

Question 1

For a fixed $L > 0$, consider

$$-y'' = \lambda y, \quad 0 \leq x \leq L, \quad y(0) = y(L) = 0.$$

An eigenvalue is a real λ for which a nonzero solution exists; any such solution is an eigenfunction. The identically zero solution does not qualify.

Tasks

1. Treat $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ separately, and find every eigenvalue and its full eigenspace.
2. Choose each eigenfunction to have $\int_0^L y^2 dx = 1$ and $y'(0) > 0$. Explain why these two conditions fix its amplitude and sign uniquely.
3. Locate all interior zeros of the n th eigenfunction. For $L = 1$, sketch the first three normalized modes and identify their interior zero counts.
4. If the interval length is doubled, determine the change in each eigenvalue and in the normalization factor. Explain why a function solving the differential equation alone need not be an eigenfunction.

Question 1 – Solution

Strategy. Enforce both homogeneous endpoint conditions in each sign case before normalizing a nonzero solution.

Step 1: Classify all real parameters. For $\lambda = -\mu^2 < 0$, $y(0) = 0$ gives $y = A \sinh(\mu x)$; since $\sinh(\mu L) > 0$, the right condition forces $A = 0$. For $\lambda = 0$, the left condition gives $y = Ax$ and the right gives $A = 0$. For $\lambda = k^2 > 0$, $y = A \sin(kx)$, so a nonzero A requires $kL = n\pi$. Thus

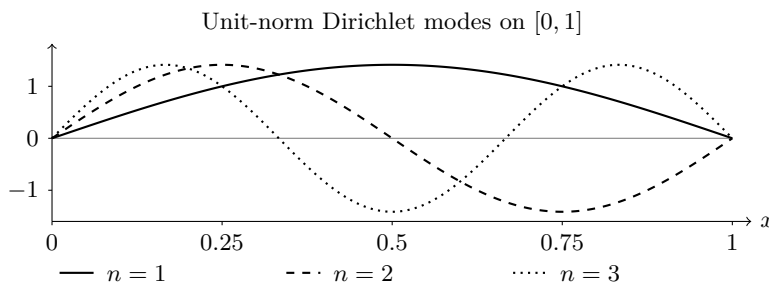
$$\lambda_n = (n\pi/L)^2, \quad E_n = \text{span}\{\sin(n\pi x/L)\}, \quad n = 1, 2, \dots$$

Step 2: Fix amplitude and sign. The identity $\sin^2 t = (1 - \cos 2t)/2$ gives $\int_0^L \sin^2(n\pi x/L) dx = L/2$. The unit norm requires $|A| = \sqrt{2/L}$, and $y'(0) = An\pi/L > 0$ selects the positive sign:

$$\phi_n(x) = \sqrt{2/L} \sin(n\pi x/L).$$

Step 3: Count the interior zeros. The zeros are $x = jL/n$, $j = 0, \dots, n$, of which exactly $n - 1$ lie inside. They are simple because the derivative there is nonzero. The first three modes on the unit interval therefore have zero, one and two interior zeros.

Step 4: Rescale the interval. Replacing L by $2L$ divides each eigenvalue by four and the normalization factor by $\sqrt{2}$. For example, $\cos(kx)$ solves the differential equation with $\lambda = k^2$ but fails the left endpoint condition. The equation, both boundary conditions and nontriviality are all part of the eigenproblem.



Question 2

Study the derivative boundary conditions

$$-y'' = \lambda y, \quad 0 \leq x \leq L, \quad y'(0) = y'(L) = 0, \quad L > 0.$$

A calculation that divides by $\sqrt{\lambda}$ can lose an important mode.

Tasks

1. Find every real eigenvalue and eigenspace, including a separate check at zero and an exclusion of negative eigenvalues.
2. Normalize each mode by $\int_0^L y^2 dx = 1$ and $y(0) > 0$. Explain why the zero mode has a different normalization factor.
3. Integrate the differential equation to prove that every nonzero-eigenvalue mode has zero mean. Decide what happens to the zero eigenspace if zero mean is imposed as an extra constraint.
4. For $L = 2$, test the claims that x is a zero-mode eigenfunction and that $1 + \cos(\pi x/2)$ is an eigenfunction. Distinguish satisfying both endpoint conditions from belonging to a single eigenspace.

Question 2 – Solution

Strategy. Preserve the constant solution by checking the zero parameter before dividing by a frequency.

Step 1: Find the complete spectrum. Integration by parts for any eigenfunction gives $\lambda \int_0^L y^2 dx = \int_0^L (y')^2 dx \geq 0$, so negative eigenvalues are impossible. At zero, $y = A + Bx$ and the derivative conditions force $B = 0$; nonzero constants are eigenfunctions. For $\lambda = k^2 > 0$, $y = A \cos(kx) + B \sin(kx)$; the left derivative condition forces $B = 0$ and the right requires $\sin(kL) = 0$. Thus

$$\lambda_n = (n\pi/L)^2, \quad E_n = \text{span}\{\cos(n\pi x/L)\}, \quad n = 0, 1, 2, \dots$$

Step 2: Normalize the modes. The constant has squared integral L , while every positive-frequency cosine has squared integral $L/2$. The positive value at zero fixes the signs:

$$\phi_0 = 1/\sqrt{L}, \quad \phi_n = \sqrt{2/L} \cos(n\pi x/L) \quad (n \geq 1).$$

Step 3: Interpret the mean constraint. Integrating gives $\lambda \int_0^L y dx = -y'(L) + y'(0) = 0$. Thus all modes with $\lambda \neq 0$ have zero mean. The only constant with zero mean is the zero function, so the additional constraint removes the zero eigenspace from the admissible nonzero functions; it preserves every higher mode.

Step 4: Test the two proposed functions. The function x has derivative one at both ends, so it fails the boundary conditions. The function $g = 1 + \cos(\pi x/2)$ satisfies both derivative conditions, but $-g'' = (\pi^2/4) \cos(\pi x/2)$ is not any constant multiple of g . Indeed, matching the constant term would require $\lambda = 0$, which fails for the cosine term. A sum from different eigenspaces need not be an eigenfunction.

Question 3

An unknown interval length $L > 0$ has mixed boundary conditions:

$$-y'' = \lambda y, \quad y(0) = 0, \quad y'(L) = 0.$$

Two measurements are reported as consecutive eigenvalues, in increasing order: $9\pi^2/16$ and $25\pi^2/16$. The report does not say they are the first two.

Tasks

1. Derive the complete real spectrum and rule out zero and negative values. Index the positive eigenvalues starting at $n = 0$.
2. Recover L and the indices of the two measured modes. Explain why taking the difference of the square roots is useful.
3. Find the true lowest eigenvalue and normalize its eigenfunction by $\int_0^L y^2 dx = 1$ and $y'(0) > 0$.
4. Drop the claim that the measurements are consecutive. Characterize every possible length and pair of indices consistent with the same two numbers. Explain exactly what information the consecutiveness assumption supplied.

Question 3 – Solution

Strategy. Use equally spaced frequencies, not equally spaced eigenvalues, to interpret the measurements.

Step 1: Classify the mixed spectrum. The endpoint term vanishes in integration by parts, so $\lambda \int y^2 = \int (y')^2 > 0$ for a nonzero admissible function; equality would force a constant and then $y = 0$. With $\lambda = k^2 > 0$, the left condition gives $y = A \sin(kx)$ and the right requires $\cos(kL) = 0$. Hence

$$\lambda_n = ((n + 1/2)\pi/L)^2, \quad n = 0, 1, \dots$$

Each eigenspace consists of multiples of the corresponding sine.

Step 2: Recover the consecutive indices. Consecutive positive frequencies differ by π/L . The measured difference is $5\pi/4 - 3\pi/4 = \pi/2$, giving $L = 2$. The first measured frequency satisfies $(n + 1/2)\pi/2 = 3\pi/4$, so its index is $n = 1$ and the next is $n = 2$. They were not the first two eigenvalues.

Step 3: Recover the missing lowest mode. The index-zero eigenvalue is $\lambda_0 = \pi^2/16$. Since $\int_0^2 \sin^2(\pi x/4) dx = 1$ and its initial slope is positive, the requested eigenfunction is $\phi_0 = \sin(\pi x/4)$.

Step 4: Identify the remaining inverse ambiguity. Let the measured indices be $r < s$. Their frequency ratio implies $5(2r + 1) = 3(2s + 1)$. Since three and five are coprime, all solutions are

$$2r + 1 = 3d, \quad 2s + 1 = 5d, \quad L = 2d, \quad r = (3d - 1)/2, \quad s = (5d - 1)/2,$$

where d is any positive odd integer. Conversely, substitution verifies every such choice. The index gap is $s - r = d$; consecutiveness forces $d = 1$ and removes this infinite ambiguity.

Question 4

Let $h \in \mathbb{R}$ control the endpoint condition

$$-y'' = \lambda y, \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y'(1) = h y(1).$$

Unlike fixed zero endpoint values, this boundary condition can permit a negative eigenvalue.

Tasks

1. Determine precisely when zero is an eigenvalue and identify its eigenspace.
2. For $\lambda = -\mu^2 < 0$, derive the equation for μ . Prove that there is exactly one negative eigenvalue when $h > 1$ and none when $h \leq 1$.
3. For $\lambda = k^2 > 0$, derive the frequency equation without losing or adding modes by division. Locate all its roots by intervals between multiples of π , including the special first interval.
4. Derive the energy identity and explain why it does not always imply nonnegative eigenvalues. For $h = 2$, bound the negative eigenvalue using $1 < \mu < 2$, and sketch the equation locating μ .

Question 4 – Solution

Strategy. Separate all three signs and prove root counts by monotonicity on the correct domains.

Step 1: Check the zero parameter directly. At $\lambda = 0$, $y = Ax$ and the right condition is $A = hA$. Thus zero is an eigenvalue exactly when $h = 1$, with eigenspace $\text{span}\{x\}$.

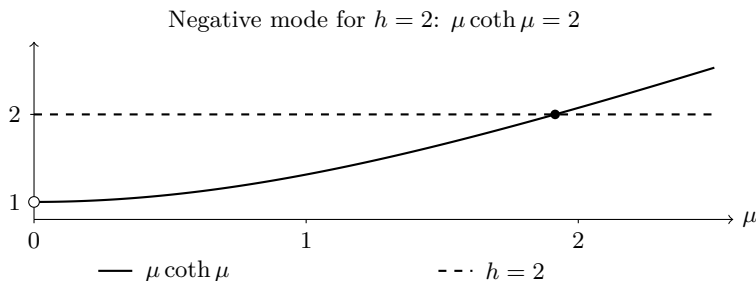
Step 2: Count the negative modes. For $\lambda = -\mu^2$, the nonzero solution is a multiple of $\sinh(\mu x)$, and its endpoint condition is $\mu \coth \mu = h$. The function $g(\mu) = \mu \coth \mu$ tends to one as $\mu \downarrow 0$ and to infinity as $\mu \rightarrow \infty$. Its derivative has numerator $\sinh \mu \cosh \mu - \mu > 0$, since this numerator starts at zero and has derivative $2 \sinh^2 \mu > 0$. Thus precisely $h > 1$ gives one negative mode.

Step 3: Locate every positive mode. Here $y = A \sin(kx)$ and $k \cos k = h \sin k$. A positive multiple of π cannot solve this equation, so division by $\sin k$ is safe at a root:

$$k \cot k = h.$$

Its derivative is $(\sin k \cos k - k)/\sin^2 k < 0$ for $k > 0$ away from poles. On each $(n\pi, (n+1)\pi)$, $n \geq 1$, it decreases from $+\infty$ to $-\infty$, giving exactly one root for every h . On $(0, \pi)$ it decreases from the unattained limit one to $-\infty$: one root if $h < 1$, none if $h \geq 1$.

Step 4: Retain the endpoint energy term. Integration by parts gives $\lambda \int_0^1 y^2 dx = \int_0^1 (y')^2 dx - hy(1)^2$, which can be negative. For $h = 2$, $g(1) < 2 < g(2)$: the first inequality follows from $e^2 > 3$ and $g(2) > 2$ follows from $\coth 2 > 1$. Thus $1 < \mu < 2$ and $-4 < \lambda_- < -1$. The graph locates the unique crossing.



Question 5

Consider a ring represented by the interval $[0, 2\pi]$ with matching values and slopes at its ends:

$$-y'' = \lambda y, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$$

Here the eigenspace dimension need not be one. No Fourier expansion is required.

Tasks

1. Rule out negative eigenvalues using the boundary terms in an energy identity, and determine the zero eigenspace directly.
2. Find all positive eigenvalues and the dimension of each eigenspace. Explain why imposing one endpoint equation alone would be insufficient.
3. At a positive eigenvalue n^2 , impose $y(0) = 0$ and $y'(0) = 1$. Find the resulting eigenfunction and decide whether these normalization data are possible at zero.
4. For the normalized positive modes, test the additional measurements $y(\pi) = 0$ and $y'(\pi) = y'(0)$. Which measurement adds no information, and which selects only part of the spectrum?

Question 5 – Solution

Strategy. Treat value and derivative matching together as a two-dimensional endpoint map.

Step 1: Handle nonpositive parameters. The boundary term $[yy']_0^{2\pi}$ is zero because both values and slopes match. Thus $\lambda \int y^2 = \int (y')^2 \geq 0$. At zero, $y = A + Bx$; value matching forces $B = 0$ and slope matching is then automatic. Hence $E_0 = \text{span}\{1\}$, of dimension one.

Step 2: Find the positive eigenspaces. For $\lambda = k^2 > 0$, write $y = A \cos(kx) + B \sin(kx)$. With $\theta = 2\pi k$, the two conditions are

$$\begin{pmatrix} \cos \theta - 1 & \sin \theta \\ -\sin \theta & \cos \theta - 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant is $2(1 - \cos \theta)$, so nonzero solutions require $k = n$, a positive integer. Then the entire matrix is zero and both constants are free:

$$\lambda_n = n^2, \quad E_n = \text{span}\{\cos nx, \sin nx\}, \quad n \geq 1.$$

Each positive eigenspace has dimension two. One equation alone would generally leave a nonzero vector even when the other endpoint equation fails.

Step 3: Apply the initial normalization. The data $y(0) = 0, y'(0) = 1$ require $A = 0, nB = 1$, giving $y_n = \sin(nx)/n$. At zero every eigenfunction is constant, so its initial derivative cannot be one. These extra data select a representative from each positive eigenspace, not from the zero eigenspace.

Step 4: Interpret the midpoint measurements. For every positive integer n , $y_n(\pi) = \sin(n\pi)/n = 0$, so that measurement adds no restriction. But $y'_n(\pi) = \cos(n\pi) = (-1)^n$ equals $y'_n(0) = 1$ exactly for even n . The second measurement selects eigenvalues $4, 16, 36, \dots$ from the original positive spectrum.

Question 6

A variable-coefficient eigenproblem uses the positive interval $[1, e^2]$:

$$-(xy')' = \lambda \frac{y}{x}, \quad y(1) = y(e^2) = 0.$$

The natural squared norm for this problem is $\int_1^{e^2} y(x)^2 dx/x$.

Tasks

1. Use $t = \log x$ and $y(x) = u(t)$ to transform the equation and both boundary conditions. Show every derivative factor.
2. Find all real eigenvalues and eigenspaces, checking all parameter signs. Normalize each mode to have weighted norm one and $y'(1) > 0$.
3. Locate the first mode's maximum and all interior zeros of the n th mode. Explain why equal spacing in the transformed coordinate does not imply equal spacing in x .
4. Replace e^2 by a general right endpoint $b > 1$. Give the eigenvalues and weighted normalization factor as functions of b . Sketch the first two normalized modes for the original interval.

Question 6 – Solution

Strategy. Transform the differential equation and its weight together, not just its apparent frequency.

Step 1: Carry out the logarithmic substitution. Since $dt/dx = 1/x$, $y' = u'/x$, $xy' = u'$ and $(xy')' = u''/x$. Multiplying the equation by x gives

$$-u'' = \lambda u, \quad u(0) = u(2) = 0.$$

Also $dx/x = dt$, so the weighted squared norm is exactly $\int_0^2 u^2 dt$.

Step 2: Classify and normalize. At zero the affine function with two zero endpoints is zero. At a negative parameter, the left-zero hyperbolic sine cannot vanish at the right endpoint. For $\lambda = k^2 > 0$, $u = A \sin(kt)$ and $\sin(2k) = 0$. Thus

$$\lambda_n = (n\pi/2)^2, \quad \phi_n(x) = \sin\left(\frac{n\pi \log x}{2}\right), \quad n \geq 1.$$

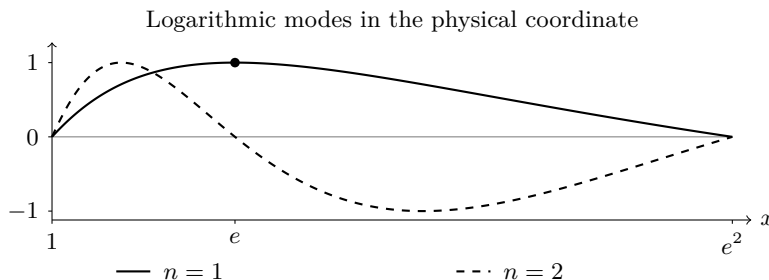
The eigenspace is the span of ϕ_n . Its weighted squared norm is one, and $\phi'_n(1) = n\pi/2 > 0$, fixing both amplitude and sign.

Step 3: Read locations in the original coordinate. For $n = 1$, the sine attains its unique maximum one when $t = 1$, hence $x = e$, the geometric rather than arithmetic midpoint of the endpoints. For mode n , the interior zeros are $x_j = e^{2j/n}$, $j = 1, \dots, n-1$. They form a geometric progression: logarithmic spacing becomes multiplicative spacing in x .

Step 4: Change the endpoint. For $[1, b]$ the transformed length is $\ell = \log b > 0$. Consequently

$$\lambda_n = (n\pi/\log b)^2, \quad \phi_n = \sqrt{\frac{2}{\log b}} \sin\left(\frac{n\pi \log x}{\log b}\right).$$

The graph for $b = e^2$ uses the physical coordinate x , with the landmark $x = e$.



Question 7

For $a \in \mathbb{R}$ and $L > 0$, study

$$-y'' + 2ay' = \lambda y, \quad y(0) = y(L) = 0.$$

The first derivative makes this look different from a sine eigenproblem. Use the weighted norm $\int_0^L e^{-2ax} y^2 dx$ when normalizing.

Tasks

1. Substitute $y = e^{ax}u$ and derive the transformed equation and endpoint conditions. Find every real eigenvalue and eigenspace.
2. Give the weighted unit-norm eigenfunction with positive initial derivative. Explain how the exponential factor affects shape without changing the zeros.
3. For $a = 1, L = \pi$, locate the maximum of the first eigenfunction with amplitude chosen as $y = e^x \sin x$. Sketch it and the underlying sine on the same axes.
4. Suppose the first two eigenvalues are known to be 5 and 8 but both a and L are unknown. Determine everything these data identify, and exhibit the remaining ambiguity.

Question 7 – Solution

Strategy. Remove the first derivative with an invertible exponential change and keep track of the shifted spectral parameter.

Step 1: Transform the operator completely. The derivatives are $y' = e^{ax}(u' + au)$ and $y'' = e^{ax}(u'' + 2au' + a^2u)$. Therefore

$$-y'' + 2ay' = e^{ax}(-u'' + a^2u), \quad \boxed{-u'' = (\lambda - a^2)u, \quad u(0) = u(L) = 0.}$$

The exponential is never zero, so this is a bijection of solution spaces. The shifted parameter must be positive: zero or negative gives only the zero Dirichlet solution. All eigenvalues and eigenfunctions are thus

$$\boxed{\lambda_n = a^2 + (n\pi/L)^2, \quad E_n = \text{span}\{e^{ax} \sin(n\pi x/L)\}, \quad n \geq 1.}$$

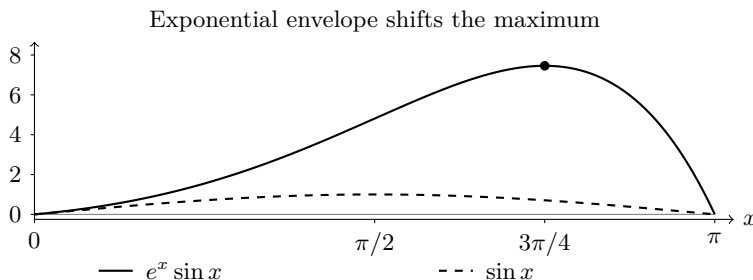
Step 2: Normalize with the correct weight. The factors e^{-2ax} and e^{2ax} cancel in the squared norm. Hence $\boxed{\phi_n = \sqrt{2/L} e^{ax} \sin(n\pi x/L)}$ has unit weighted norm and positive initial derivative. The envelope changes heights and the locations of extrema, but its positivity leaves the zeros jL/n unchanged.

Step 3: Locate the displaced maximum. For $y = e^x \sin x$, $y' = e^x(\sin x + \cos x)$ is positive up to $3\pi/4$ and negative afterwards on $(0, \pi)$. Thus

$$\boxed{x_{\max} = 3\pi/4, \quad y_{\max} = e^{3\pi/4}/\sqrt{2}.}$$

The sine itself peaks at $\pi/2$; multiplying by the increasing envelope shifts the peak without moving either zero.

Step 4: Solve the inverse spectral data. The gap $8 - 5 = 3$ equals $3\pi^2/L^2$, so $L = \pi$. Then $a^2 + 1 = 5$, giving $\boxed{a = 2 \text{ or } a = -2}$. Both choices give the entire same spectrum $4 + n^2$, yet their exponential envelopes point in opposite directions. The eigenvalues alone cannot recover the sign of the drift parameter.



Question 8

Compare an eigenproblem with its forced counterpart on $[0, 1]$:

$$-y'' - \lambda y = \sin(\pi x), \quad y(0) = y(1) = 0.$$

The homogeneous Dirichlet eigenvalues of $-d^2/dx^2$ are $n^2\pi^2$, $n \geq 1$. The identity

$$\int_0^1 \sin(n\pi x) \sin(\pi x) dx = 0 \quad (n \geq 2)$$

may be verified by $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.

Tasks

1. Solve the forced problem for every real λ outside the homogeneous spectrum and prove uniqueness.
2. At $\lambda = \pi^2$, prove nonexistence using integration by parts against $\sin(\pi x)$, rather than merely observing a vanishing denominator.
3. At $\lambda = n^2\pi^2$ with $n \geq 2$, find all solutions. Determine which is selected by $\int_0^1 y \sin(n\pi x) dx = 0$.
4. Compare the uniform size of the solution as $\lambda \rightarrow \pi^2$ with its limit as $\lambda \rightarrow 4\pi^2$ through nonspectral values. Explain why a finite limiting particular solution does not imply uniqueness at the limiting parameter.

Question 8 – Solution

Strategy. Separate the existence of a particular solution from the nontrivial homogeneous space at a spectral parameter.

Step 1: Solve away from the spectrum. Since $-(\sin \pi x)'' = \pi^2 \sin \pi x$,

$$y = \frac{\sin \pi x}{\pi^2 - \lambda}$$

solves the equation and both endpoints whenever $\lambda \neq \pi^2$. If λ is outside the entire spectrum, any difference of two solutions is a zero-endpoint homogeneous solution and is therefore zero. This proves uniqueness, including all zero and negative parameter values.

Step 2: Prove failure of compatibility at the first eigenvalue. Let $s = \sin \pi x$. For a proposed solution at $\lambda = \pi^2$, twice integrating by parts gives

$$\int_0^1 s(-y'' - \pi^2 y) dx = [-sy' + s'y]_0^1 + \int_0^1 (-s'' - \pi^2 s)y dx = 0.$$

Both s and y vanish at the endpoints. The forcing instead requires this integral to equal $\int_0^1 s^2 dx = 1/2$, a contradiction. No solution exists.

Step 3: Find every compatible resonant solution. For $n \geq 2$, the particular solution above is valid and the homogeneous space is spanned by $\sin(n\pi x)$. Therefore all solutions are

$$y = \frac{\sin \pi x}{(1 - n^2)\pi^2} + C \sin(n\pi x), \quad C \in \mathbb{R}.$$

The supplied product identity and $\int_0^1 \sin^2(n\pi x) dx = 1/2$ show that the extra integral condition is $C/2 = 0$, selecting exactly $C = 0$.

Step 4: Distinguish two limiting behaviors. Outside the spectrum, $\max |y| = 1/|\pi^2 - \lambda|$, which diverges at the first eigenvalue. As $\lambda \rightarrow 4\pi^2$, the same solution converges uniformly to $-\sin(\pi x)/(3\pi^2)$. This is only the $C = 0$ member of the infinite family at $4\pi^2$. A finite chosen limit does not remove the homogeneous freedom.

Question 9

An inverse design seeks a continuous potential q on $[0, 1]$ for

$$-y'' + q(x)y = \lambda y, \quad y(0) = y(1) = 0.$$

For the constructive part, prescribe $\lambda = \pi^2$ and $y = \sin \pi x + \frac{1}{4} \sin 2\pi x$. Eigenfunctions are required to be C^2 .

Tasks

1. Show that the alternative target $y = x(1 - x)$ cannot be an eigenfunction for any continuous q and any finite real λ . Explain the endpoint obstruction.
2. For the trigonometric target, recover q explicitly and extend it continuously to both endpoints. Compute $q(0)$ and $q(1)$ and verify the differential equation.
3. Prove that the target is positive on $(0, 1)$ and has simple endpoint zeros. For any C^2 test function v with zero endpoints, put $z = v/y$ inside the interval and derive $\int_0^1 [v'^2 + (q - \lambda)v^2]dx = \int_0^1 y^2 z'^2 dx$.
4. Use this identity to prove that no real eigenvalue can lie below π^2 , and that every eigenfunction with eigenvalue π^2 is a multiple of the designed target. Address the boundary term in the identity.

Question 9 – Solution

Strategy. Recover the potential where the target is nonzero, then use endpoint regularity and a factored energy identity to certify the design.

Step 1: Reject an incompatible target. For $y = x(1 - x)$, division inside the interval would require $q = \lambda + y''/y = \lambda - 2/[x(1 - x)]$, which is unbounded at both ends. Equivalently, continuity of the equation at zero would require $-y''(0) + q(0)y(0) = 2 = \lambda y(0) = 0$. No continuous potential can work.

Step 2: Construct a regular potential. Write $c = \cos \pi x$. The target is $y = \sin \pi x(1 + c/2)$ and $y'' = -\pi^2 \sin \pi x(1 + 2c)$. Consequently

$$q(x) = -\frac{3\pi^2 c}{2 + c}, \quad q(0) = -\pi^2, \quad q(1) = 3\pi^2.$$

Since $2 + c \geq 1$, the formula is continuous on the closed interval. Substitution gives $qy = (\lambda y + y'')$, verifying the equation inside; continuity verifies it at the endpoints as well.

Step 3: Factor the energy. The factor $1 + c/2$ lies between $1/2$ and $3/2$, so $y > 0$ inside. The endpoint derivatives are $y'(0) = 3\pi/2$ and $y'(1) = -\pi/2$, both nonzero. With $v = yz$ and $y'' = (q - \lambda)y$, expansion gives

$$v'^2 + (q - \lambda)v^2 = y^2 z'^2 + (yy' z^2)'$$

For zero-endpoint C^2 functions v , the simple zeros of y imply that $z = v/y$ and z' stay bounded near either endpoint, by Taylor expansion. Thus $[yy' z^2]_0^1 = 0$ and integration proves the requested identity.

Step 4: Certify the lowest eigenvalue and its eigenspace. If v is a nonzero eigenfunction with eigenvalue ν , integration by parts and the identity yield

$$(\nu - \pi^2) \int_0^1 v^2 dx = \int_0^1 y^2 z'^2 dx \geq 0.$$

Thus $\nu \geq \pi^2$, and the constructed nonzero y attains this bound. Equality forces $z' = 0$ inside, hence $v = Cy$. Therefore π^2 is the lowest real eigenvalue and its eigenspace is precisely $\text{span}\{y\}$.

Question 10

A two-material interval has stiffness

$$p(x) = \begin{cases} 1, & 0 < x < 1/2, \\ 4, & 1/2 < x < 1, \end{cases} \quad -(py')' = \lambda y, \quad y(0) = y(1) = 0.$$

Require y and the flux py' to be continuous at $x = 1/2$. Solutions are C^2 on each side. Ordinary slopes need not be continuous across the interface.

Tasks

1. Use integration by parts on the two subintervals to show that every real eigenvalue is positive. Explain cancellation of the interface terms.
2. Set $k = \sqrt{\lambda}$ and write the endpoint-compatible sine on each side. Derive the two interface equations for their amplitudes and a determinant condition.
3. With $t = k/4$, factor the determinant and characterize all positive eigenvalues. Treat $\sin t = 0$ separately to avoid losing modes when solving for the amplitude ratio.
4. Identify the smallest eigenvalue and give its eigenfunction with left amplitude one. Verify continuity and the slope ratio at the interface, and sketch the mode with its physical corner.

Question 10 – Solution

Strategy. Match displacement and stiffness-weighted slope, then preserve every branch of the determinant equation.

Step 1: Exclude nonpositive eigenvalues. Adding the two integrations by parts gives $\lambda \int_0^1 y^2 dx = \int_0^1 p(y')^2 dx$: endpoint values vanish and the interface terms cancel because both y and py' match. The right side is positive for a nonzero admissible function; equality would make it constant on both sides, hence identically zero. Therefore $\lambda > 0$.

Step 2: Form the interface system. The endpoint-compatible pieces are $A \sin(kx)$ on the left and $B \sin(k(1-x)/2)$ on the right. With $t = k/4$, matching gives

$$A \sin 2t = B \sin t, \quad A \cos 2t = -2B \cos t.$$

The second equation uses flux, not ordinary derivative. The determinant vanishes exactly when $2 \sin 2t \cos t + \cos 2t \sin t = 0$.

Step 3: Preserve both spectral branches. Factoring yields $\sin t(6 \cos^2 t - 1) = 0$. Thus the complete spectrum is

$$\lambda = 16t^2, \quad t > 0, \quad t = m\pi \ (m \geq 1) \text{ or } \cos^2 t = 1/6.$$

At $t = m\pi$, the first matching equation is zero and the second gives $B = (-1)^{m+1}A/2$. On the other branch, $\sin t \neq 0$ and $B = 2A \cos t$. Each branch gives a nonzero solution and a one-dimensional eigenspace. The apparent root $t = 0$ is excluded by Step 1.

Step 4: Construct the first mode and its corner. The smallest positive root is $t_0 = \arccos(1/\sqrt{6}) \in (0, \pi/2)$. Hence $\lambda_1 = 16t_0^2$ and, taking $A = 1$,

$$y = \begin{cases} \sin(4t_0 x), & 0 \leq x \leq 1/2, \\ \sqrt{2/3} \sin(2t_0(1-x)), & 1/2 \leq x \leq 1. \end{cases}$$

At the join both values equal $\sqrt{5}/3$. The one-sided slopes are $-8t_0/3$ and $-2t_0/3$, so $y'(1/2-) = 4y'(1/2+)$ as required by flux continuity. The slope change is physical; smoothing it would violate the model.

