

Question 1

Let $f(x) = x$ on $[0, \pi]$. Define its sine coefficients and partial sums by

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx, \quad S_N = \sum_{n=1}^N b_n \sin(nx).$$

For a piecewise smooth odd periodic extension, the Fourier series converges to the average of its one-sided limits, including at periodic joins.

Tasks

1. Derive b_n by integration by parts. Describe the odd 2π -periodic extension, including its jump at odd multiples of π .
2. Determine the series sum at every $x \in [0, \pi]$. Distinguish the assigned endpoint value $f(\pi)$ from the series limit.
3. Decide whether S_N converges uniformly to f on $[0, \pi]$, or to x on the open interval $(0, \pi)$. Give a quantitative obstruction for both claims.
4. Use the value at $x = \pi/2$ to evaluate $1 - 1/3 + 1/5 - \dots$. Sketch S_{12} with f and explain why the endpoint mismatch does not invalidate the interior representation.

Question 1 – Solution

Strategy. Determine the odd extension before applying the convergence theorem at its endpoints.

Step 1: Compute the coefficients and extension. Integration by parts yields

$$\int_0^{\pi} x \sin(nx) dx = [-x \cos(nx)/n]_0^{\pi} + [\sin(nx)/n^2]_0^{\pi} = -\pi(-1)^n/n.$$

Hence $b_n = 2(-1)^{n+1}/n$. The odd extension equals x on $(-\pi, \pi)$ and repeats every 2π . At an odd multiple of π its left and right limits are π and $-\pi$. The join value may be chosen zero for an odd periodic representative.

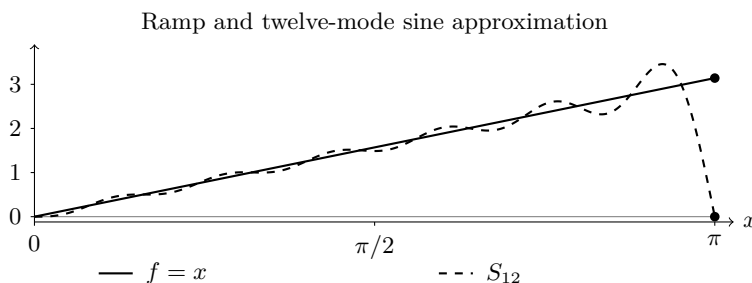
Step 2: Apply the convergence theorem. The series equals x for $0 < x < \pi$. At zero it equals zero by continuity of the extension, and at π it equals the jump average zero:

$$\sum_{n \geq 1} \frac{2(-1)^{n+1}}{n} \sin(nx) = x \quad (0 \leq x < \pi), \quad \text{sum at } \pi = 0.$$

This does not reproduce the separately assigned value $f(\pi) = \pi$.

Step 3: Rule out uniform convergence near the join. On the closed interval, $|f(\pi) - S_N(\pi)| = \pi$ for every N . On the open interval, continuity of each finite sum gives $\lim_{x \uparrow \pi} |x - S_N(x)| = \pi$, so the supremum error is at least π . Neither asserted uniform convergence holds, despite pointwise convergence inside.

Step 4: Evaluate an interior numerical series. At $x = \pi/2$, only odd modes remain and their sine values alternate: $\pi/2 = 2(1 - 1/3 + 1/5 - \dots)$. Therefore the convergent series equals $\pi/4$. The graph displays a finite approximation; the convergence theorem, not endpoint agreement of the original data, justifies the interior sum.



Question 2

Let $f(x) = 1$ for $0 < x < \pi$, with assigned endpoint values zero. The odd 2π -periodic extension is a square wave. Define its first M nonzero modes by

$$P_M(x) = \frac{4}{\pi} \sum_{j=0}^{M-1} \frac{\sin((2j+1)x)}{2j+1}.$$

You may use convergence to the average of the one-sided limits for piecewise smooth periodic functions.

Tasks

1. Derive the sine coefficients and verify the definition of P_M . State the limiting values inside the interval and at its endpoints.
2. Sum the finite cosine progression to prove $P'_M(x) = (2/\pi) \sin(2Mx)/\sin x$ for $0 < x < \pi$. Locate the first local maximum to the right of zero.
3. Express the height of that maximum as an integral. Rescale the integral and prove that the heights tend to $(2/\pi) \int_0^\pi (\sin t)/t \, dt$.
4. Prove that this limiting height is greater than one, using $\sin t/t \geq 1 - t^2/6$ for $0 \leq t \leq \pi/2$ and $1/t \geq 1/\pi$ for $\pi/2 \leq t \leq \pi$. Sketch P_{10} and explain why pointwise convergence does not remove the near-jump overshoot.

Question 2 – Solution

Strategy. Analyze a moving maximum of a finite sum rather than substitute a moving point into a pointwise convergence claim.

Step 1: Find the nonzero modes. Direct integration gives $b_n = 2(1 - (-1)^n)/(\pi n)$: zero for even n and $4/(\pi n)$ for odd n . Thus P_M is the stated truncation. The odd extension has jump averages zero at both endpoints; its sum is one on $(0, \pi)$.

Step 2: Locate the first maximum. A finite geometric sum, or telescoping $2 \sin x \cos((2j+1)x)$, gives $\sum_{j=0}^{M-1} \cos((2j+1)x) = \sin(2Mx)/(2 \sin x)$. Differentiating the finite sum is legitimate, so

$$P'_M(x) = \frac{2}{\pi} \frac{\sin(2Mx)}{\sin x}, \quad x_M = \frac{\pi}{2M}.$$

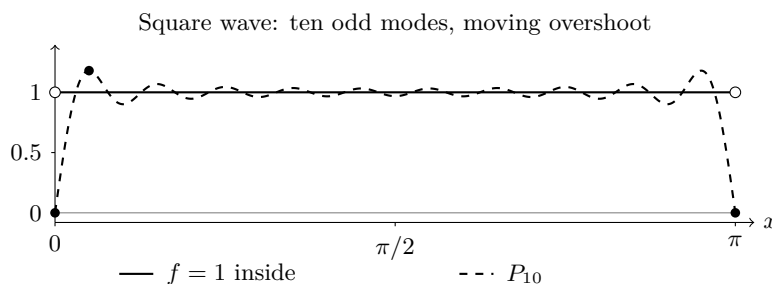
The derivative is positive before x_M and negative just after it, for $M \geq 1$.

Step 3: Pass to the moving-peak limit correctly. Since $P_M(0) = 0$, substitution $t = 2Mx$ gives

$$P_M(x_M) = \frac{2}{\pi} \int_0^\pi \frac{\sin t}{2M \sin(t/(2M))} dt \longrightarrow \boxed{\frac{2}{\pi} \int_0^\pi \frac{\sin t}{t} dt}.$$

Write the integrand as $(\sin t/t)(z/\sin z)$, where $z = t/(2M)$. Since $z/\sin z$ extends continuously with value one at zero, this factor tends uniformly to one for $0 \leq t \leq \pi$. Also $|\sin t/t| \leq 1$. These facts justify the integral limit; its value is approximately 1.179.

Step 4: Prove a persistent excess. The supplied inequalities imply $\int_0^\pi \sin t/t dt \geq \pi/2 - \pi^3/144 + 1/\pi > \pi/2$, since $\pi^4 < 144$. The limiting peak is strictly above one. The peak location moves toward the jump, so convergence at each fixed interior point is consistent with this persistent overshoot. The square wave jumps by two across zero.



Question 3

A tent on $[0, 1]$ has height $H > 0$ and peak at $a \in (0, 1)$:

$$f(x) = \begin{cases} Hx/a, & 0 \leq x \leq a, \\ H(1-x)/(1-a), & a \leq x \leq 1. \end{cases}$$

Its sine coefficients are $b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx$.

Tasks

1. Integrate separately on the two sides of the corner to derive b_n . Explain why the corner contributes even though both endpoint values vanish.
2. For $a = 1/3, H = 1$, identify all missing modes and prove that the sine series converges uniformly to the tent on the entire closed interval.
3. Suppose the only known coefficient data are $b_1 > 0$ and $b_2/b_1 = 1/4$, and the signal is known to be a tent of this form. Recover a, H uniquely in terms of b_1 .
4. Compare a tent with its reflection $f(1-x)$. State how each coefficient changes and explain why the absolute values of all coefficients cannot distinguish the two peak locations. Sketch the $a = 1/3, H = 1$ tent and its first twelve-mode sum.

Question 3 – Solution

Strategy. Retain the change in slope at the interior corner when integrating by parts.

Step 1: Compute the corner contribution. Put $k = n\pi$. One integration by parts removes the zero endpoint terms; integrating each constant slope then gives

$$\int_0^1 f \sin(kx) dx = \frac{1}{k^2} \left(\frac{H}{a} + \frac{H}{1-a} \right) \sin(ka).$$

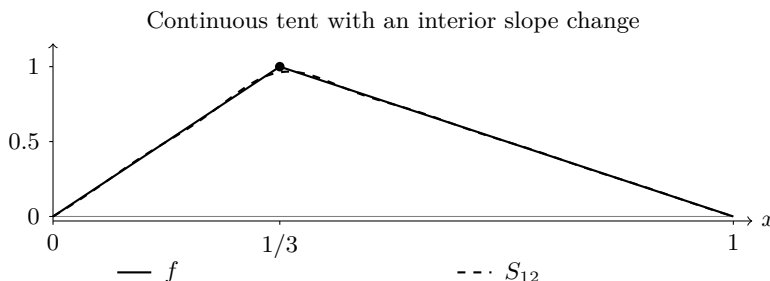
Thus $b_n = 2H \sin(n\pi a) / [\pi^2 a(1-a)n^2]$. The nonzero slope change, not an endpoint displacement, supplies these terms.

Step 2: Identify missing modes and convergence. For $a = 1/3, H = 1$, $b_n = 9 \sin(n\pi/3) / (\pi^2 n^2)$. Exactly the multiples of three vanish. The bound $|b_n| \leq 9/(\pi^2 n^2)$ gives uniform absolute convergence by the Weierstrass test. The continuous, piecewise linear odd extension satisfies the Fourier convergence theorem, so the sum is the tent at every point, including both endpoints and the corner.

Step 3: Recover the unknown tent. Because $\sin(\pi a) > 0$, $b_2/b_1 = \sin(2\pi a)/(4 \sin \pi a) = \frac{1}{2} \cos \pi a$. The measured ratio gives $\cos \pi a = 1/2$, hence $a = 1/3$ uniquely on $(0, 1)$. Substituting into b_1 gives

$$H = \frac{2\pi^2 b_1}{9\sqrt{3}}.$$

Step 4: Interpret reflection and lost sign information. Under $x \mapsto 1-x$, $\sin(n\pi(1-x)) = (-1)^{n+1} \sin(n\pi x)$. A change of variable gives $b_n^{\text{reflected}} = (-1)^{n+1} b_n$. Odd modes are unchanged and even modes reverse sign. All magnitudes therefore coincide, so they cannot distinguish a from $1-a$ when those locations differ.



Question 4

On $[0, \pi]$, let f equal one on $(\pi/3, 2\pi/3)$ and zero elsewhere, including the two jump points. Let S_N denote its first N sine modes.

Tasks

1. Derive every sine coefficient from the defining integral. Use symmetry about $\pi/2$ to explain why all even modes vanish.
2. Determine whether any odd modes vanish. State the sine-series limit at every point of $[0, \pi]$, paying attention to the assigned values at the jumps.
3. Change the assigned values at the two jumps to arbitrary real numbers. Explain what happens to the coefficients, the series and its pointwise limits.
4. Prove that no continuous function can approximate this assigned target with uniform error less than $1/2$. Explain the implication for sine partial sums and sketch S_{15} alongside the pulse.

Question 4 – Solution

Strategy. Use symmetry for coefficient selection, but use one-sided limits to determine convergence at jumps.

Step 1: Compute and simplify the coefficients. Integration over the nonzero portion gives

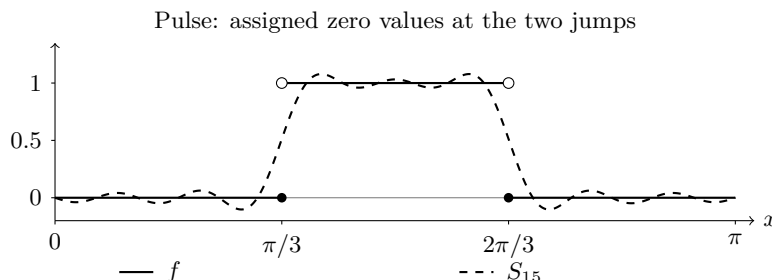
$$b_n = \frac{2}{\pi n} [\cos(n\pi/3) - \cos(2n\pi/3)].$$

The reflection $f(\pi - x) = f(x)$ implies $b_n = (-1)^{n+1}b_n$, so even modes are zero. For odd n , $\cos(2n\pi/3) = -\cos(n\pi/3)$, giving $b_n = 4\cos(n\pi/3)/(\pi n)$.

Step 2: Account for every mode and every limit. For odd integers n , $\cos(n\pi/3)$ is $1/2$ or -1 , never zero. Thus no odd mode vanishes. The odd periodic extension is piecewise constant; the Fourier convergence theorem gives one inside the pulse, zero outside, and $1/2$ at $\pi/3$ and $2\pi/3$. Both endpoints have limit zero. The jump-point sums differ from the assigned target values zero.

Step 3: Change only the assigned values. Changing finitely many points does not change any coefficient integral. The partial sums and their limits are therefore unchanged. The series records the jump averages, not arbitrary isolated assignments at the jumps.

Step 4: Establish the uniform-error obstruction. Let g be continuous and suppose $\sup |f - g| \leq E$. Taking one-sided limits at the first jump gives $|g(\pi/3)| \leq E$ and $|1 - g(\pi/3)| \leq E$. The triangle inequality yields $1 \leq 2E$, so $E \geq 1/2$. Every finite sine sum is continuous and obeys this lower bound. Thus pointwise convergence away from the jumps cannot become uniform convergence to this target on the whole interval. The graph shows the finite-sum transition and oscillations.



Question 5

Let $f(x) = x(\pi - x)$ on $[0, \pi]$ and $S_N = \sum_{n=1}^N b_n \sin(nx)$. You may use Parseval's identity $\int_0^\pi f^2 dx = (\pi/2) \sum_{n \geq 1} b_n^2$, and the standard convergence theorem for piecewise smooth odd periodic extensions.

Tasks

1. Derive all coefficients by integration by parts, and identify the missing modes.
2. Prove a uniform tail bound $\|f - S_N\|_\infty \leq 4/(\pi N^2)$ for $N \geq 1$. State why uniform convergence includes both endpoints here.
3. Find the smallest integer N certified by that bound to give error below 10^{-3} . Distinguish a sufficient certificate from the actual smallest successful truncation.
4. Evaluate $\int_0^\pi f^2 dx$ exactly and use Parseval to derive the sum of the reciprocal sixth powers of the positive odd integers.

Question 5 – Solution

Strategy. Use the zero endpoint values and the constant second derivative to obtain rapidly decaying coefficients.

Step 1: Integrate twice and retain the parity. Because $f(0) = f(\pi) = 0$ and $f'' = -2$, integration by parts gives $\int_0^\pi f \sin(nx) dx = 2(1 - (-1)^n)/n^3$. Therefore

$$b_n = \frac{4(1 - (-1)^n)}{\pi n^3} = \begin{cases} 8/(\pi n^3), & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

Step 2: Bound the entire tail uniformly. The coefficient series is absolutely summable, so it converges uniformly. The continuous odd periodic extension and the convergence theorem identify its sum with f everywhere on $[0, \pi]$. For every x ,

$$|f(x) - S_N(x)| \leq \frac{8}{\pi} \sum_{n>N} \frac{1}{n^3} \leq \frac{8}{\pi} \int_N^\infty t^{-3} dt = \boxed{\frac{4}{\pi N^2}}.$$

Both endpoint values of the target and all partial sums are zero; there is no endpoint mismatch or jump obstructing uniform convergence.

Step 3: Choose a certified truncation. The stated bound is below 10^{-3} when $N > \sqrt{4000/\pi}$. Since this threshold lies between 35 and 36, $\boxed{N = 36}$ is the smallest integer certified by this bound. The estimate discarded the missing even modes and possible cancellation; it does not prove that every smaller N fails.

Step 4: Extract an exact numerical sum. Expanding the polynomial square yields $\int_0^\pi x^2(\pi - x)^2 dx = \pi^5/30$. Parseval gives

$$\frac{\pi^5}{30} = \frac{32}{\pi} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^6}, \quad \boxed{\sum_{j=0}^{\infty} \frac{1}{(2j+1)^6} = \frac{\pi^6}{960}}.$$

This conclusion uses the complete series and Parseval, not a finite numerical fit.

Question 6

For $0 < x < \pi$, the sine series of $f(x) = x$ is

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

Its partial sums converge to f in $L^2(0, \pi)$ as well as pointwise inside. Here $\|g\|_2 = (\int_0^\pi |g|^2 dx)^{1/2}$.

Tasks

1. Form the series obtained by formally differentiating once. Test the necessary term-to-zero condition at $x = \pi/2$ and decide whether the formal differentiation is valid there.
2. Integrate each finite partial sum from zero to x . Use Cauchy–Schwarz and the given L^2 convergence to justify passing to the limit uniformly for $0 \leq x \leq \pi$.
3. Derive the resulting cosine identity for $x^2/2$, including both endpoints. Give a uniform tail bound for this integrated series.
4. Evaluate the identity at $x = \pi$ to sum the reciprocal squares of the positive odd integers. Explain why integration is justified here while the attempted differentiation fails.

Question 6 – Solution

Strategy. Test the operation itself: differentiating weakens coefficient decay, while integration improves it.

Step 1: Reject the formal derivative at a specific point. Formal differentiation produces

$$2 \sum_{n \geq 1} (-1)^{n+1} \cos(nx).$$

At $x = \pi/2$, the even-indexed terms have magnitude two and do not tend to zero. Thus this series diverges there and cannot be a valid ordinary termwise derivative representation of $f' = 1$.

Step 2: Pass from finite integrals to the limit. For each finite sum S_N , direct integration gives

$$\int_0^x S_N(t) dt = 2 \sum_{n=1}^N \frac{(-1)^{n+1}}{n^2} (1 - \cos nx).$$

Cauchy-Schwarz yields $|\int_0^x (S_N - f) dt| \leq \sqrt{x} \|S_N - f\|_2 \leq \sqrt{\pi} \|S_N - f\|_2 \rightarrow 0$. This bound is independent of x , proving uniform convergence of the integrals.

Step 3: State the integrated identity and its error. Consequently

$$\boxed{\frac{x^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} (1 - \cos nx), \quad 0 \leq x \leq \pi.}$$

The tail is bounded uniformly by $4 \sum_{n > N} n^{-2} \leq 4/N$. This also proves absolute uniform convergence of the integrated series. At zero both sides vanish; the right endpoint is included by the uniform integral argument, despite the original sine mismatch.

Step 4: Evaluate at the right endpoint. At $x = \pi$ the even terms vanish and each odd term contributes $4/n^2$. Hence $\pi^2/2 = 4 \sum_{j \geq 0} (2j+1)^{-2}$, giving $\boxed{\sum_{j \geq 0} (2j+1)^{-2} = \pi^2/8}$. The integration has a proved limiting argument and summable coefficients; the differentiated series fails even a necessary condition for convergence.

Question 7

On $[0, 1]$, approximate $f(x) = e^x$ using sine functions $\sin(n\pi x)$. To preserve the endpoint data, also consider

$$\ell(x) = 1 + (e - 1)x, \quad r(x) = e^x - \ell(x), \quad A_N(x) = \ell(x) + \sum_{n=1}^N c_n \sin(n\pi x),$$

where c_n are the sine coefficients of r .

Tasks

1. Compute the sine coefficients b_n of e^x exactly. State the ordinary sine-series limits at zero and one.
2. Compute the sine coefficients of ℓ and subtract them to derive c_n . Explain the improvement in decay.
3. Prove that A_N converges uniformly to e^x on $[0, 1]$, with error at most $(1 + e)/(\pi^3 N^2)$. Find an integer N certified to give error below 10^{-3} .
4. Compare endpoint errors of the raw sine partial sum and A_N . Sketch their errors for $N = 12$, and explain why adding the boundary interpolant changes the approximation problem rather than contradicting the endpoint limitation of pure sine sums.

Question 7 – Solution

Strategy. Remove the nonzero boundary values before asking a sine series for uniform approximation.

Step 1: Compute the raw coefficients. For $k = n\pi$, an antiderivative is $e^x(\sin kx - k \cos kx)/(1 + k^2)$. Evaluation gives

$$b_n = \frac{2k(1 - (-1)^n e)}{1 + k^2}.$$

The odd periodic extension has opposite one-sided values at each endpoint, so its jump averages are zero. Every raw sine partial sum is also zero there.

Step 2: Cancel the leading boundary contribution. Integration of the linear interpolant gives $b_n^\ell = 2(1 - (-1)^n e)/k$. Therefore

$$c_n = b_n - b_n^\ell = -\frac{2(1 - (-1)^n e)}{k(1 + k^2)}.$$

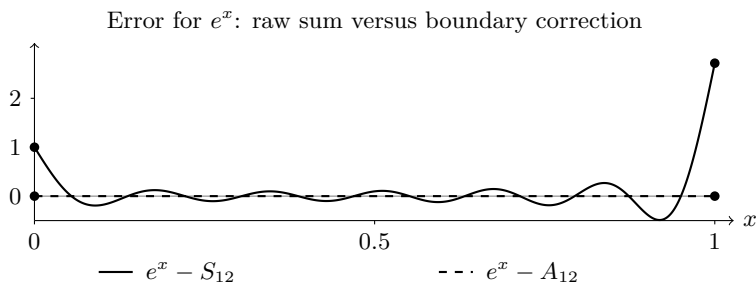
The raw coefficients are of order $1/n$, while c_n are of order $1/n^3$. The residual has zero values at both endpoints; the leading boundary term cancels.

Step 3: Establish a uniform certificate. Since $|c_n| \leq 2(1 + e)/(\pi^3 n^3)$, the residual series converges absolutely and uniformly. Its continuous piecewise smooth odd extension identifies the sum with r throughout the closed interval. Thus

$$\|e^x - A_N\|_\infty \leq \frac{1 + e}{\pi^3 N^2}.$$

For $N = 12$, the bound is below 10^{-3} , so twelve modes suffice.

Step 4: Interpret the endpoint comparison. For the raw sum S_N , $f - S_N$ equals one at zero and e at one. For A_N , both endpoint errors are exactly zero, and the entire error has the small bound above. The graph shows these two errors on the same scale. The corrected approximant includes the non-sine function ℓ ; it is not a pure sine sum and therefore is not constrained to vanish at the endpoints.



Question 8

Solve the boundary-value problem

$$-u'' = x, \quad 0 \leq x \leq \pi, \quad u(0) = u(\pi) = 0$$

using the sine coefficients $b_n = 2(-1)^{n+1}/n$ of the forcing x . You may use L^2 convergence of its sine series and the standard theorem allowing differentiation when the series of derivatives converges uniformly and the original series converges at one point.

Tasks

1. Derive the proposed sine coefficients of u from the action of $-d^2/dx^2$ on a sine mode. Independently solve the differential equation in closed form.
2. Verify that the closed-form solution has the proposed sine coefficients using integration by parts and the boundary values.
3. Justify one termwise differentiation of the solution series. Explain why a second differentiation cannot produce the forcing correctly at the right endpoint as an everywhere pointwise identity.
4. State precisely how the second derivatives of the finite solution sums converge in L^2 . Prove a uniform error bound for the solution itself and explain the smoothing effect of solving the boundary-value problem.

Question 8 – Solution

Strategy. Divide each forcing coefficient by its eigenvalue, then use an independent closed-form solution to verify the spectral construction.

Step 1: Obtain two descriptions of the solution. Since $-(\sin nx)'' = n^2 \sin nx$, the proposed coefficients are $a_n = b_n/n^2 = 2(-1)^{n+1}/n^3$. Direct integration of $u'' = -x$ and the two endpoint conditions gives

$$u(x) = \frac{\pi^2 x - x^3}{6}.$$

Its second derivative is $-x$ and both endpoint values vanish.

Step 2: Verify the actual coefficients. Let $\hat{u}_n = (2/\pi) \int_0^\pi u \sin nx \, dx$. Twice integrating by parts, using zero endpoint values of u and of the sine, gives $(2/\pi) \int_0^\pi (-u'') \sin nx \, dx = n^2 \hat{u}_n$. The left side is b_n , proving $\hat{u}_n = a_n$. The continuous odd extension and the convergence theorem identify the series with this polynomial, including at the endpoints.

Step 3: Justify the first derivative and limit the second. The derivative coefficients have magnitude $2/n^2$, so their cosine series converges uniformly. The original series converges at zero, where it is zero; the stated differentiation theorem therefore applies once. The formal second derivative is $-\sum b_n \sin nx$. At $x = \pi$ every term is zero, whereas $u''(\pi) = -\pi$. Thus the second derivative series cannot equal u'' pointwise everywhere on the closed interval.

Step 4: State the valid convergence and smoothing. For $U_N = \sum_{n=1}^N a_n \sin nx$, $U_N'' = -S_N$ exactly. The supplied forcing convergence gives $\|U_N'' + x\|_2 \rightarrow 0$; isolated endpoint disagreement does not change this norm. Moreover

$$\|u - U_N\|_\infty \leq 2 \sum_{n>N} n^{-3} \leq 1/N^2.$$

Dividing by n^2 suppresses high-frequency forcing coefficients and gives a uniformly convergent displacement with a uniformly convergent first derivative.

Question 9

For $0 < \varepsilon < \pi/2$, define a unit-area narrow pulse centered at $\pi/2$:

$$f_\varepsilon(x) = \begin{cases} (2\varepsilon)^{-1}, & |x - \pi/2| < \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \quad 0 \leq x \leq \pi.$$

Use $b_n(\varepsilon) = (2/\pi) \int_0^\pi f_\varepsilon \sin nx \, dx$. You may use Parseval and Bessel's inequality for square-integrable functions.

Tasks

1. Verify the area and derive all sine coefficients. Interpret the value of $\sin z/z$ at $z = 0$ by its limit.
2. Find the limit of each fixed coefficient as $\varepsilon \downarrow 0$. Prove that the limiting coefficient sequence cannot be the sine coefficients of any $L^2(0, \pi)$ function.
3. Compute $\|f_\varepsilon\|_2^2$ and use Parseval to find $\sum_{n \geq 1} b_n(\varepsilon)^2$. Explain why a point-mass limit is not an ordinary square-integrable function.
4. Use $|\sin z/z - 1| \leq z^2/6$ to bound the largest coefficient error among $1 \leq n \leq N$. Derive this inequality from $\sin z/z = \int_0^1 \cos(zt) \, dt$ and explain why the resulting estimate is not uniform over all mode numbers.

Question 9 – Solution

Strategy. Separate a limit at each fixed frequency from a limit of the complete coefficient sequence in squared norm.

Step 1: Integrate the narrow pulse. The height times the width is one. Writing $a = \pi/2$, integration gives

$$b_n(\varepsilon) = \frac{\cos(n(a - \varepsilon)) - \cos(n(a + \varepsilon))}{\pi\varepsilon n} = \boxed{\frac{2}{\pi} \sin(n\pi/2) \frac{\sin(n\varepsilon)}{n\varepsilon}}.$$

The factor $\sin z/z$ has limiting value one at zero. Even modes vanish.

Step 2: Examine the fixed-mode limit. For each fixed n , $b_n(\varepsilon) \rightarrow \beta_n = (2/\pi) \sin(n\pi/2)$. Every odd-indexed β_n has magnitude $2/\pi$, so $\sum \beta_n^2$ diverges. Bessel's inequality would bound this sum for any L^2 function. Hence no square-integrable function has all these limiting coefficients.

Step 3: Track the diverging energy. Direct integration gives $\boxed{\|f_\varepsilon\|_2^2 = 1/(2\varepsilon)}$. Parseval therefore implies

$$\boxed{\sum_{n \geq 1} b_n(\varepsilon)^2 = \frac{1}{\pi\varepsilon}}.$$

The energies grow without bound while area stays one. Indeed, integration against a continuous test function tends to its value at $\pi/2$, by continuity and averaging over the shrinking interval. This is the point-mass limit, not an L^2 function or an ordinary pointwise sine representation of one.

Step 4: Quantify finite-band convergence. From $|1 - \cos s| \leq s^2/2$ and the supplied integral representation, $|\sin z/z - 1| \leq \int_0^1 z^2 t^2/2 dt = z^2/6$. Thus

$$\boxed{\max_{1 \leq n \leq N} |b_n(\varepsilon) - \beta_n| \leq \frac{N^2 \varepsilon^2}{3\pi}}.$$

For each fixed N this tends to zero. It grows with N , so it does not justify uniform convergence across all frequencies or interchange with the infinite energy sum.

Question 10

A signal on $[0, \pi]$ is

$$f(x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + 2 \sin 10x.$$

An approximation may retain at most three sine modes, with arbitrary real coefficients. Compare this freedom with the usual truncation to the first three frequencies. Use the ordinary squared integral error on $[0, \pi]$.

Tasks

1. Verify all sine coefficients by orthogonality and compute $\|f\|_2^2$.
2. Find the first-three-frequency partial sum and its exact squared error.
3. Find the best approximation using any three sine modes. Prove both the optimal choice of frequencies and their coefficients, and compare its error with the ordinary low-frequency truncation.
4. Suppose only the first three coefficients and the total squared norm are known, without the displayed formula for f . Determine the unseen energy and explain whether these data locate it at frequency ten. Give two distinct compatible signals.

Question 10 – Solution

Strategy. Orthogonality makes the cost of omitting a mode explicit, so optimal selection depends on coefficient size rather than frequency alone.

Step 1: Identify the coefficient energy. The sine functions have squared norm $\pi/2$ and are mutually orthogonal. Thus $b_1 = 1, b_2 = 1/2, b_3 = 1/3, b_{10} = 2$, with all other coefficients zero. Consequently

$$\|f\|_2^2 = \frac{\pi}{2} \left(1 + \frac{1}{4} + \frac{1}{9} + 4 \right) = \frac{193\pi}{72}.$$

Step 2: Evaluate the conventional truncation. The first-three-frequency sum is $S_3 = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$. Its residual is $2 \sin 10x$, so $\|f - S_3\|_2^2 = 2\pi$. Frequency order alone discards the largest-amplitude component here.

Step 3: Optimize the three selected modes. For any selected frequencies, orthogonality expresses the error as $\pi/2$ times the sum of squared coefficient differences on selected modes plus the squared coefficients of omitted modes. Each retained coefficient must therefore equal the true coefficient. To minimize the omitted energy, retain the three largest magnitudes, uniquely 2, 1, 1/2, at frequencies 10, 1, 2. Hence

$$g_* = 2 \sin 10x + \sin x + \frac{1}{2} \sin 2x, \quad \|f - g_*\|_2^2 = \frac{\pi}{18}.$$

The squared error of S_3 is 36 times larger. Adding a zero-coefficient mode cannot improve the selection and using fewer than three omits additional energy.

Step 4: Interpret incomplete spectral information. The measured first-three energy is $(\pi/2)(1 + 1/4 + 1/9) = 49\pi/72$. Subtracting it from the known total leaves $\boxed{2\pi}$ of unseen energy. Its location is not determined: $S_3 + 2 \sin 10x$ and $S_3 + 2 \sin 100x$ have the same given coefficients and norm. Total energy constrains the squared size of the tail, not its frequency distribution.