

Question 1

For $f(x) = x$ on $[0, \pi]$, use the convention

$$C_N = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \quad (n \geq 0).$$

For piecewise smooth periodic extensions, the Fourier series converges to the average of its one-sided limits.

Tasks

1. Compute a_0 and all a_n , and identify the actual constant term. Explain the error caused by confusing a_0 with the mean.
2. Describe the even 2π -periodic extension and determine the sum at the endpoints and inside the interval.
3. Prove uniform convergence on $[0, \pi]$ and give an explicit uniform tail bound. Sketch C_{12} and f .
4. Every C_N has zero endpoint derivatives. Does the limit inherit these slopes? Give a quantitative obstruction to uniform convergence of the derivatives to $f' = 1$.

Question 1 – Solution

Strategy. Separate the mean convention from positive-frequency normalization, and distinguish function convergence from derivative convergence.

Step 1: Compute the mean and coefficients. The zero-frequency integral gives $a_0 = \pi$, so the mean term is $\pi/2$. Integration by parts for $n \geq 1$ gives

$$\int_0^\pi x \cos nx \, dx = \frac{(-1)^n - 1}{n^2}, \quad \boxed{a_n = \frac{2((-1)^n - 1)}{\pi n^2}}.$$

Even modes vanish; odd coefficients are $-4/(\pi n^2)$. Using a_0 in place of $a_0/2$ adds the erroneous constant $\pi/2$.

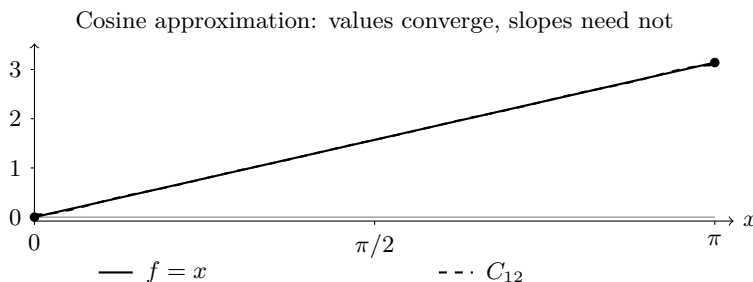
Step 2: Identify the extension and its limits. The extension is $|x|$ on $[-\pi, \pi]$, repeated with period 2π . It is continuous, with corners at multiples of π . At zero its two limits are zero; at π both are π . Therefore

$$\boxed{x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^2}, \quad 0 \leq x \leq \pi.}$$

Step 3: Bound the uniform error. The bound $|a_n| \leq 4/(\pi n^2)$ gives absolute uniform convergence by the Weierstrass test. The Fourier convergence theorem identifies the sum with f , including the endpoints. For $N \geq 1$,

$$\boxed{\|f - C_N\|_\infty \leq \frac{4}{\pi} \sum_{n>N} n^{-2} \leq \frac{4}{\pi N}}.$$

Step 4: Reject an unjustified derivative limit. Every finite derivative is a sine sum, so $C'_N(0) = C'_N(\pi) = 0$. The target has one-sided endpoint derivatives one. Thus $\boxed{\sup_{[0,\pi]} |C'_N - 1| \geq 1}$ for every N . Uniform convergence of functions alone does not permit differentiating the limit or passing endpoint derivative conditions to it.



Question 2

An interior cusp has unknown location $a \in (0, \pi)$:

$$f_a(x) = |x - a|, \quad 0 \leq x \leq \pi.$$

Write $f_a \sim m + \sum_{n \geq 1} a_n \cos nx$, with $m = \pi^{-1} \int_0^\pi f_a$ and $a_n = (2/\pi) \int_0^\pi f_a \cos nx$.

Tasks

1. Compute the mean and all coefficients by splitting at a . Identify the endpoint-slope and interior-corner contributions separately.
2. At $a = \pi/2$, determine exactly which modes survive and find the fundamental period of the even periodic extension. Sketch the target and its first twelve-mode sum.
3. Recover a uniquely from the signed first coefficient a_1 . State the admissible range of that measurement.
4. If only the mean and all coefficient magnitudes are known, is the location unique? Explain the reflection ambiguity and its exceptional symmetric case.

Question 2 – Solution

Strategy. Retain the interior slope jump when integrating a continuous function with a corner.

Step 1: Integrate the linear pieces. The triangular areas give $m = [a^2 + (\pi - a)^2]/(2\pi)$. Twice integrating the pieces separately gives

$$a_n = \frac{2}{\pi n^2} [1 + (-1)^n - 2 \cos(na)], \quad n \geq 1.$$

The endpoint slopes $-1, 1$ contribute $1 + (-1)^n$, and the interior slope jump two contributes $-2 \cos(na)$. None of these terms can be dropped.

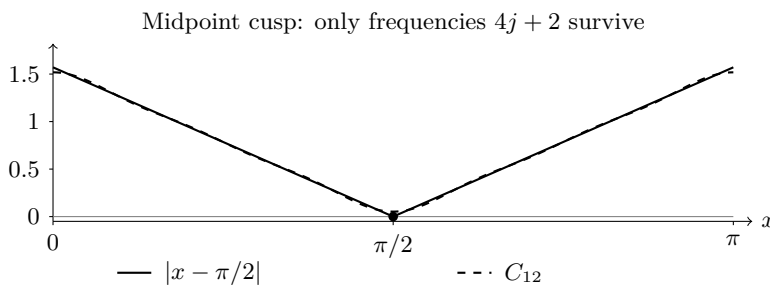
Step 2: Specialize to the midpoint cusp. At $a = \pi/2$ the mean is $\pi/4$. Odd modes vanish, and among the even modes exactly $n = 4j + 2$ survive, with coefficient $8/(\pi n^2)$. The extension has period π ; its maxima occur precisely at multiples of π , so every positive period must be a positive multiple of π . Its fundamental period is therefore $[\pi]$. Absolute coefficient summability gives uniform convergence, and the continuous piecewise linear even extension identifies the sum at all points, including cusps.

Step 3: Invert the first coefficient. The general formula gives $a_1 = -4 \cos a/\pi$. Strict monotonicity of cosine on $(0, \pi)$ yields

$$a = \arccos(-\pi a_1/4), \quad -4/\pi < a_1 < 4/\pi.$$

The open measurement range corresponds exactly to interior cusp locations.

Step 4: Describe what magnitudes lose. Reflection gives $f_{\pi-a}(x) = f_a(\pi-x)$ and $a_n^{\text{reflected}} = (-1)^n a_n$, with unchanged mean. Thus all magnitudes coincide for a and $\pi - a$. In fact $m = \pi/4 + (a - \pi/2)^2/\pi$, so the mean leaves precisely those two possibilities unless $a = \pi/2$, when they coincide. Signed odd coefficients resolve the nonsymmetric ambiguity.



Question 3

Consider the normalized Neumann problem

$$-u'' = x - \pi/2, \quad u'(0) = u'(\pi) = 0, \quad \int_0^\pi u(x)dx = 0.$$

Use the convention $u = a_0/2 + \sum_{n \geq 1} a_n \cos nx$.

Tasks

1. Derive the integral compatibility condition for a general forcing f with these derivative conditions. Explain why replacing the forcing by x makes the problem impossible.
2. Find the cosine coefficients of the stated forcing. Explain how the zero mode differs from every positive-frequency mode when solving for u .
3. Solve in closed form and as a cosine series. Verify the equation, both derivative conditions and the mean.
4. Prove uniqueness with the normalization, and state the full family without it. Explain why adding a constant cannot repair an incompatible forcing.

Question 3 – Solution

Strategy. Check the forcing mean before dividing by eigenvalues; the constant mode has eigenvalue zero.

Step 1: Check compatibility before solving. Integration gives $\int_0^\pi f = -u'(\pi) + u'(0) = 0$. The stated forcing satisfies this condition, whereas $\int_0^\pi x = \pi^2/2 \neq 0$. The replacement is impossible regardless of the additive level of u .

Step 2: Solve the mode equations. The forcing has mean zero and coefficients $F_n = 2((-1)^n - 1)/(\pi n^2)$. Since $-(\cos nx)'' = n^2 \cos nx$,

$$a_n = \frac{2((-1)^n - 1)}{\pi n^4} \quad (n \geq 1).$$

The zero-mode equation is $0 = 0$ and leaves the mean free. The normalization sets $a_0 = 0$; dividing this equation by zero would be invalid.

Step 3: Verify an independent polynomial solution. Direct integration gives

$$u = -\frac{x^3}{6} + \frac{\pi x^2}{4} - \frac{\pi^3}{24} = -\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^4}.$$

Its derivative is $x(\pi - x)/2$, zero at both ends, and $-u'' = x - \pi/2$. The mean of the first two terms is $\pi^3/24$, canceled by the constant. Twice integrating its coefficient integral by parts using $u' = 0$ at the endpoints gives $n^2 a_n = F_n$, verifying the coefficients. The continuous piecewise smooth even extension identifies the series sum.

Step 4: Separate the free constant from compatibility. A difference satisfies $w'' = 0$ and zero endpoint derivatives, hence is constant. Its zero mean forces it to vanish. Without normalization, all solutions are the displayed polynomial plus any real constant. Adding a constant changes neither the equation nor its derivative conditions, so it cannot fix the integrated forcing balance.

Question 4

For $f = e^x$ on $[0, 1]$, improve a cosine approximation by matching slopes:

$$\ell = x + \frac{e-1}{2}x^2, \quad r = e^x - \ell, \quad A_N = \ell + m_r + \sum_{n=1}^N c_n \cos(n\pi x),$$

where $m_r = \int_0^1 r$ and $c_n = 2 \int_0^1 r(x) \cos(n\pi x) dx$.

Tasks

1. Compute the raw mean and cosine coefficients of e^x . Find the endpoint slopes of every raw cosine partial sum.
2. Verify the slopes of ℓ , then compute m_r and all c_n . Explain the improvement from order n^{-2} to order n^{-4} .
3. Prove $\|e^x - A_N\|_\infty \leq 2(1+e)/(3\pi^4 N^3)$ and give an integer N certified to make the bound less than 10^{-4} .
4. Justify one differentiation of the residual series and verify the slopes of every A_N . Sketch the derivative errors of raw and corrected approximants for $N = 8$. State what slope matching does not guarantee.

Question 4 – Solution

Strategy. Remove the endpoint-slope contribution while retaining the residual's generally nonzero mean.

Step 1: Compute the raw data. For $k = n\pi$, direct integration gives

$$\text{mean} = e - 1, \quad a_n = \frac{2(e(-1)^n - 1)}{1 + k^2}.$$

Every raw finite cosine sum has zero endpoint derivatives, disagreeing with $f'(0) = 1$ and $f'(1) = e$.

Step 2: Cancel the slope contribution. The derivative $\ell' = 1 + (e - 1)x$ matches both target endpoint slopes. Two integrations by parts give its positive-frequency coefficient $2(e(-1)^n - 1)/k^2$; the constant second derivative has zero cosine integral. Subtracting yields

$$m_r = \frac{5}{6}(e - 1) - \frac{1}{2}, \quad c_n = -\frac{2(e(-1)^n - 1)}{k^2(1 + k^2)}.$$

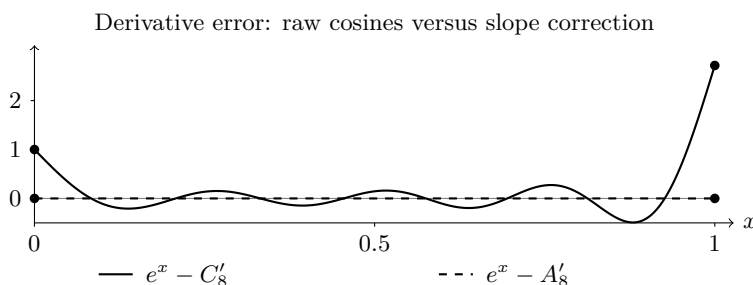
The residual has zero endpoint derivatives, so the leading slope term cancels and fourth-power decay remains. Its mean must still be included.

Step 3: Certify the function error. The bound $|c_n| \leq 2(1 + e)/(\pi^4 n^4)$ gives uniform absolute convergence. The continuous piecewise smooth even extension identifies the sum with r . Consequently

$$\|e^x - A_N\|_\infty \leq \frac{2(1 + e)}{\pi^4} \int_N^\infty t^{-4} dt = \frac{2(1 + e)}{3\pi^4 N^3}.$$

The choice $N = 7$ makes this bound less than 10^{-4} .

Step 4: Verify the derivative behavior. Derivative coefficients are bounded by a constant times n^{-3} , so that series converges uniformly. Together with convergence at one point, this justifies differentiation. Every residual partial sum has zero endpoint slopes; adding ℓ gives $A'_N(0) = 1, A'_N(1) = e$ exactly. Slope matching does not force exact endpoint values at finite N ; those are controlled by the function-error bound and become exact in the limit.



Question 5

On the shifted interval $[1, 4]$, expand $f(x) = (x - 1)(4 - x)$ using a cosine basis adapted to its left endpoint. Put $t = x - 1 \in [0, 3]$.

Tasks

1. State the correct shifted basis and coefficient normalization. Compute the mean and all positive-frequency coefficients.
2. Use reflection about $x = 5/2$ to explain the missing odd modes. Determine the series sum at the endpoints and midpoint.
3. Someone uses $1, \cos(\pi x/3), \cos(2\pi x/3), \dots$ with the usual orthogonal coefficient formula. Compute the inner product of the first two listed functions and diagnose the failure.
4. Derive a uniform truncation bound for the correct series. Explain why interval length alone does not specify the phase origin required by standard cosine orthogonality.

Question 5 – Solution

Strategy. Shift the coordinate in the basis as well as in the target, so the coefficient formulas retain orthogonality.

Step 1: Compute the translated expansion. The basis is $1, \cos(n\pi(x-1)/3)$. The mean is $m = (1/3) \int_1^4 f = 3/2$, and $a_n = (2/3) \int_1^4 f \cos(n\pi(x-1)/3)$. For $f = t(3-t)$, two integrations by parts yield

$$a_n = -\frac{18(1 + (-1)^n)}{\pi^2 n^2}.$$

The endpoint slopes are $3, -3$; the constant second derivative integrates to zero against each positive-frequency cosine.

Step 2: Use symmetry and continuity. Reflection sends t to $3-t$, multiplying mode n by $(-1)^n$ while leaving the target unchanged. Thus odd modes vanish. Coefficient summability and the continuous piecewise smooth even extension give uniform convergence to f . The sums are zero at $x = 1, 4$ and $\boxed{9/4}$ at $x = 5/2$.

Step 3: Diagnose the unshifted family. The constant and first unshifted cosine have inner product

$$\int_1^4 \cos(\pi x/3) dx = \frac{3}{\pi} \left(\sin \frac{4\pi}{3} - \sin \frac{\pi}{3} \right) = -\frac{3\sqrt{3}}{\pi} \neq 0.$$

They are not orthogonal. Dividing independent moments by the usual cosine norms therefore does not compute orthogonal projection coefficients here. The unshifted phase is not an innocuous substitution in that formula.

Step 4: Give a uniform certificate. For the correctly shifted partial sum,

$$\|f - C_N\|_\infty \leq \frac{36}{\pi^2} \sum_{n>N} n^{-2} \leq \frac{36}{\pi^2 N}.$$

Length sets the frequency spacing $\pi/3$; the left endpoint sets the phase. Both are used in translating the standard half-range expansion.

Question 6

Let $f = 1$ on $[0, \pi/2)$, $f(\pi/2) = 1/2$, and $f = 0$ on $(\pi/2, \pi]$. Write $C_N = a_0/2 + \sum_{n=1}^N a_n \cos nx$, including $C_0 = a_0/2$, and define

$$\sigma_N = \frac{1}{N+1} \sum_{j=0}^N C_j.$$

Let F be the even 2π -periodic extension. The ordinary Fourier convergence theorem may be used for this piecewise constant function.

Tasks

1. Compute the mean and coefficients. Show that C_1 becomes negative although the target is nonnegative.
2. Derive the coefficient weights in σ_N . Expand $K_N(t) = (N+1)^{-1} |\sum_{j=0}^N e^{ijt}|^2$ and prove nonnegativity and integral 2π over one period.
3. Verify $\sigma_N(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} F(x-t) K_N(t) dt$, then prove $0 \leq \sigma_N \leq 1$ everywhere.
4. Determine all limits of σ_N on $[0, \pi]$ by averaging convergent sequences of partial sums. Sketch C_9, σ_9 and the target, and explain the tradeoff made by averaging.

Question 6 – Solution

Strategy. Use an average with a nonnegative kernel and unit total mass to preserve the range of the data.

Step 1: Exhibit a negative partial sum. The mean is $1/2$ and $a_n = 2 \sin(n\pi/2)/(\pi n)$. Thus $C_1 = 1/2 + (2/\pi) \cos x$ has minimum $1/2 - 2/\pi < 0$ at π , despite $0 \leq f \leq 1$.

Step 2: Compute the weights and kernel. Mode n appears in $N + 1 - n$ of the partial sums, giving

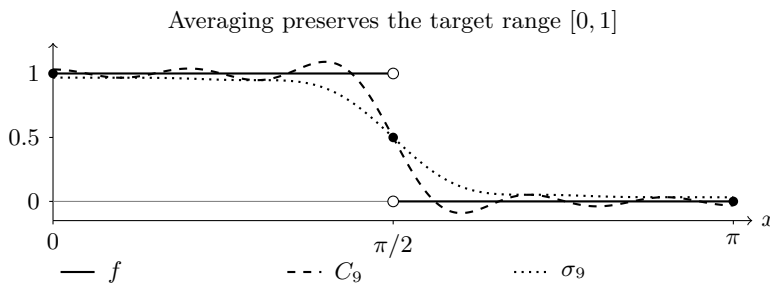
$$\sigma_N = \frac{1}{2} + \sum_{n=1}^N \left(1 - \frac{n}{N+1}\right) a_n \cos nx.$$

Counting pairs with index difference n in the squared modulus gives $K_N = 1 + 2 \sum_{n=1}^N (1 - n/(N+1)) \cos nt$. The squared-modulus definition proves $K_N \geq 0$, and integrating the finite expansion gives $\int_{-\pi}^{\pi} K_N = 2\pi$.

Step 3: Prove preservation of the range. Insert the finite kernel expansion into the convolution. The average integral of $F(x-t) \cos nt$ is $a_n \cos nx/2$; after shifting variables, its sine component vanishes by evenness of F . The constant term gives mean $1/2$. This proves the formula. Because $0 \leq F \leq 1$ and the kernel is nonnegative with mass 2π ,

$$0 \leq \sigma_N(x) \leq 1.$$

Step 4: Identify the limits and tradeoff. Ordinary partial sums tend to one left of the jump, zero right of it, and $1/2$ at the jump. The extension is continuous at zero and π , with values one and zero. Arithmetic means of a convergent sequence have the same limit, so σ_N has exactly these limits. Averaging damps the higher modes and prevents overshoot, but spreads the finite transition instead of fitting the edge as sharply.



Question 7

For $0 < \rho < 1$, define the positive profile

$$P_\rho(x) = \frac{1 - \rho^2}{1 - 2\rho \cos x + \rho^2}, \quad 0 \leq x \leq \pi.$$

Use $P_\rho = a_0/2 + \sum_{n \geq 1} a_n \cos nx$ and, when needed, Parseval: $\int_0^\pi f^2 = \pi(a_0/2)^2 + (\pi/2) \sum_{n \geq 1} a_n^2$.

Tasks

1. Derive the cosine expansion from a geometric series, justifying uniform convergence for fixed ρ . Identify the mean and every coefficient.
2. Compute the integral and squared norm exactly. Locate the maximum and minimum and give their values.
3. Determine the pointwise limit as $\rho \uparrow 1$. Explain why integrating it does not recover the constant mass.
4. For every continuous φ on $[0, \pi]$, prove $\int_0^\pi P_\rho \varphi \rightarrow \pi \varphi(0)$ by splitting at a small fixed $\delta > 0$. Sketch $P_{1/2}$ and $P_{4/5}$ and interpret this limit.

Question 7 – Solution

Strategy. Sum the geometric coefficients exactly, then locate the mass accumulating near the endpoint.

Step 1: Derive the uniformly convergent expansion. The geometric identity gives $1 + 2 \operatorname{Re}[\rho e^{ix}/(1 - \rho e^{ix})] = P_\rho(x)$. Since $\sum \rho^n < \infty$, its cosine series converges absolutely and uniformly:

$$P_\rho = 1 + 2 \sum_{n=1}^{\infty} \rho^n \cos nx, \quad a_0 = 2, \quad a_n = 2\rho^n.$$

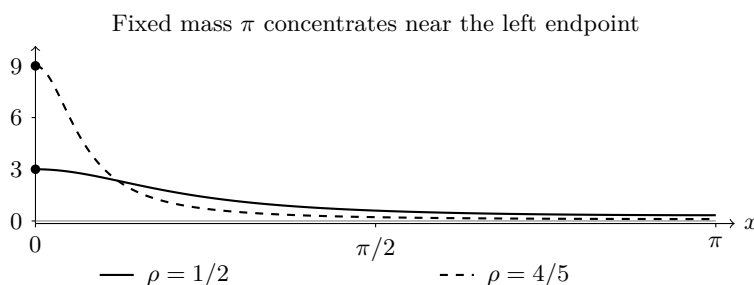
Step 2: Compute mass, energy and extrema. Uniform integration gives $\int_0^\pi P_\rho = \pi$. Parseval and the geometric sum give

$$\int_0^\pi P_\rho^2 = \pi \frac{1 + \rho^2}{1 - \rho^2}.$$

The denominator increases with x . The maximum at zero is $(1 + \rho)/(1 - \rho)$; the minimum at π is $(1 - \rho)/(1 + \rho)$.

Step 3: Explain the apparent loss of mass. For fixed $x > 0$, the numerator tends to zero and the denominator to $2(1 - \cos x) > 0$, so $P_\rho(x) \rightarrow 0$. At zero it tends to infinity. The almost-everywhere limit is zero, yet the integral stays π . Uniform convergence and interchange of the integral with this limit are not available; the exact squared norm also diverges.

Step 4: Prove endpoint concentration. Given $\eta > 0$, choose $\delta > 0$ so $|\varphi(x) - \varphi(0)| < \eta$ on $[0, \delta]$. That part of $\int P_\rho(\varphi - \varphi(0))$ is at most $\pi\eta$ in magnitude. On $[\delta, \pi]$, $P_\rho \leq (1 - \rho^2)/[2\rho(1 - \cos \delta)] \rightarrow 0$ uniformly, so the remaining part tends to zero. Letting $\eta \downarrow 0$ proves the claim. The limit is mass π concentrated at zero, not an ordinary integrable profile.



Question 8

Initially f is known to lie in $\text{span}\{1, \cos x, \cos 2x\}$ on $[0, \pi]$. Measurements give

$$\int_0^\pi f = \pi, \quad \int_0^\pi f \cos x = \pi/4, \quad \int_0^\pi f^2 = 5\pi/4.$$

There is also a requirement $f \geq 0$.

Tasks

1. Recover all functions in the stated span satisfying the measurements. Identify where a sign ambiguity enters.
2. Decide whether nonnegativity eliminates either candidate. Prove the answer on the entire interval using $t = \cos x \in [-1, 1]$.
3. A fourth measurement is $f(\pi) = 0$. Find the unique candidate in the stated span and verify all the data.
4. Remove the restriction to that span. Construct another nonnegative continuous function with all four measurements, and explain the limitation of the earlier uniqueness result.

Question 8 – Solution

Strategy. Linear moments preserve coefficient signs, while a squared-energy measurement can lose them.

Step 1: Solve the coefficient equations. Write $f = A + B \cos x + C \cos 2x$. Orthogonality gives $A = 1$, $B = 1/2$ and $\pi[A^2 + (B^2 + C^2)/2] = 5\pi/4$. Thus $C^2 = 1/4$, leaving

$$f_{\pm} = 1 + \frac{1}{2} \cos x \pm \frac{1}{2} \cos 2x.$$

The energy measurement determines the magnitude, but not the sign, of C .

Step 2: Test positivity without sampling. With $t = \cos x$, $f_+ = t^2 + t/2 + 1/2 = (t + 1/4)^2 + 7/16 > 0$. The other candidate is $f_- = 3/2 + t/2 - t^2$. This concave quadratic reaches its minimum on $[-1, 1]$ at an endpoint, with values zero and one. Thus both candidates are nonnegative; positivity does not resolve the ambiguity.

Step 3: Apply the endpoint measurement. At π , $f_+(\pi) = 1$ while $f_-(\pi) = 0$. Hence the unique candidate within the specified span is $f = f_-$. Its mean and first moment are the prescribed ones, and its squared norm is $\pi(1 + 1/8 + 1/8) = 5\pi/4$, verifying all four data.

Step 4: Remove the model restriction. The distinct function

$$g = 1 + \frac{1}{2} \cos x - \frac{1}{2} \cos 4x$$

has the same mean, first cosine moment and squared norm by orthogonality. Also $g(\pi) = 0$ and $g \geq 1 - 1/2 - 1/2 = 0$ everywhere. Thus these measurements and nonnegativity do not give uniqueness among all continuous functions. The finite-span hypothesis was essential.

Question 9

For $\alpha > 0$, consider

$$-u''_{\alpha} + \alpha^2 u_{\alpha} = 1 + \cos x, \quad u'_{\alpha}(0) = u'_{\alpha}(\pi) = 0.$$

The zeroth cosine mode has eigenvalue zero for $-d^2/dx^2$.

Tasks

1. Solve by cosine modes, treating the constant separately. Verify the equation and both boundary conditions.
2. Prove uniqueness for $\alpha > 0$ by an energy identity. Determine the mean and its behavior as $\alpha \downarrow 0$.
3. Subtract the mean to define $v_{\alpha} = u_{\alpha} - \pi^{-1} \int_0^{\pi} u_{\alpha}$. Find its limit and prove uniform convergence of the functions and their first two derivatives.
4. Explain why the unregularized problem with forcing $1 + \cos x$ has no solution, whereas the limit of v_{α} solves a different normalized problem. State that problem precisely.

Question 9 – Solution

Strategy. Track the constant mode separately: it becomes singular as the regularization vanishes.

Step 1: Solve the forced modes. The constant is divided by α^2 , while the first cosine is divided by $1 + \alpha^2$. Therefore

$$u_\alpha = \frac{1}{\alpha^2} + \frac{\cos x}{1 + \alpha^2}.$$

Its derivative is $-\sin x/(1 + \alpha^2)$, zero at both endpoints. Substitution recovers both the constant forcing one and the cosine term.

Step 2: Establish uniqueness and inspect the mean. For a homogeneous difference, integration by parts gives $\int_0^\pi [(w')^2 + \alpha^2 w^2] dx = 0$, so $w = 0$. The mean is $\boxed{1/\alpha^2}$ and diverges as $\alpha \downarrow 0$; the full family cannot have a finite uniform limit.

Step 3: Control the normalized family. Subtracting the mean gives $v_\alpha = \cos x/(1 + \alpha^2)$. The first two derivatives of cosine, as well as cosine itself, have supremum norm one. Thus for $j = 0, 1, 2$,

$$\|v_\alpha^{(j)} - (\cos x)^{(j)}\|_\infty = \frac{\alpha^2}{1 + \alpha^2} \rightarrow 0.$$

This proves function and derivative convergence on the whole interval.

Step 4: Identify the correct limiting problem. The original forcing has integral $\pi \neq 0$, violating the Neumann condition $\int f = 0$ when $\alpha = 0$. Hence the unregularized original problem is impossible. Subtracting the mean also removes the constant forcing: $-v_\alpha'' + \alpha^2 v_\alpha = \cos x$. The limit solves

$$-v'' = \cos x, \quad v'(0) = v'(\pi) = 0, \quad \int_0^\pi v = 0, \quad v = \cos x.$$

It is the unique zero-mean solution of this compatible problem.

Question 10

Samples at $x_0 = \pi/6$, $x_1 = \pi/2$, $x_2 = 5\pi/6$ give the values $(2, 0, 1)$. Initially assume $f = A + B \cos x + C \cos 2x$ and define $\langle u, v \rangle_d = \frac{1}{3} \sum_{j=0}^2 u(x_j)v(x_j)$.

Tasks

1. List the sample vectors of $1, \cos x, \cos 2x$. Verify discrete orthogonality and compute their squared norms.
2. Recover A, B, C from the discrete moments. Verify every sample and prove uniqueness within the assumed span.
3. Drop the span restriction and exhibit infinitely many smooth functions with the same samples by adding an invisible cosine mode.
4. Do nonnegative samples imply that the reconstructed function is nonnegative throughout $[0, \pi]$? Find its exact minimum using $t = \cos x$ and explain what the samples do not certify.

Question 10 – Solution

Strategy. Distinguish reconstruction in a finite discrete basis from claims about unmeasured modes or intermediate values.

Step 1: Compute the discrete Gram data. The vectors are

$$(1, 1, 1), \quad (\sqrt{3}/2, 0, -\sqrt{3}/2), \quad (1/2, -1, 1/2).$$

Their distinct dot products are zero. Dividing squared lengths by three gives squared norms $\boxed{1, 1/2, 1/2}$.

Step 2: Recover and verify the coefficients. The sample mean gives $A = 1$. Dividing each other discrete moment by $1/2$ gives

$$\boxed{B = 1/\sqrt{3}, \quad C = 1, \quad f = 1 + \frac{1}{\sqrt{3}} \cos x + \cos 2x.}$$

The values at the three nodes are $1 + 1/2 + 1/2 = 2$, $1 - 1 = 0$ and $1 - 1/2 + 1/2 = 1$. Three nonzero orthogonal vectors are independent, so the coefficient triple is unique within the specified span.

Step 3: Exhibit the invisible smooth freedom. The function $\cos 3x$ vanishes at all nodes, whose tripled arguments are $\pi/2, 3\pi/2, 5\pi/2$. Hence $\boxed{f_D = f + D \cos 3x, \quad D \in \mathbb{R}}$ has the same samples. These are distinct smooth functions; reconstruction is not unique among all smooth functions.

Step 4: Check between-node positivity exactly. Since $\cos 2x = 2t^2 - 1$, the reconstructed function is $2t^2 + t/\sqrt{3}$. Completing the square gives

$$\boxed{f = 2 \left(t + \frac{1}{4\sqrt{3}} \right)^2 - \frac{1}{24}.}$$

Its minimum is $-1/24$, attained at the interior point $x = \arccos(-1/(4\sqrt{3}))$. Nonnegative samples therefore fail to certify nonnegativity between the nodes, even for this unique finite-span reconstruction.