform convergence on each closed subinterval avoiding $2\pi\mathbb{Z}$ for every $p > 0$.
4. For $1/2 < p \leq 1$, explain rigorously how an L^2 function can have these Fourier partial sums although they diverge at zero. Can redefining the limit's value at zero change that divergence?

Question 9 – Solution

Strategy. The thresholds for square summability, pointwise convergence at a common phase, and uniform convergence need not coincide.

Step 1: Classify mean-square convergence. For $M > N$, orthogonality gives

$$\|F_{p,M} - F_{p,N}\|_2^2 = \pi \sum_{n=N+1}^M n^{-2p}.$$

This tends to zero uniformly in $M > N$ exactly when $2p > 1$. Completeness proves existence of an L^2 limit in that case. For $p \leq 1/2$, the norms $\pi \sum_{n=1}^N n^{-2p}$ are unbounded, so no L^2 limit is possible. Thus $p > 1/2$ is necessary and sufficient.

Step 2: Classify each pointwise series. At zero modulo 2π the series is $\sum n^{-p}$, converging exactly when $p > 1$; otherwise its partial sums tend to $+\infty$. At any other fixed x , the geometric-sum bound gives bounded partial sums of $\cos nx$. The weights n^{-p} decrease to zero for every $p > 0$, so Dirichlet's test proves convergence at every such point.

Step 3: Distinguish global and local uniform convergence. For $p > 1$, the Weierstrass test proves absolute uniform convergence globally. For $0 < p \leq 1$, divergence at zero precludes uniform convergence on the full period. Away from $2\pi\mathbb{Z}$, the cosine block sums are bounded by $1/\sin(\delta/2)$. Summation by parts with weights n^{-p} gives

$$\sup_{\delta \leq x \leq 2\pi - \delta} \left| \sum_{n > N} \frac{\cos nx}{n^p} \right| \leq \frac{(N+1)^{-p}}{\sin(\delta/2)}.$$

Hence local uniform convergence holds for every $p > 0$.

Step 4: Interpret the L^2 limit. For $1/2 < p \leq 1$, continuity of inner products shows that the L^2 limit has mean zero, cosine coefficients n^{-p} and zero sine coefficients. Thus these really are its Fourier partial sums, not arbitrary approximations. An L^2 function is defined only up to changes on sets of measure zero; a single divergent point does not contradict convergence in the integral norm. Assigning any value to the limit at zero changes neither coefficients nor partial sums, and therefore cannot remove the divergence there.

Question 10

Let $f \in L^2(-\pi, \pi)$ be real and periodic with mean m and Fourier coefficients a_n, b_n . For $t > 0$, define the damped expansion

$$U_t(x) = m + \sum_{n=1}^{\infty} e^{-n^2 t} (a_n \cos nx + b_n \sin nx).$$

Use Parseval and L^2 convergence of Fourier partial sums. This problem concerns the series itself; no partial differential equation is assumed.

Tasks

1. Prove that for each fixed $t > 0$ the series, and every series obtained by differentiating in x finitely many times, converge uniformly.
2. Prove $U_t \rightarrow f$ in L^2 as $t \downarrow 0$ using a finite-head and small-tail argument, with an exact coefficient formula for the error.
3. If $\sum n^2(a_n^2 + b_n^2) \leq M$, prove $\|U_t - f\|_2 \leq \sqrt{\pi M t}$.
4. Show that the mean is preserved and that $\|U_t - m\|_2 \leq \|f - m\|_2$. Explain why the general L^2 assumption alone cannot guarantee uniform convergence to the assigned function f as $t \downarrow 0$.

Question 10 – Solution

Strategy. Damping controls all high frequencies for positive t ; removing it requires a separate convergence argument.

Step 1: Prove smoothness for positive damping. Parseval gives $E = \sum (a_n^2 + b_n^2) < \infty$. After k derivatives, the absolute value of the n th harmonic is at most $n^k e^{-n^2 t} \sqrt{a_n^2 + b_n^2}$. Cauchy-Schwarz bounds the sum by

$$\sqrt{E} \left(\sum_{n \geq 1} n^{2k} e^{-2n^2 t} \right)^{1/2} < \infty.$$

The exponential dominates every power, so the Weierstrass test applies for each fixed $t > 0$ and each integer $k \geq 0$. Repeated use of the uniform derivative theorem justifies differentiation; U_t is smooth and periodic.

Step 2: Remove damping in the integral norm. The Fourier coefficients of the difference give

$$\|U_t - f\|_2^2 = \pi \sum_{n \geq 1} (1 - e^{-n^2 t})^2 (a_n^2 + b_n^2).$$

Given $\varepsilon > 0$, choose K so that $\pi \sum_{n > K} (a_n^2 + b_n^2) < \varepsilon/2$. Since $0 \leq 1 - e^{-n^2 t} \leq 1$, this bounds the error tail for every t . The remaining finite sum tends to zero, hence is below $\varepsilon/2$ for sufficiently small t . This proves the limit.

Step 3: Obtain a quantitative rate under extra information. For $z \geq 0$, $1 - e^{-z} \leq \min(z, 1)$, so $(1 - e^{-z})^2 \leq z$. Applying this with $z = n^2 t$ gives

$$\|U_t - f\|_2^2 \leq \pi t \sum n^2 (a_n^2 + b_n^2) \leq \pi M t.$$

Taking square roots yields the requested rate; this uses a stronger hypothesis than mere square summability.

Step 4: Separate smoothing from uniform recovery. Uniform integration of the damped series preserves the mean m . Parseval gives $\|U_t - m\|_2^2 = \pi \sum e^{-2n^2 t} (a_n^2 + b_n^2) \leq \|f - m\|_2^2$. Every U_t is continuous. Choose an L^2 periodic step function with a jump: it cannot be a uniform limit of these continuous functions. Even isolated redefinitions of f are invisible to the coefficients. Thus smooth approximants and mean-square recovery do not alone guarantee uniform recovery of an arbitrarily assigned representative.