

Question 1

A positive number T is a period of f if $f(x + T) = f(x)$ for every real x . A fundamental period is the smallest positive period, when one exists. Consider

$$f(x) = \sin 6x + \cos 10x.$$

You may use linear independence of sines and cosines at distinct positive integer frequencies; justify that fact here by integrating products on $[0, 2\pi]$.

Tasks

1. Find a common period of the two summands. Explain why a common period alone does not prove that the sum has that fundamental period.
2. Expand $f(x + T) - f(x)$ and use independence to characterize every positive period of f . Determine the fundamental period exactly.
3. Find the fundamental period of $g(x) = 3f(2x - 1) + 7$. Explain the effects of translation, nonzero amplitude scaling and a constant offset.
4. Decide whether a nonzero constant function and the zero function have fundamental periods. Explain why the definition cannot simply assign either function period zero as a fundamental period.

Question 1 – Solution

Strategy. Prove that a period of the sum must preserve each of its distinct frequency components.

Step 1: Find a candidate period. The summands have periods $\pi/3$ and $\pi/5$, and π is a common period. Thus π is a period of f . In general, cancellation can give a sum a shorter period or even make it constant, so minimality still needs proof.

Step 2: Prove necessity and minimality. Product-to-sum identities show that distinct positive-integer sine/cosine frequencies have zero product integral on $[0, 2\pi]$, while each squared integral equals π . Multiplying a zero linear combination by each component and integrating therefore forces every coefficient to vanish. Expanding the shift of f now gives

$$\cos 6T = 1, \quad \sin 6T = 0, \quad \cos 10T = 1, \quad \sin 10T = 0.$$

Equivalently, $6T = 2\pi m$ and $10T = 2\pi n$ for integers m, n . Then $5m = 3n$, so $m = 3j, n = 5j$ and $T = j\pi$. Conversely every such positive shift works. Hence

Positive periods: $j\pi$ ($j = 1, 2, \dots$), $T_0 = \pi$.

Step 3: Transform the independent variable. Since $2x - 1$ ranges over all real numbers, a shift $T > 0$ preserves g exactly when $2T$ preserves f . Thus $T_g = \pi/2$. Translation, multiplication by a nonzero constant and addition of a constant do not change the set of periods; the argument dilation rescales it.

Step 4: Treat constant functions separately. Every positive number is a period of a constant function, including the zero function. There is no smallest positive number, so neither has a fundamental period. Zero is excluded by the definition and cannot be used to repair this lack of a least positive period.

Question 2

Consider a signal with incommensurate frequencies:

$$f(x) = \cos x + \cos(\sqrt{2}x).$$

A graph over a long interval may suggest repeated patterns without establishing an exact period. Recall that $\sqrt{2}$ is irrational.

Tasks

1. Show that $f(x) \leq 2$ and determine every point where equality holds. Use this to prove that f has no positive period.
2. For a second proof, apply $D^2 + 2$ to f , where $D = d/dx$. Explain why a period of a smooth function is also a period of this derivative combination, and derive a contradiction.
3. Set $T = 140\pi$. Prove a uniform bound on $|f(x+T) - f(x)|$ using $99^2 - 2 \cdot 70^2 = 1$ and $|\cos(u+v) - \cos u| \leq |v|$.
4. Decide whether this small uniform discrepancy makes T an exact period. Contrast the result with $h(x) = \cos x + \cos(3x/2)$, and determine the fundamental period of h .

Question 2 – Solution

Strategy. Distinguish exact simultaneous phase return from a uniformly small phase mismatch.

Step 1: Use the exact maximum. Each cosine is at most one. Equality $f(x) = 2$ requires $x = 2\pi m$ and $\sqrt{2}x = 2\pi n$ for integers m, n . Irrationality forces $m = n = 0$, so the maximum is attained only at $x = 0$. If $T > 0$ were a period, then $f(T) = f(0) = 2$, contradicting this uniqueness.

Step 2: Isolate the frequencies by differentiation. Direct differentiation gives $(D^2 + 2)f = \cos x$. Differentiating an identity $f(x+T) = f(x)$ twice preserves it, so T would also be a period of $\cos x$. Subtracting this cosine from f shows that T preserves $\cos(\sqrt{2}x)$ too. Evaluating both shifted cosines at zero requires $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$, impossible for positive T .

Step 3: Bound an approximate return. At $T = 140\pi$, the first cosine returns exactly. The second phase shift is $2\pi(70\sqrt{2}) = 2\pi \cdot 99 - \delta$, where

$$\delta = 2\pi(99 - 70\sqrt{2}) = \frac{2\pi}{99 + 70\sqrt{2}}.$$

The stated cosine inequality therefore yields

$$\sup_{x \in \mathbb{R}} |f(x + 140\pi) - f(x)| \leq \frac{2\pi}{99 + 70\sqrt{2}} < 0.032.$$

Step 4: Contrast approximate and exact recurrence. The discrepancy bound is positive and does not make T a period; Step 1 rules out every positive exact period. For h , any period must send its value two at zero to another value two. Thus $T = 2\pi m$ and $3T/2 = 2\pi n$, forcing m even. Conversely 4π preserves both summands. Hence $\boxed{T_h = 4\pi}$, an exact fundamental period.

Question 3

Define the two-periodic extension of a ramp by the unique representation

$$x = 2m + r, \quad m \in \mathbb{Z}, \quad -1 \leq r < 1, \quad s(x) = r.$$

The half-open convention specifies the actual function values at the joins. Also let p be the one-periodic extension of $x(1 - x)$ from $[0, 1)$.

Tasks

1. Prove that s has fundamental period two. Find its value and one-sided limits at every odd integer, and sketch it on $[-3, 3]$ with correct open and closed points.
2. Determine whether $s(-x) = -s(x)$ holds everywhere. Change only its join values to produce an odd, two-periodic function, and decide whether that change can make it continuous.
3. Prove that $\int_a^{a+2} s(x) dx = 0$ for every real a . Explain why changing the join values does not affect this statement.
4. Determine the continuity and differentiability of p at integers. Explain why matching function values at the ends of a defining interval is weaker than matching first derivatives.

Question 3 – Solution

Strategy. Separate periodicity, assigned endpoint values, one-sided limits and integral properties.

Step 1: Locate the joins and establish minimality. The representation gives $s(x+2) = s(x)$. Its discontinuities occur precisely at odd integers. Any period must preserve this set, so a positive period is a positive even integer; two is therefore fundamental. At $c = 2m + 1$,

$$s(c) = -1, \quad s(c-) = 1, \quad s(c+) = -1.$$

Each increasing segment has its left endpoint included and its right endpoint excluded. The graph also includes the assigned value $s(3) = -1$.

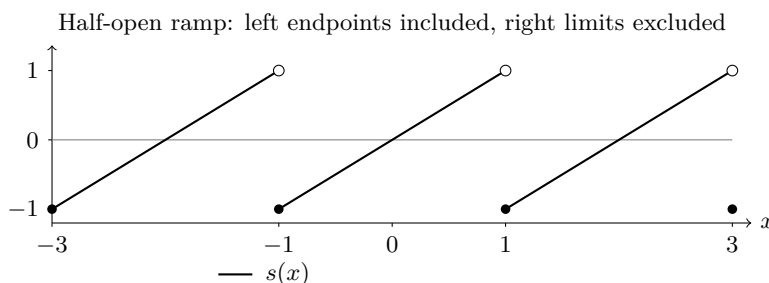
Step 2: Repair oddness, not continuity. Away from odd integers, $s(-x) = -s(x)$. At an odd integer both $s(x)$ and $s(-x)$ equal -1 , so pointwise oddness fails. Assigning zero at every odd integer produces an odd function and preserves two-periodicity. No assigned join value can reconcile the unequal one-sided limits, so the jumps remain.

Step 3: Integrate over any full period. The integral on $[-1, 1]$ is $\int_{-1}^1 x \, dx = 0$. Reduce a modulo two to $b \in [-1, 1)$. Splitting at one and shifting the remaining piece by two gives

$$\int_a^{a+2} s = \int_b^1 s + \int_1^{b+2} s = \int_b^1 s + \int_{-1}^b s = 0.$$

Changing finitely many join values in any bounded interval leaves these Riemann integrals unchanged. Integral cancellation does not require pointwise oddness.

Step 4: Check a continuous periodic join. For p , both endpoint limits of $x(1-x)$ are zero, matching the assigned value at each integer. Thus p is continuous everywhere. The left derivative at an integer is -1 , while the right derivative is 1 , so it is not differentiable there. Value matching alone does not give derivative matching.



Question 4

Orthogonality depends on the interval used in the inner product. For $L > 0$, let

$$\langle u, v \rangle_{a,L} = \int_a^{a+L} u(x)v(x)dx.$$

Examine $u = \sin x$ and $v = \cos x$.

Tasks

1. Compute $\langle u, v \rangle_{a,L}$ exactly. Characterize all lengths L for which u and v are orthogonal for every starting point a .
2. When $L = \pi/2$, find all starting points that give orthogonality. Explain why the same length can work at one location and fail at another.
3. On $[0, \pi/2]$, compute both squared norms and the inner product of the individually normalized functions. Decide whether normalization creates orthogonality.
4. On a window of length $m\pi$, m a positive integer, give an orthonormal pair obtained by rescaling u, v . Sketch their product on $[0, \pi]$ to explain cancellation without treating the graph as proof.

Question 4 – Solution

Strategy. Compute the product integral before deciding whether a pair is orthogonal or merely normalized.

Step 1: Determine the window dependence. Using $\sin x \cos x = \frac{1}{2} \sin 2x$ gives

$$\langle u, v \rangle_{a,L} = \frac{\cos 2a - \cos(2a + 2L)}{4} = \boxed{\frac{1}{2} \sin L \sin(2a + L)}.$$

It vanishes for all a exactly when $\sin L = 0$. The positive lengths are $\boxed{L = m\pi, m = 1, 2, \dots}$. If $\sin L \neq 0$, one can choose a with $\sin(2a + L) = 1$, proving necessity.

Step 2: Locate the shorter orthogonal windows. For $L = \pi/2$, orthogonality requires $\sin(2a + \pi/2) = 0$. Hence

$$\boxed{a = -\pi/4 + j\pi/2, \quad j \in \mathbb{Z}.$$

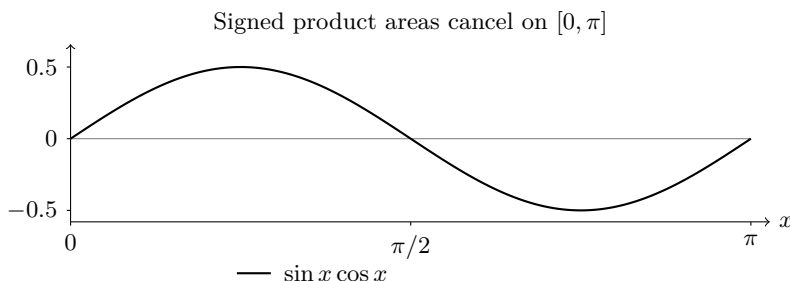
The product changes sign; shifting the window can balance its signed areas. At $a = 0$, however, the product is positive inside the interval.

Step 3: Test individual normalization. On $[0, \pi/2]$, $\|u\|^2 = \|v\|^2 = \pi/4$ and $\langle u, v \rangle = 1/2$. The unit-norm functions are $\hat{u} = 2 \sin x / \sqrt{\pi}$ and $\hat{v} = 2 \cos x / \sqrt{\pi}$, with

$$\boxed{\langle \hat{u}, \hat{v} \rangle = 2/\pi \neq 0}.$$

Nonzero scaling cannot turn a nonzero inner product into zero.

Step 4: Normalize on the correct windows. Over any length $m\pi$, the oscillatory term in either squared integral cancels, so both squared norms equal $m\pi/2$. Thus $\boxed{\sqrt{2/(m\pi)} \sin x, \sqrt{2/(m\pi)} \cos x}$ are orthonormal on every such window. The graph shows equal positive and negative product areas on $[0, \pi]$; the integral calculation establishes their exact cancellation.



Question 5

Use $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. Construct monic orthogonal polynomials from $1, x, x^2, x^3$; monic means the leading coefficient is one.

Tasks

1. Explain which inner products vanish by parity. Starting with $p_0 = 1$ and $p_1 = x$, find monic p_2, p_3 so that all four polynomials are mutually orthogonal.
2. Compute their squared norms exactly and turn them into an orthonormal set. Distinguish monic normalization from unit-norm normalization.
3. Find the unique quadratic-or-lower polynomial minimizing $\int_{-1}^1 [x^3 - q(x)]^2 dx$. Give the exact minimum error and justify uniqueness.
4. A proposed shortcut merely divides each monomial by its norm. Give an explicit inner product showing why this does not produce an orthogonal family.

Question 5 – Solution

Strategy. Use parity to eliminate unnecessary projections, then compute the remaining moments exactly.

Step 1: Construct the monic family. An odd integrand integrates to zero on $[-1, 1]$, so every even polynomial is orthogonal to every odd polynomial. For $p_2 = x^2 - c$, the equation $\langle p_2, 1 \rangle = 2/3 - 2c = 0$ gives $c = 1/3$. For $p_3 = x^3 - dx$, $\langle p_3, x \rangle = 2/5 - 2d/3 = 0$ gives $d = 3/5$. Thus

$$p_0 = 1, \quad p_1 = x, \quad p_2 = x^2 - 1/3, \quad p_3 = x^3 - 3x/5.$$

Parity and these two moment equations verify every distinct pair.

Step 2: Compute and apply the norms. Integrating the squares term by term yields

$$\|p_0\|^2 = 2, \quad \|p_1\|^2 = 2/3, \quad \|p_2\|^2 = 8/45, \quad \|p_3\|^2 = 8/175.$$

An orthonormal set is therefore $\left[p_0/\sqrt{2}, \sqrt{3/2} p_1, \sqrt{45/8} p_2, \sqrt{175/8} p_3 \right]$. Monic normalization fixes the leading coefficient; unit normalization fixes the squared integral and generally changes that coefficient.

Step 3: Find the best quadratic approximation. The decomposition $x^3 = 3x/5 + p_3$ has its residual orthogonal to every quadratic. For any other quadratic q , the cross term vanishes, giving

$$\|x^3 - q\|^2 = \|p_3\|^2 + \|3x/5 - q\|^2.$$

Hence $\left[q_* = 3x/5, \quad \min \|x^3 - q\|^2 = 8/175 \right]$. Equality forces the continuous polynomial $q - 3x/5$ to have zero norm and therefore vanish identically, proving uniqueness.

Step 4: Reject normalization without orthogonalization. The normalized versions of 1 and x^2 are $1/\sqrt{2}$ and $\sqrt{5/2} x^2$. Their inner product is $\sqrt{5}/3 \neq 0$. Normalizing lengths alone does not remove projections onto the preceding functions.

Question 6

On continuous real functions on $[0, 1]$, define

$$\langle f, g \rangle_w = \int_0^1 x f(x) g(x) dx.$$

The weight vanishes at one endpoint. Seek a monic quadratic $q(x) = x^2 + ax + b$ orthogonal to both 1 and x in this inner product.

Tasks

1. Prove positive definiteness despite the zero weight at $x = 0$. Explain the role of continuity of the functions.
2. Determine a, b from the two weighted moment equations. Verify both equations and prove uniqueness of the monic quadratic.
3. Compute $\|q\|_w^2$, give its unit-norm version with positive leading coefficient, and locate its roots. Sketch q on $[0, 1]$.
4. Test whether the same q is orthogonal to 1 for unweighted integration. Also explain why replacing the weight by $x - 1/2$ would fail to define an inner product on this space.

Question 6 – Solution

Strategy. Include the weight in every moment, and distinguish an isolated zero of a nonnegative weight from a sign-changing weight.

Step 1: Establish positive definiteness. Symmetry and bilinearity follow from integration, and $\int_0^1 x f^2 dx \geq 0$. If a continuous f is not identically zero, continuity supplies an interval of positive length inside $(0, 1]$ on which $f^2 > 0$, even if the first known nonzero value is at zero. The weighted integral is then positive. Thus zero weighted norm forces $f = 0$ everywhere.

Step 2: Solve the weighted moment equations. The two conditions are

$$\frac{1}{4} + \frac{a}{3} + \frac{b}{2} = 0, \quad \frac{1}{5} + \frac{a}{4} + \frac{b}{3} = 0.$$

Their coefficient determinant is $1/9 - 1/8 = -1/72 \neq 0$, so the solution is unique:

$$a = -6/5, \quad b = 3/10, \quad q = x^2 - 6x/5 + 3/10.$$

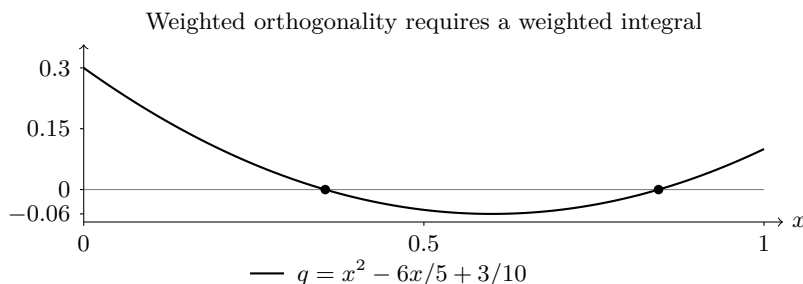
Substitution gives $1/4 - 2/5 + 3/20 = 0$ and $1/5 - 3/10 + 1/10 = 0$.

Step 3: Normalize and locate the sign changes. Expansion and integration give

$$\int_0^1 x q^2 dx = \frac{1}{6} - \frac{12}{25} + \frac{51}{100} - \frac{6}{25} + \frac{9}{200} = \boxed{\frac{1}{600}}.$$

Thus the unit-norm version is $\boxed{\sqrt{600}q}$. The roots are $\boxed{(6 - \sqrt{6})/10, (6 + \sqrt{6})/10}$, both strictly inside $(0, 1)$. The quadratic is positive outside these roots and negative between them.

Step 4: Check two changes of weight. Without the weight, $\int_0^1 q dx = 1/3 - 3/5 + 3/10 = \boxed{1/30 \neq 0}$. Orthogonality therefore changes with the weight. For the proposed signed weight, the nonzero constant function one would have squared “norm” $\int_0^1 (x - 1/2) dx = 0$. This violates positive definiteness, so that bilinear form is not an inner product on the stated space.



Question 7

Let $p \in C^1[a, b]$ be positive, let q, w be continuous with $w > 0$, and let u, v be nonzero real C^2 eigenfunctions:

$$-(pu')' + qu = \lambda wu, \quad -(pv')' + qv = \mu wv.$$

At each endpoint $c = a, b$, both satisfy the same condition $A_c y(c) + B_c p(c) y'(c) = 0$, with real $(A_c, B_c) \neq (0, 0)$.

Tasks

1. Derive an identity relating $(\lambda - \mu) \int_a^b wuv \, dx$ to endpoint terms by multiplying and subtracting the differential equations.
2. Prove the endpoint terms vanish for all the stated boundary conditions, including $B_c = 0$. Deduce weighted orthogonality when $\lambda \neq \mu$.
3. Apply the result to $u = \sin \pi x$ and $v = \sin 2\pi x$ on $[0, 1]$ with zero endpoint values. Give an orthonormal pair.
4. Explain why equal eigenvalues do not force orthogonality. On $[0, 2\pi]$ with periodic value-and-slope matching, use $u = \sin x$ and $v = \sin x + \cos x$ as a counterexample, then orthogonalize and normalize this pair.

Question 7 – Solution

Strategy. Track the boundary expression before dividing by an eigenvalue difference.

Step 1: Derive the boundary identity. Multiply the equation for u by v and that for v by u , then subtract. The q terms cancel, and integration gives

$$(\lambda - \mu) \int_a^b wuv \, dx = [p(uv' - u'v)]_a^b.$$

This identity is valid before any orthogonality conclusion is drawn.

Step 2: Use the actual endpoint conditions. At an endpoint with $B_c = 0$, $A_c \neq 0$ and both function values are zero, so the boundary expression is zero. If $B_c \neq 0$, then $pu' = -A_c u/B_c$ and $p'v = -A_c v/B_c$;

substitution again gives zero. Therefore $\lambda \neq \mu$ implies $\int_a^b wuv \, dx = 0$. Equal eigenvalues give only the identity $0 = 0$.

Step 3: Normalize two distinct Dirichlet modes. Here $p = w = 1, q = 0$, with eigenvalues π^2 and $4\pi^2$. The theorem gives orthogonality, and direct integration of each square gives $1/2$. Thus

$\sqrt{2} \sin \pi x, \sqrt{2} \sin 2\pi x$ are orthonormal. The normalization is separate from the boundary proof of orthogonality.

Step 4: Handle a repeated eigenvalue. Both $u = \sin x$ and $v = \sin x + \cos x$ have eigenvalue one and satisfy periodic matching on $[0, 2\pi]$. Nevertheless, $\langle u, v \rangle = \pi \neq 0$. Since $\|u\|^2 = \pi$, subtracting the projection gives $v - \langle v, u \rangle u / \|u\|^2 = \cos x$. The resulting orthonormal pair is

$$\sin x / \sqrt{\pi}, \quad \cos x / \sqrt{\pi}.$$

Distinct eigenvalues ensure orthogonality under the hypotheses; a repeated eigenspace may instead require an orthogonal choice of basis.

Question 8

Approximate $f(x) = |x|$ on $[-1, 1]$ by a polynomial $g(x) = a + bx^2$. Use the squared error $\|f - g\|_2^2 = \int_{-1}^1 (f - g)^2 dx$. For this inner product, 1 and $p_2 = x^2 - 1/3$ are orthogonal, with squared norms 2 and $8/45$ respectively.

Tasks

1. Compute the projection coefficients onto 1, p_2 and find the unique polynomial g_* minimizing the squared integral error.
2. Verify directly that the residual is orthogonal to both 1 and x^2 . Prove the error decomposition that establishes the minimum and uniqueness.
3. Compute the exact minimum squared error. Sketch f and g_* , and identify where the largest absolute residual occurs.
4. Does the least-squares optimum also minimize the maximum absolute error? Compare it with $h(x) = 1/8 + x^2$ and justify your conclusion by exact uniform error bounds.

Question 8 – Solution

Strategy. Use an orthogonal basis for the integral objective, then test the distinct uniform-error objective separately.

Step 1: Compute the projection. The needed moments are $\langle f, 1 \rangle = 1$ and $\langle f, p_2 \rangle = 1/2 - 1/3 = 1/6$. Thus

$$g_* = \frac{1}{2} + \frac{1/6}{8/45}(x^2 - 1/3) = \boxed{\frac{3}{16} + \frac{15}{16}x^2}.$$

Step 2: Verify residual orthogonality and uniqueness. For $r = f - g_*$,

$$\int_{-1}^1 r \, dx = 1 - \frac{3}{8} - \frac{5}{8} = 0, \quad \int_{-1}^1 rx^2 \, dx = \frac{1}{2} - \frac{1}{8} - \frac{3}{8} = 0.$$

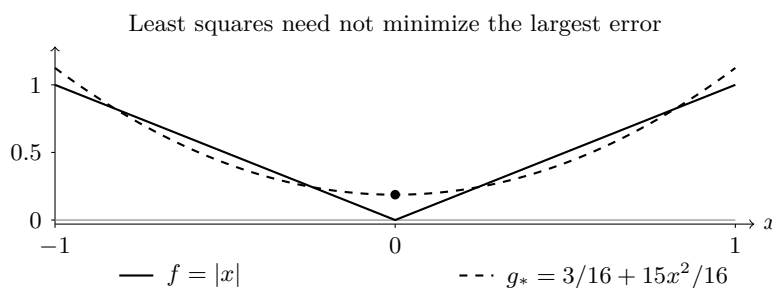
For any other $g = a + bx^2$, the cross term vanishes, so $\|f - g\|_2^2 = \|r\|_2^2 + \|g_* - g\|_2^2$. The second term is zero only when the two continuous polynomials agree, proving both optimality and uniqueness.

Step 3: Evaluate the two errors for the optimum. Subtracting the squared projection norm from $\|f\|_2^2 = 2/3$ gives

$$\|r\|_2^2 = \frac{2}{3} - \frac{1}{2} - \frac{(1/6)^2}{8/45} = \frac{1}{96}.$$

On $[0, 1]$, $r = x - 3/16 - 15x^2/16$ has maximum $19/240$ at $x = 8/15$, and endpoint values $-3/16, -1/8$. By evenness, the largest absolute error is $\boxed{\|r\|_\infty = 3/16}$, attained at zero.

Step 4: Disprove uniform optimality. For $h = 1/8 + x^2$, put $t = |x| \in [0, 1]$. Then $f - h = t(1-t) - 1/8 \in [-1/8, 1/8]$, with both extremes attained. Hence $\boxed{\|f - h\|_\infty = 1/8 < 3/16}$. This competitor proves that the unique least-squares solution is not a uniform best approximation. Different error objectives need not select the same function.



Question 9

A continuous signal f on $[-\pi, \pi]$ is known only through three moments:

$$\int_{-\pi}^{\pi} f \, dx = 2\pi, \quad \int_{-\pi}^{\pi} f \cos x \, dx = \pi, \quad \int_{-\pi}^{\pi} f \sin x \, dx = -2\pi.$$

Use the ordinary integral inner product. The three measurement functions $1, \cos x, \sin x$ are mutually orthogonal.

Tasks

1. Find the unique function in their span that matches all three measurements. Compute its squared norm.
2. Characterize every continuous signal with those measurements as a determined part plus an undetermined residual. Prove a sharp lower bound for $\int_{-\pi}^{\pi} f^2 dx$ and characterize equality.
3. Exhibit infinitely many distinct compatible signals explicitly, and compute their squared norms. Explain why linear independence of the measurement functions does not imply recovery of an arbitrary signal.
4. Suppose a fourth measurement gives $\int_{-\pi}^{\pi} f \cos 2x \, dx = 0$. Does that make the signal unique? Give an explicit counterexample and distinguish a finite orthogonal family from a complete representation.

Question 9 – Solution

Strategy. Orthogonal measurements determine a finite projection, while all perpendicular components remain unseen.

Step 1: Recover the measured component. The squared norms of $1, \cos x, \sin x$ are $2\pi, \pi, \pi$. Dividing each moment by its corresponding squared norm gives

$$p(x) = 1 + \cos x - 2 \sin x, \quad \|p\|^2 = 7\pi.$$

Orthogonality makes these coefficients unique within the three-dimensional span.

Step 2: Derive the sharp energy bound. Every compatible f has the form $f = p + r$, where r is continuous and orthogonal to all three measurement functions. Conversely every such residual preserves the measurements. Since $\langle p, r \rangle = 0$,

$$\|f\|^2 = 7\pi + \|r\|^2 \geq 7\pi.$$

Equality holds exactly when $r = 0$ everywhere: a continuous function with zero squared integral is identically zero. Thus p is the unique minimum-energy signal.

Step 3: Exhibit the invisible freedom. Product-to-sum identities give zero inner products between $\cos 2x$ and each of $1, \cos x, \sin x$. Therefore

$$f_C = p + C \cos 2x, \quad C \in \mathbb{R}, \quad \|f_C\|^2 = 7\pi + C^2\pi.$$

The measurement functions are independent, but their span does not contain all continuous functions. The data determine the projection, not its orthogonal complement.

Step 4: Test the fourth measurement. The added measurement removes the particular $\cos 2x$ freedom above, but $p + D \sin 2x$ satisfies all four conditions for every real D . The required product integrals again vanish by parity or product-to-sum. A finite orthogonal family supplies independent coordinates for its own span; it is not a complete representation of all continuous signals on the interval.

Question 10

Let $u(x) = \sin x$ and $v(x) = \sin 5x$ on $[0, 2\pi]$. Compare the continuous average inner product with its equally spaced sampling version:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} fg \, dx, \quad \langle f, g \rangle_N = \frac{1}{N} \sum_{j=0}^{N-1} f(2\pi j/N)g(2\pi j/N).$$

The sampled form may fail to be positive definite on functions even though it is positive definite on the vectors of sampled values.

Tasks

1. Compute the continuous Gram matrix of u, v ; a Gram matrix has entries consisting of all pairwise inner products.
2. For $N = 4$, list both sample vectors and compute the sampled Gram matrix. Exhibit a nonzero continuous function in their span with zero sampled norm, and sketch the curves with the common sampled points.
3. For integer m , derive $S_m = N^{-1} \sum_{j=0}^{N-1} \cos(2\pi m j/N)$ using a finite geometric sum. Express every entry of the sampled Gram matrix in terms of S_2, S_4, S_6, S_{10} .
4. Find the smallest integer $N \geq 3$ for which the sampled Gram matrix equals the continuous one. Check every smaller candidate and explain why exactness for this pair does not imply exactness for all continuous functions.

Question 10 – Solution

Strategy. Distinguish continuous orthogonality from orthogonality of a finite set of sampled values.

Step 1: Compute the continuous matrix. Product-to-sum gives $\langle u, v \rangle = 0$, while each sine has average square $1/2$. Hence $G = \frac{1}{2}I_2$.

Step 2: Identify the four-point alias. At $0, \pi/2, \pi, 3\pi/2$, both functions give $(0, 1, 0, -1)$. Thus

$$G_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The nonzero function $u - v$ vanishes at every sample, so its sampled norm is zero, although its continuous squared norm is one. The samples have lost the distinction.

Step 3: Derive the exact sampling formulas. Put $z = e^{2\pi im/N}$. If N divides m , every summand equals one. Otherwise $z \neq 1$ and $\sum_{j=0}^{N-1} z^j = (1 - z^N)/(1 - z) = 0$. Taking real parts yields $S_m = 1$ when $N \mid m$, and $S_m = 0$ otherwise. Product-to-sum now gives

$$G_N = \frac{1}{2} \begin{pmatrix} 1 - S_2 & S_4 - S_6 \\ S_4 - S_6 & 1 - S_{10} \end{pmatrix}.$$

Step 4: Find the smallest exact sample count. For $N = 3$ the off-diagonal entry is $-1/2$; for $N = 4$ it is $1/2$. For $N = 5$, the second diagonal entry is zero. For $N = 6$ the off-diagonal entry is $-1/2$. None matches G . For $N = 7$, none of $2, 4, 6, 10$ is divisible by seven, so $G_7 = G$ and seven is the smallest allowed count. This is exact only for the tested products: for example, $\sin 7x$ vanishes at all seven sample points but has continuous average square $1/2$.

