

Question 1

Let $f(x) = x + x^2$ for $-\pi < x < \pi$, extend with period 2π , and assign $f(\pi + 2k\pi) = \pi^2$. Use

$$S_N = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx), \quad (a_n, b_n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)(\cos nx, \sin nx) dx.$$

For piecewise smooth periodic functions, the Fourier series converges to the average of the one-sided limits.

Tasks

1. Separate the even and odd contributions and derive all coefficients, including a_0 .
2. Determine the sum at every point, explaining why the two endpoint limits differ despite periodicity. Sketch f and S_8 on one period.
3. Find the best approximation to f in $\text{span}\{1, \cos x, \sin x\}$ for squared integral error. Prove optimality and uniqueness.
4. Compute that minimum error exactly by orthogonality and direct integration. Explain why changing the assigned endpoint value would not change this approximation.

Question 1 – Solution

Strategy. Use parity to separate coefficients, then use orthogonal projection to measure the combined approximation.

Step 1: Separate the two symmetries. The even part is x^2 and the odd part is x . Odd integrands give zero in the cross terms. Integration by parts gives

$$a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{4(-1)^n}{n^2}, \quad b_n = \frac{2(-1)^{n+1}}{n}.$$

For example, $\int_0^\pi x \sin nx = -\pi(-1)^n/n$ and $\int_0^\pi x^2 \cos nx = 2\pi(-1)^n/n^2$.

Step 2: Identify the periodic join. Inside the base interval the sum is $x + x^2$. At the join, the left limit is $\pi^2 + \pi$ and the right limit is $\pi^2 - \pi$, so the sum is π^2 . Periodicity identifies the location of the join, not its two one-sided limits. The assigned value equals this average; the target still has a jump.

Step 3: Prove the projection claim. Orthogonality gives

$$p(x) = \pi^2/3 - 4\cos x + 2\sin x.$$

The residual $f - p$ is orthogonal to the three basis functions. For any other $q = A + B\cos x + C\sin x$,

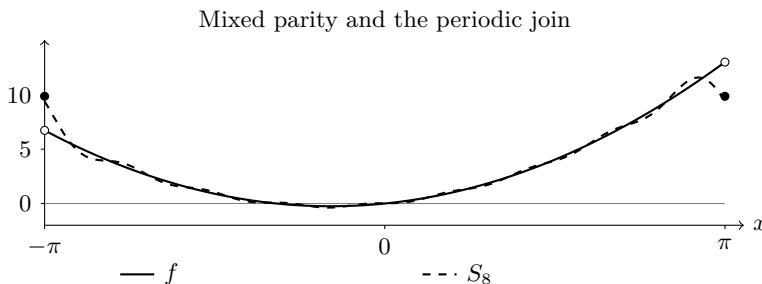
$$\|f - q\|_2^2 = \|f - p\|_2^2 + 2\pi(A - \pi^2/3)^2 + \pi(B + 4)^2 + \pi(C - 2)^2.$$

Thus p is the unique minimizer in this span.

Step 4: Compute the error independently. The odd cross term integrates to zero, giving $\|f\|_2^2 = 2\pi^5/5 + 2\pi^3/3$, while $\|p\|_2^2 = 2\pi^5/9 + 20\pi$. Consequently

$$\min \|f - q\|_2^2 = \frac{8\pi^5}{45} + \frac{2\pi^3}{3} - 20\pi.$$

Changing isolated endpoint values changes neither integrals nor coefficients. It may change whether the assigned values equal the series sums.



Question 2

A 2π -periodic pulse has height one on the circular interval $c - \pi/3 < x < c + \pi/3$ (modulo 2π), height zero elsewhere, and value $1/2$ at its two edges. The unknown center c is defined modulo 2π . Use the real Fourier convention $f = a_0/2 + \sum_{n \geq 1} (a_n \cos nx + b_n \sin nx)$, with coefficients integrated over any full period. The series has the usual one-sided-average value at a jump.

Tasks

1. Derive the mean and all coefficients by centering the integration interval at c .
2. Determine exactly which entire harmonics vanish. Distinguish a missing harmonic from a zero cosine coefficient caused only by phase.
3. Recover c modulo 2π from the signed pair (a_1, b_1) . State an exact consistency condition for a measured pair under this pulse model.
4. Show what is lost if only the harmonic amplitudes $\sqrt{a_n^2 + b_n^2}$ are measured. For $c = \pi/4$, sketch the pulse and its degree-12 partial sum, marking the edge values.

Question 2 – Solution

Strategy. Centering exposes the pulse width; translating back rotates each sine/cosine coefficient pair.

Step 1: Integrate the centered pulse. The area is $2\pi/3$, so the mean is $1/3$ and $a_0 = 2/3$. Put $x = c + t$. The sine terms odd in t integrate to zero, leaving

$$a_n = \frac{2 \sin(n\pi/3)}{\pi n} \cos nc, \quad b_n = \frac{2 \sin(n\pi/3)}{\pi n} \sin nc.$$

These formulas also apply when the pulse crosses the chosen period boundary: integrate instead over a period centered at c .

Step 2: Identify genuine missing modes. The amplitude is

$$R_n = \sqrt{a_n^2 + b_n^2} = \frac{2|\sin(n\pi/3)|}{\pi n}.$$

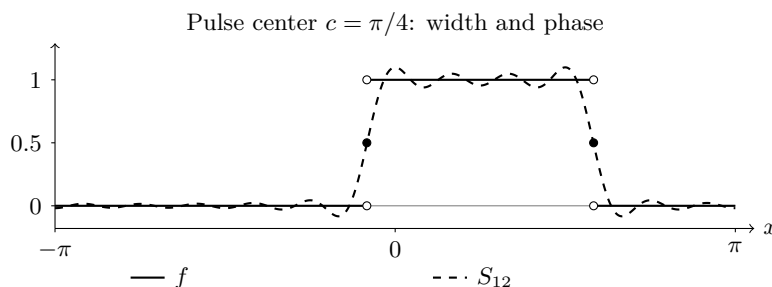
Hence the entire harmonic vanishes exactly when 3 divides n . For other n , a_n alone can vanish because $\cos nc = 0$, while $b_n \neq 0$. Discarding such a sine term would remove a real harmonic.

Step 3: Recover the circular center. For $n = 1$ the prefactor is positive:

$$(a_1, b_1) = \frac{\sqrt{3}}{\pi}(\cos c, \sin c), \quad c = \text{atan2}(b_1, a_1) \pmod{2\pi}.$$

Consistency is exactly $a_1^2 + b_1^2 = 3/\pi^2$ for real measured coefficients. Every pair on that circle corresponds to precisely one center modulo 2π . Using only an ordinary arctangent of b_1/a_1 loses quadrant information.

Step 4: Separate location from amplitude. Every R_n is independent of c , so amplitude measurements leave every center possible, even if all harmonics are known. At $c = \pi/4$, the pulse edges in $[-\pi, \pi]$ are $-\pi/12$ and $7\pi/12$. The sum is one between them, zero outside, and $1/2$ at each edge. The plotted partial sum uses both coefficient families; the finite oscillations do not alter these limiting edge averages.



Question 3

Consider a half-wave rectifier with output $h(x) = \max(\sin x, 0)$, extended with period 2π . Use the mean-term convention $a_0/2$. You may use Parseval:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n \geq 1} (a_n^2 + b_n^2).$$

Tasks

1. Express h using $\sin x$ and $|\sin x|$. Derive its complete real Fourier series.
2. Explain why all sine coefficients except b_1 vanish, and why only even positive cosine frequencies occur.
3. Compute the average squared output and its division into the constant term, the fundamental harmonic, and all higher harmonics.
4. Determine the fundamental periods of h and $|\sin x|$, with a proof that the proposed periods are minimal. Sketch h and its degree-8 Fourier sum.

Question 3 – Solution

Strategy. Rectification combines an odd fundamental with an even function of half the period.

Step 1: Derive the expansion. Since $h = (\sin x + |\sin x|)/2$, its mean is $1/\pi$. Direct integration on $(0, \pi)$ gives $b_1 = 1/2$ and $b_n = 0$ for $n \neq 1$. For $k \geq 1$, product-to-sum integration gives

$$\frac{1}{\pi} \int_0^\pi \sin x \cos(2kx) dx = -\frac{2}{\pi(4k^2 - 1)}.$$

Odd cosine coefficients vanish, including a_1 . Thus

$$h(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{4k^2 - 1}.$$

The coefficients are absolutely summable; the standard piecewise smooth Fourier theorem identifies the sum with this continuous target, including its corners.

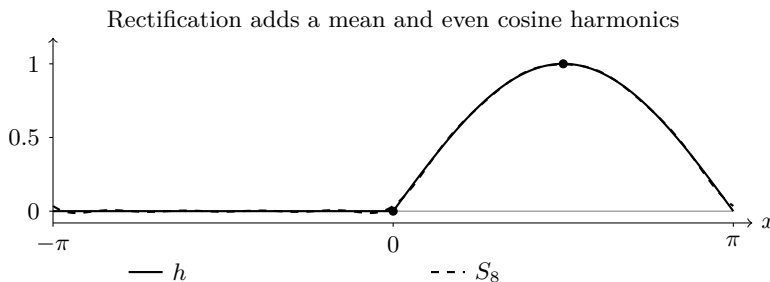
Step 2: Explain the selection rules. The function $|\sin x|$ is even, so contributes only cosines. It is also π -periodic, eliminating odd cosine frequencies in a 2π expansion. The remaining odd part is exactly $\sin x/2$, so it contributes only b_1 . The full rectified waveform is neither even nor odd.

Step 3: Account for the average power. Direct integration gives $(2\pi)^{-1} \int h^2 = 1/4$. Orthogonality separates the nonnegative contributions:

$$P_{\text{constant}} = \frac{1}{\pi^2}, \quad P_{\text{fundamental}} = \frac{1}{8}, \quad P_{\text{higher}} = \frac{1}{8} - \frac{1}{\pi^2}.$$

They sum to $1/4$. The higher-harmonic contribution is positive; the mean squared is only the constant contribution, not the total average squared output.

Step 4: Prove the fundamental periods. The maxima of h occur exactly at $\pi/2 + 2k\pi$. Any positive period must take one maximum to another, hence be a positive multiple of 2π ; 2π works. The maxima of $|\sin x|$ occur at $\pi/2 + k\pi$, so its least positive period is π . Thus the full output has fundamental period 2π despite its even-frequency cosine terms: its nonzero sine fundamental matters.



Question 4

Let $f(x) = e^x$ for $0 < x < 2$ and extend periodically with period 2. At each join assign the average $(1+e^2)/2$. Write

$$f \sim m + \sum_{n \geq 1} (a_n \cos(n\pi x) + b_n \sin(n\pi x)), \quad m = \frac{1}{2} \int_0^2 f, \quad (a_n, b_n) = \int_0^2 f(\cos(n\pi x), \sin(n\pi x)).$$

Set $E = e^2 - 1$.

Tasks

1. Derive m, a_n, b_n . Identify the series sum at the periodic join.
2. Use integration by parts on $f' = f$ inside $(0, 2)$ to obtain two coefficient relations. Explain why omitting the boundary jump would incorrectly force every positive-frequency coefficient to zero.
3. Find a continuous 2-periodic function F with $F' = f - m$ away from the joins and $F(0) = 0$. Prove uniqueness under this normalization.
4. Compute the mean and all Fourier coefficients of F directly. Verify that differentiating its positive-frequency terms formally gives those of $f - m$, and explain why f itself cannot have a continuous periodic primitive.

Question 4 – Solution

Strategy. Keep the endpoint jump in integration by parts, and remove the mean before seeking a periodic primitive.

Step 1: Compute the full coefficients. For $k = n\pi$, exponential-trigonometric antiderivatives at $0, 2$ give

$$m = \frac{E}{2}, \quad a_n = \frac{E}{1+k^2}, \quad b_n = -\frac{kE}{1+k^2}.$$

The Fourier sum at a join is $(1+e^2)/2$, since its two limits are e^2 and 1 . The assigned join value does not enter any integral.

Step 2: Retain the boundary contribution. On the open interval, $f' = f$. Integration by parts gives

$$\int_0^2 f' \cos kx = E + k \int_0^2 f \sin kx, \quad \int_0^2 f' \sin kx = -k \int_0^2 f \cos kx.$$

Thus $a_n = E + kb_n$ and $b_n = -ka_n$, yielding the coefficients above. Dropping E would give $(1+k^2)a_n = 0$ and then $b_n = 0$. The periodic extension has a jump, so it is not a globally smooth periodic solution of $f' = f$.

Step 3: Construct the periodic primitive. Integrating the mean-zero forcing gives, on $[0, 2]$,

$$F(x) = e^x - 1 - \frac{E}{2}x.$$

Both endpoint values are zero, so its periodic extension is continuous. Two such primitives differ by a constant on each open period; continuity connects those constants, and $F(0) = 0$ fixes the remaining freedom.

Step 4: Check the transformed coefficients. Direct integration gives mean $M = -1$. If A_n, B_n denote the coefficients of F , the polynomial terms have zero cosine integral and $\int_0^2 x \sin kx = -2/k$. Consequently

$$A_n = \frac{E}{1+k^2}, \quad B_n = \frac{E}{k(1+k^2)}.$$

Indeed $kB_n = a_n$ and $-kA_n = b_n$. These are coefficient identities; the derivative equation is verified by the closed form on each open period. Finally, a continuous piecewise smooth periodic primitive of f would have net change $\int_0^2 f = E \neq 0$, which is impossible. Subtracting the mean is essential.

Question 5

A change of origin replaces $f(x)$ by $f(t + c)$, where c is defined modulo 2π . Consider

$$f(x) = 2 \cos x + 2 \sin x + \cos 2x.$$

Tasks

1. Derive how a general coefficient pair (a_n, b_n) changes under $x = t + c$. Prove that its amplitude is unchanged.
2. Determine every origin c for which $f(t + c)$ is even, or prove that none exists.
3. Determine every origin c for which $f(t + c)$ is odd. Give the shifted expansion for one such origin.
4. Replace $\cos 2x$ by $\sin 2x$ to obtain g . Find all origins making $g(t + c)$ even and give one resulting cosine-only expansion. Explain why separately aligning each harmonic is not a valid test for a single common symmetry center.

Question 5 – Solution

Strategy. A common origin must satisfy the phase conditions of every nonzero harmonic simultaneously.

Step 1: Derive the coefficient rotation. Angle addition gives the new coefficients

$$a'_n = a_n \cos nc + b_n \sin nc, \quad b'_n = -a_n \sin nc + b_n \cos nc.$$

Squaring and adding cancels the cross terms, so $(a'_n)^2 + (b'_n)^2 = a_n^2 + b_n^2$. For a trigonometric polynomial, evenness is equivalent to every b'_n being zero; oddness requires a zero mean and every a'_n zero. Necessity follows from orthogonality, and sufficiency follows from sine/cosine parity.

Step 2: Test all evenness conditions. At frequency one, $b'_1 = 2(\cos c - \sin c) = 0$ implies $c = \pi/4 + k\pi$. At frequency two, $b'_2 = -\sin 2c$ would then equal -1 . Hence

$$\text{No origin makes } f(t+c) \text{ even.}$$

The first frequency can be aligned, but that shift makes the second purely sine.

Step 3: Find every oddness origin. The first condition $a'_1 = 2(\cos c + \sin c) = 0$ gives $c = 3\pi/4 + k\pi$. At each of these, $a'_2 = \cos 2c = 0$, and the mean is already zero. These are all solutions. At $c = 3\pi/4$,

$$f(t + 3\pi/4) = -2\sqrt{2} \sin t + \sin 2t.$$

Modulo 2π the two origins are $3\pi/4$ and $7\pi/4$.

Step 4: Change the relative harmonic phase. For $g = 2 \cos x + 2 \sin x + \sin 2x$, the first-frequency evenness condition is again $c = \pi/4 + k\pi$. Now the second is $b'_2 = \cos 2c = 0$, which is satisfied. At $c = \pi/4$,

$$g(t + \pi/4) = 2\sqrt{2} \cos t + \cos 2t.$$

All evenness origins are $\pi/4 + k\pi$. Each harmonic admits its own phase alignment, but a symmetry of the full function requires the same c for all frequencies. The two examples have identical harmonic amplitudes and different answers to the evenness question.

Question 6

Find the continuous 2π -periodic steady response of

$$y' + 2y = 1 + q(x), \quad q(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & \pi < x < 2\pi, \end{cases}$$

where q is periodically extended and assigned zero at its jumps. The differential equation is required on the open intervals between jumps. Solutions are continuous and piecewise continuously differentiable.

Tasks

1. Derive the full Fourier coefficients of q , including its mean.
2. Obtain the mean and all cosine and sine coefficients of the periodic response by mode equations.
3. Independently solve on the two intervals and enforce continuity and periodicity. Prove that this response is unique.
4. Verify its mean, extrema and behavior at the switches. Sketch the exact response and its degree-9 Fourier approximation. Explain why a sine-only forcing produces cosine terms in the response.

Question 6 – Solution

Strategy. Solve the harmonic response and the piecewise differential equation independently.

Step 1: Expand the input. The periodic extension of q is odd. Direct integration gives zero mean and zero cosine coefficients, with

$$b_n^{(q)} = \frac{2(1 - (-1)^n)}{\pi n} = \begin{cases} 4/(\pi n), & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The constant in $1 + q$ is one.

Step 2: Solve the coupled mode equations. Write $y = m + \sum (A_n \cos nx + B_n \sin nx)$. Integration by parts is legitimate piecewise: continuity and periodicity cancel the internal and endpoint terms. Thus $2m = 1$, $nB_n + 2A_n = 0$, and $-nA_n + 2B_n = b_n^{(q)}$. Hence

$$m = \frac{1}{2}, \quad A_n = -\frac{4}{\pi(n^2 + 4)}, \quad B_n = \frac{8}{\pi n(n^2 + 4)} \quad (n \text{ odd}),$$

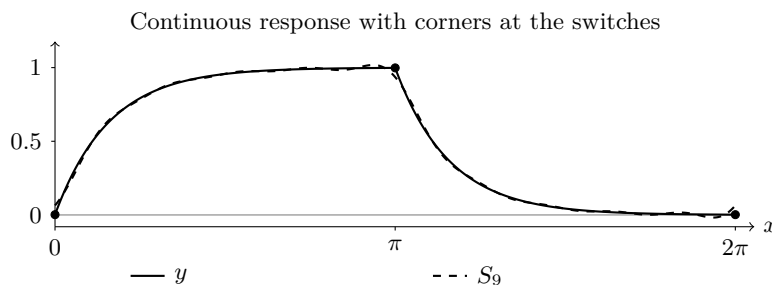
and both coefficients vanish for even n .

Step 3: Verify a closed-form periodic response. Put $r = e^{-2\pi}$. Solving the constant-forcing equations and matching gives

$$y(x) = \begin{cases} 1 - \frac{e^{-2x}}{1+r}, & 0 \leq x \leq \pi, \\ \frac{e^{-2(x-\pi)}}{1+r}, & \pi \leq x \leq 2\pi. \end{cases}$$

Both expressions give $y(\pi) = 1/(1+r)$, and $y(0) = y(2\pi) = r/(1+r)$. A difference of two responses is Ce^{-2x} across the whole period by continuity; periodicity forces $C = 0$. The verified closed form therefore has the coefficients determined above.

Step 4: Interpret the response. The first branch increases and the second decreases, so the extrema are $r/(1+r)$ and $1/(1+r)$. Integrating the equation over a period gives mean $1/2$. The response is continuous, but y' jumps by -2 at π and by 2 at 0 modulo 2π ; the assigned input values do not require a derivative there. Differentiation couples sine and cosine coefficients, creating a phase shift. Absolute coefficient summability and the Fourier theorem identify the plotted series with the continuous response.



Question 7

For real parameters α, β , seek a twice continuously differentiable 2π -periodic solution of

$$y'' + 4y = 3 + \cos x + \alpha \cos 2x + \beta \sin 2x.$$

Tasks

1. Derive necessary conditions on α, β by multiplying by each resonant mode and integrating over one period.
2. Prove those conditions are sufficient and find all periodic solutions when they hold.
3. Under the same conditions, impose $y(0) = y'(0) = 0$. Find the resulting solution and verify all conditions directly.
4. Explain why initial conditions cannot restore periodic solvability when a resonant forcing coefficient is nonzero. Relate this to the failed division by $4 - n^2$ in the Fourier equations.

Question 7 – Solution

Strategy. Test the forcing against the periodic homogeneous modes before dividing by a spectral factor.

Step 1: Derive the compatibility conditions. For a periodic C^2 function, both y and y' match at the period endpoints. Two integrations by parts therefore give

$$\int_0^{2\pi} (y'' + 4y) \cos 2x = 0, \quad \int_0^{2\pi} (y'' + 4y) \sin 2x = 0.$$

Orthogonality makes the corresponding right sides $\pi\alpha$ and $\pi\beta$. Thus

$$\alpha = \beta = 0$$

is necessary. The constant and first harmonic do not contribute to these tests.

Step 2: Construct every compatible solution. With these parameters zero, a particular solution is $3/4 + \cos x/3$. The homogeneous equation has basis $\cos 2x, \sin 2x$, both periodic. Thus

$$y(x) = \frac{3}{4} + \frac{1}{3} \cos x + A \cos 2x + B \sin 2x, \quad A, B \in \mathbb{R}.$$

This proves sufficiency as well as completeness: subtract the particular solution from any solution and solve the homogeneous equation.

Step 3: Enforce the additional normalization. The initial values give $3/4 + 1/3 + A = 0$ and $2B = 0$. Hence

$$y(x) = \frac{3}{4} + \frac{1}{3} \cos x - \frac{13}{12} \cos 2x.$$

It has $y(0) = 0$, $y'(0) = 0$, and is 2π -periodic together with all derivatives. Substitution gives $y'' + 4y = 3 + \cos x$ because the frequency-two term is annihilated.

Step 4: Identify the resonance obstruction. For frequency n , the mode equation multiplies each coefficient of y by $4 - n^2$. At $n = 2$ it reads $0 = \alpha$ or $0 = \beta$: it is a compatibility condition, not a formula for an unknown coefficient. When it holds, the two coefficients are free until extra conditions are supplied. When it fails, no periodic C^2 solution exists. Although an initial-value problem still has a unique solution, its resonant response cannot be periodic; altering homogeneous constants does not change either integral obstruction.

Question 8

Let $f(x) = 1 + \cos x + 2 \sin x + \cos 2x$. Its periodic autocorrelation is

$$R(\tau) = \frac{1}{2\pi} \int_0^{2\pi} f(x)f(x+\tau) dx.$$

Tasks

1. Derive R using orthogonality. Explain why sine terms disappear from the correlation even though f has a nonzero sine coefficient.
2. Find the exact maximum and minimum of R , including all shifts attaining them modulo 2π .
3. Construct a real trigonometric polynomial g with the same mean and the same R which is not a translate of f . Prove both claims.
4. Derive the identity relating R to the squared distance between f and its translate. Determine every shift leaving f unchanged and connect this with the fundamental period.

Question 8 – Solution

Strategy. Correlation retains harmonic energies but discards their individual phases.

Step 1: Evaluate the correlation. Different frequencies are orthogonal. At frequency n , a pair $a_n \cos nx + b_n \sin nx$ contributes $(a_n^2 + b_n^2) \cos(n\tau)/2$; the two mixed sine terms cancel. The constant contributes the square of the mean. Thus

$$R(\tau) = 1 + \frac{5}{2} \cos \tau + \frac{1}{2} \cos 2\tau.$$

In particular, $R(0) = 4$ is the mean square of f , not the square of its mean. The correlation is even regardless of the parity of f .

Step 2: Optimize over all shifts. Put $t = \cos \tau \in [-1, 1]$. Then $R = t^2 + (5/2)t + 1/2$, whose derivative $2t + 5/2$ is positive throughout this interval. Hence

$$\max R = 4 \quad (\tau = 0 \bmod 2\pi), \quad \min R = -1 \quad (\tau = \pi \bmod 2\pi).$$

A correlation can be negative here; the original signal is not required to be nonnegative.

Step 3: Exhibit phase ambiguity beyond translation. Take $g(x) = 1 + \sqrt{5} \cos x + \cos 2x$. Its mean is one and its harmonic squared amplitudes are 5 and 1, so it gives exactly the same R . If $g(x) = f(x + c)$, equality at frequency two requires $\cos 2c = 1$ and $\sin 2c = 0$, so $c = k\pi$. But then the first sine coefficient of $f(x + c)$ is $2(-1)^k \neq 0$, unlike g . Thus g is not a translate. Correlation does not determine relative harmonic phases.

Step 4: Recover the exact translation invariance. Translation preserves $\int f^2$. Expanding the squared difference gives

$$\frac{1}{2\pi} \int_0^{2\pi} [f(x + \tau) - f(x)]^2 dx = 2(R(0) - R(\tau)).$$

This is zero precisely at $\tau = 0$ modulo 2π . Zero integral of a continuous nonnegative square means equality at every point. Therefore the invariant shifts are exactly $2k\pi$, and the fundamental period is 2π .

Question 9

For real a, b , a modulated waveform is

$$s_{a,b}(x) = (1 + a \cos x) \cos 3x + b \sin x \sin 3x.$$

Use a 2π Fourier expansion, and define average power as $(2\pi)^{-1} \int_0^{2\pi} s_{a,b}^2$.

Tasks

1. Derive the complete Fourier spectrum using product identities. Identify the frequencies surrounding the carrier frequency 3.
2. Recover a, b from measured cosine coefficients C_2, C_4 . Determine all choices canceling the frequency-4 term while leaving a prescribed frequency-2 coefficient d .
3. For $d = 1/2$ with frequency 4 canceled, find the fundamental period. Prove minimality even though the frequency-1 coefficient is zero.
4. Compute the average power for general a, b , then for this special case. Prove a sharp global bound on the special waveform's absolute value and identify points where equality holds.

Question 9 – Solution

Strategy. Resolve the products into actual harmonics before inferring frequency content, period or energy.

Step 1: Resolve the two products. The identities $\cos x \cos 3x = (\cos 2x + \cos 4x)/2$ and $\sin x \sin 3x = (\cos 2x - \cos 4x)/2$ give

$$s_{a,b}(x) = \cos 3x + \frac{a+b}{2} \cos 2x + \frac{a-b}{2} \cos 4x.$$

The mean and every sine coefficient are zero. The neighboring frequencies, often called sidebands, are 2 and 4, not frequency 1 from the modulation alone.

Step 2: Invert the coefficient measurements. The linear equations give

$$a = C_2 + C_4, \quad b = C_2 - C_4.$$

Requiring $C_4 = 0$ and $C_2 = d$ fixes $a = b = d$ uniquely. The two modulation terms reinforce the lower frequency and cancel the upper frequency when their amplitudes agree.

Step 3: Establish the least period. For $d = 1/2$ the signal is $s = \cos 3x + \frac{1}{2} \cos 2x$. If $T > 0$ is a period, translating this finite expansion and using orthogonality forces $3T$ and $2T$ each to be integer multiples of 2π . Subtracting gives T an integer multiple of 2π . Conversely 2π works. Thus the fundamental period is $\boxed{2\pi}$. A nonzero first harmonic is sufficient for this period in a 2π polynomial, but is not necessary: frequencies 2 and 3 already enforce it.

Step 4: Measure power and peak amplitude. Orthogonality gives

$$P = \frac{1}{2} \left(1 + \frac{(a+b)^2}{4} + \frac{(a-b)^2}{4} \right) = \frac{1}{2} + \frac{a^2 + b^2}{4}.$$

At $a = b = 1/2$, $P = 5/8$. Also $|s| \leq |\cos 3x| + \frac{1}{2} |\cos 2x| \leq 3/2$, with equality at $x = 2k\pi$. For negative equality both cosines would have to be -1 ; the equations $3x = (2j+1)\pi$, $2x = (2k+1)\pi$ are incompatible by parity. Positive equality requires both to be 1, giving exactly $x = 2k\pi$.

Question 10

Two periodic signals are $f(x) = \cos x + \cos 2x$ and $g(x) = \sin x + \cos 2x$. To align them by one delay, minimize

$$D(\delta) = \int_0^{2\pi} [f(x - \delta) - g(x)]^2 dx, \quad \delta \in \mathbb{R}/(2\pi\mathbb{Z}).$$

Tasks

1. Use orthogonality to derive $D(\delta)$ without expanding a long pointwise square.
2. Determine all globally optimal delays and the exact minimum error. Prove global optimality.
3. Compare delays 0 and $\pi/2$. Explain why exactly aligning the first harmonic need not give the best alignment of the full signal.
4. Decide whether any delay makes the two functions identical. Sketch $D(\delta)/\pi$ over one period, marking every minimizer, and explain the nonuniqueness of the best delay.

Question 10 – Solution

Strategy. One delay rotates each harmonic by a different multiple of the same angle, creating a coupled optimization.

Step 1: Compute distance through correlation. Both signals have squared norm 2π . Angle addition and orthogonality give

$$\int_0^{2\pi} f(x-\delta)g(x) dx = \pi(\sin \delta + \cos 2\delta).$$

Therefore

$$\frac{D(\delta)}{\pi} = 4 - 2(\sin \delta + \cos 2\delta).$$

The sine sign follows from $\cos(x-\delta) = \cos x \cos \delta + \sin x \sin \delta$.

Step 2: Optimize over the whole circle. Put $t = \sin \delta \in [-1, 1]$. Since $\cos 2\delta = 1 - 2t^2$,

$$\frac{D(\delta)}{\pi} = \frac{7}{4} + 4 \left(t - \frac{1}{4} \right)^2.$$

The feasible value $t = 1/4$ is the unique minimizing sine value. Consequently

$$D_{\min} = \frac{7\pi}{4}, \quad \delta = \arcsin(1/4) \text{ or } \pi - \arcsin(1/4) \pmod{2\pi}.$$

Completing the square proves these are all global minimizers.

Step 3: Compare two tempting alignments. At $\delta = 0$, the second harmonics agree and $D = 2\pi$. At $\delta = \pi/2$, the first harmonic matches $\sin x$, but the second becomes $-\cos 2x$, so $D = 4\pi$. Exactly matching one harmonic can worsen the full error. The optimum balances their contributions and improves on both choices.

Step 4: Interpret the two minima. The minimum is strictly positive, so no delay makes the signals identical. The objective depends only on $\sin \delta$; the two distinct delays with sine $1/4$ give the same compromise. They do not produce the same shifted function: their first cosine coefficients are opposite and nonzero. Thus nonuniqueness of the best delay does not imply exact agreement or identical aligned waveforms. The graph shows both minima in $[0, 2\pi)$.

