

Question 1

A boundary-value problem prescribes values at two different points:

$$y'' = -6x, \quad y(0) = 1, \quad y(1) = 2.$$

Compare it with the family of initial-value problems having $y(0) = 1$ and $y'(0) = s$, where s is a freely adjustable initial slope.

Tasks

1. Solve the initial-value family and determine the unique slope that meets the right boundary condition. Explain the distinction between shooting for an endpoint and prescribing an initial slope.
2. Verify the resulting differential equation and both endpoint values. Find the maximum and minimum on $[0, 1]$ and determine whether endpoint values bound the entire response.
3. Replace the boundary values by arbitrary α, β . Prove existence and uniqueness for every pair and give the solution explicitly.
4. If the endpoint values change by δ_0, δ_1 while the forcing stays fixed, find the exact change in the solution and a sharp uniform bound. Sketch the original solution and the line joining its endpoints.

Question 1 – Solution

Strategy. Solve the initial-value family first, then use the endpoint equation to select its slope.

Step 1: Select the shooting slope. Integrating twice gives $y_s = 1 + sx - x^3$. The right endpoint is $y_s(1) = s$, so $s = 2$ is the unique successful slope. An initial-value problem fixes s directly; the boundary-value problem determines it from a condition at another point.

Step 2: Verify the solution and its extrema. The solution $y = 1 + 2x - x^3$ has $y'' = -6x$, $y(0) = 1$, $y(1) = 2$. Since $y' = 2 - 3x^2$, its maximum occurs at $r = \sqrt{2/3}$:

$$y_{\max} = 1 + \frac{4}{3}\sqrt{\frac{2}{3}} > 2, \quad y_{\min} = y(0) = 1.$$

Thus the endpoint values do not bound this forced response from above.

Step 3: Allow arbitrary endpoint values. Direct integration gives

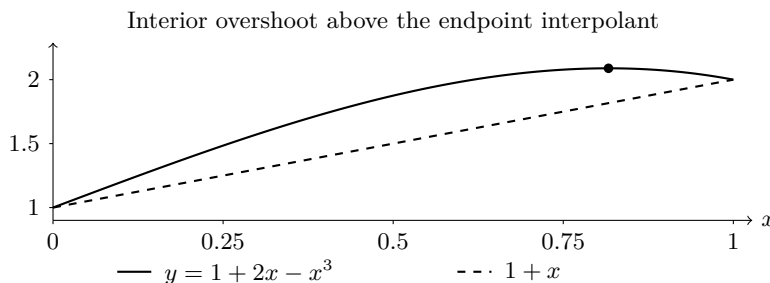
$$y_{\alpha,\beta}(x) = \alpha + (\beta - \alpha + 1)x - x^3.$$

This works for every α, β . The difference of two solutions with the same data satisfies $w'' = 0$, $w(0) = w(1) = 0$, so it is zero, proving uniqueness.

Step 4: Quantify boundary-data sensitivity. The exact response change is $\Delta y = (1 - x)\delta_0 + x\delta_1$. Therefore

$$\max_{0 \leq x \leq 1} |\Delta y| = \max(|\delta_0|, |\delta_1|).$$

The inequality follows from convex weights; equality holds at an endpoint. The original curve exceeds its endpoint line $1 + x$ by $x - x^3 > 0$ inside $(0, 1)$.



Question 2

Consider the fixed-interval problem

$$y'' + y = 1, \quad 0 \leq x \leq \pi, \quad y(0) = \alpha, \quad y(\pi) = \beta.$$

All functions are real. Counting two endpoint conditions does not settle whether a second-order boundary-value problem has exactly one solution.

Tasks

1. Find the complete solution family before applying the boundary conditions. Derive the necessary and sufficient compatibility condition on α, β .
2. Classify the number of solutions for every pair of boundary values. Explain the role of the nonzero homogeneous solution that vanishes at both endpoints.
3. When the problem is compatible, impose $\int_0^\pi y(x) dx = m$. Determine whether this extra condition selects a unique solution for every real m .
4. Replace the right endpoint by $\pi/2$, retaining prescribed values at both ends. Explain why uniqueness returns. For the original interval and $\alpha = \beta = 1$, sketch three distinct solutions with identical endpoint values.

Question 2 – Solution

Strategy. Test whether the endpoint equations independently determine the homogeneous constants.

Step 1: Derive compatibility. The general solution is $y = 1 + A \cos x + B \sin x$. The left endpoint gives $A = \alpha - 1$, while the right endpoint gives $\beta = 1 - A = 2 - \alpha$. Thus

$$\boxed{\text{A solution exists exactly when } \alpha + \beta = 2.}$$

Step 2: Count the compatible solutions. When compatible, the entire family is

$$\boxed{y = 1 + (\alpha - 1) \cos x + B \sin x, \quad B \in \mathbb{R}.}$$

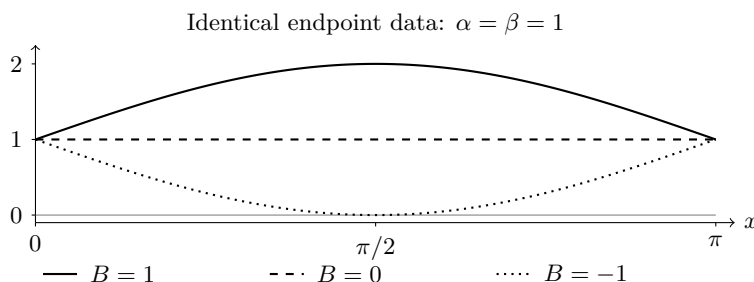
There are infinitely many solutions; otherwise there are none. There is never exactly one with just these two conditions. The nonzero homogeneous solution $\sin x$ vanishes at both endpoints and can be added without changing the data.

Step 3: Use an integral normalization. Since the cosine integrates to zero and the sine integrates to two,

$$\int_0^\pi y \, dx = \pi + 2B = m, \quad \boxed{B = (m - \pi)/2.}$$

For compatible endpoints, every real m selects exactly one solution. If the endpoints are incompatible, an extra integral condition cannot fix that.

Step 4: Change the interval and compare. On $[0, \pi/2]$, the right endpoint gives $B = \beta - 1$ independently of $A = \alpha - 1$, so every pair gives one solution. Independence of the endpoint equations, not their number alone, makes the difference. The graph on the original interval shows $y = 1 + B \sin x$ for $B = -1, 0, 1$.



Question 3

Let the signed flux be $q(x) = -(1+x)y'(x)$ on $[0, 1]$. A variable-conductivity boundary-value problem is

$$-((1+x)y')' = 2x - 1, \quad q(0) = q_0, \quad q(1) = q_1.$$

The sign convention uses the positive x direction at both endpoints.

Tasks

1. Derive the necessary and sufficient compatibility condition on the two prescribed fluxes. Interpret it as a balance over the interval.
2. For $q_0 = q_1 = 0$, find every solution and verify the equation and both flux conditions directly.
3. Add the normalization $\int_0^1 y(x) dx = 0$. Find the resulting constant exactly and prove uniqueness of the normalized solution.
4. Determine whether this normalized solution is increasing, and compute its endpoint values. Explain why flux data alone cannot fix its absolute level and why incompatible fluxes cannot be repaired by adding a constant.

Question 3 – Solution

Strategy. Integrate the flux balance before trying to determine the additive level of the response.

Step 1: Check the interval balance. The equation is $q' = 2x - 1$, so

$$q(x) = q_0 + x^2 - x, \quad q_1 - q_0 = \int_0^1 (2x - 1) dx = 0.$$

Thus $\boxed{q_1 = q_0}$ is necessary. It is sufficient because division by $1 + x > 0$ and one integration produce a solution for every such pair.

Step 2: Solve the zero-flux case. Here $y' = (x - x^2)/(1 + x) = -x + 2 - 2/(1 + x)$, hence

$$\boxed{y = C - x^2/2 + 2x - 2\log(1 + x)}.$$

Multiplication by $-(1 + x)$ gives $q = x^2 - x$, whose derivative is $2x - 1$ and whose values at zero and one are zero. This checks the equation and flux data.

Step 3: Fix the additive constant. Using $\int_0^1 \log(1 + x) dx = 2\log 2 - 1$, we obtain

$$\int_0^1 y dx = C + \frac{17}{6} - 4\log 2.$$

Thus $\boxed{C = 4\log 2 - 17/6}$. The difference of two solutions with the same fluxes has $((1 + x)w')' = 0$ and zero endpoint flux, so $w' = 0$. It is constant; the zero-mean condition forces that constant to vanish.

Step 4: Interpret the resulting profile. Since $y' = x(1 - x)/(1 + x) > 0$ for $0 < x < 1$, the response is strictly increasing. Its endpoint values are

$$\boxed{y(0) = 4\log 2 - 17/6, \quad y(1) = 2\log 2 - 4/3}.$$

Fluxes involve derivatives and therefore leave a constant level undetermined. Adding a constant changes neither flux nor the integrated balance, so it cannot repair a violation of $q_1 = q_0$.

Question 4

A mixed boundary condition couples endpoint value and slope:

$$y'' = 2, \quad y(0) = 0, \quad y'(1) - h y(1) = b,$$

where h, b are real parameters.

Tasks

1. Solve the problem for $h \neq 1$ and verify all data. Classify the number of solutions at $h = 1$ for every b .
2. For fixed $h \neq 1$, determine the exact response to a perturbation $b \mapsto b + \delta$. Find its uniform size on $[0, 1]$.
3. Follow the path $b = 1$ as $h \rightarrow 1$. Explain how a well-defined limit along this path coexists with nonuniqueness at the limiting parameters.
4. For arbitrary real k , use $h_n = 1 - 1/n$ and $b_n = 1 + k/n$ to find the limiting solution. Decide whether the boundary-value solution extends continuously as a single-valued function of (h, b) at $(1, 1)$.

Question 4 – Solution

Strategy. Inspect the denominator in the endpoint equation and distinguish a chosen parameter path from a unique limiting response.

Step 1: Solve and classify the singular case. The left boundary gives $y = x^2 + cx$. The mixed condition becomes $(1 - h)c = b - 2 + h$. For $h \neq 1$,

$$y = x^2 + \left(\frac{b-1}{1-h} - 1 \right) x.$$

Its second derivative is two, its left value is zero, and substitution verifies the mixed condition. At $h = 1$, the condition reduces to $b = 1$ independently of c : infinitely many solutions if $b = 1$, and none otherwise.

Step 2: Quantify sensitivity. At fixed $h \neq 1$, the change in the response is

$$\Delta y(x) = \frac{\delta x}{1-h}, \quad \max_{[0,1]} |\Delta y| = \frac{|\delta|}{|1-h|}.$$

Thus arbitrarily small boundary errors can be strongly amplified near $h = 1$.

Step 3: Examine one compatible path. If $b = 1$ and $h \neq 1$, then $c = -1$, so every solution on this path is $y = x^2 - x$. Its limit is one member of the infinite family at $(h, b) = (1, 1)$. The path selects this member; the limiting boundary conditions alone do not.

Step 4: Compare different approaches. For the proposed sequences, $(b_n - 1)/(1 - h_n) = k$, so

$$y_n = x^2 + (k-1)x \quad \text{for every } n.$$

All parameter pairs approach $(1, 1)$, but their solution limits depend on k . They range over the entire limiting family. Consequently no choice of a single solution at $(1, 1)$ can make the solution map continuous there along all paths.

Question 5

A dimensionless fourth-order beam model has constant load $y^{(4)} = 24$ on $[0, 1]$. Compare two endpoint designs:

$$\begin{array}{ll} \text{Clamped:} & y(0) = y'(0) = y(1) = y'(1) = 0, \\ \text{Simply supported:} & y(0) = y''(0) = y(1) = y''(1) = 0. \end{array}$$

Tasks

1. Find the solution for each design by integrating the fourth-order equation. Verify all four conditions separately for each solution.
2. Prove uniqueness for both designs by applying integration by parts to the difference of two solutions. Track the boundary terms explicitly.
3. Find each maximum displacement and its location. Explain why symmetry alone would not be a complete proof that the midpoint is a maximum.
4. Compare the two profiles pointwise and compute the ratio of their maximum displacements. Sketch both on the same axes and explain why changing derivative conditions changes the response under the same load.

Question 5 – Solution

Strategy. Determine the integration constants from the correct derivative data, then use an energy identity to check uniqueness.

Step 1: Solve both sets of endpoint equations. Starting from $y = x^4 + ax^3 + bx^2 + cx + d$, the two solutions are

$$y_C = x^2(1-x)^2, \quad y_S = x^4 - 2x^3 + x.$$

Both fourth derivatives equal 24 and both endpoint values are zero. Also $y'_C = 2x(1-x)(1-2x)$ vanishes at both ends, while $y'_S = 12x(x-1)$ vanishes at both ends. These verify the distinct boundary data.

Step 2: Prove uniqueness in each case. A difference w has $w^{(4)} = 0$ and homogeneous boundary conditions. Twice integrating by parts gives

$$0 = \int_0^1 ww^{(4)} dx = [ww''' - w'w'']_0^1 + \int_0^1 (w'')^2 dx.$$

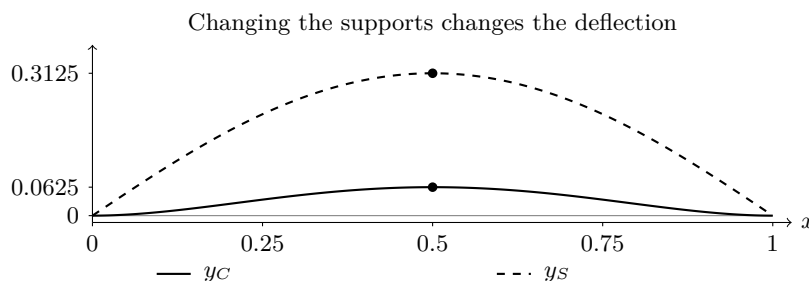
The boundary term is zero for clamped data because $w = w' = 0$, and for simply supported data because $w = w'' = 0$. Thus $w'' = 0$; its zero endpoint values imply $w = 0$. Each design has exactly one solution.

Step 3: Locate and compare the maxima. The derivative of y_C changes from positive to negative at $1/2$. For the other design, $y'_S = (2x-1)(2x^2-2x-1)$, whose second factor is negative on $[0, 1]$. Thus both have their unique maximum at $1/2$, with

$$y_C(1/2) = 1/16, \quad y_S(1/2) = 5/16.$$

The derivative signs establish maxima; symmetry by itself only suggests a stationary midpoint.

Step 4: Compare the complete profiles. The difference is $y_S - y_C = x(1-x) > 0$ inside the interval, and the maximum ratio is five. The same differential load is compatible with different responses because fixing slopes differs from fixing second derivatives at the boundaries.



Question 6

A third-order model is to satisfy four endpoint measurements:

$$y''' = q, \quad y(0) = y'(0) = 0, \quad y(1) = a, \quad y'(1) = b.$$

The constant load q may either be prescribed or be an unknown to infer.

Tasks

1. For a fixed prescribed q , derive the necessary and sufficient relation between a, b for a solution to exist. Determine the number of compatible solutions.
2. Instead regard q as unknown. Recover both the load and the complete profile uniquely for arbitrary real a, b .
3. Apply the inverse problem to $a = 1, b = 0$. Verify the load and all four endpoint conditions, and prove that the resulting displacement is increasing on $(0, 1)$.
4. For a nominal load $q = 6$ and right displacement $a = 1$, determine the required right slope. Explain why the fourth boundary condition can be inconsistent when q is fixed but informative when q is unknown.

Question 6 – Solution

Strategy. Count the integration constants together with any unknown model parameter, and then solve the actual endpoint equations.

Step 1: Find compatibility for a fixed load. The left conditions give $y = qx^3/6 + cx^2/2$. The right conditions require $a = q/6 + c/2$ and $b = q/2 + c$. Subtraction yields

$$b - 2a = q/6.$$

This condition is necessary and sufficient: if it holds, $c = 2a - q/3$ gives the unique solution; otherwise no solution satisfies all the prescribed data.

Step 2: Recover an unknown load. When q is free, the same two equations determine both q and c :

$$q = 6(b - 2a), \quad y = (b - 2a)x^3 + (3a - b)x^2.$$

There is one solution for every pair (a, b) . Direct differentiation verifies the constant load and the two endpoint slopes and values.

Step 3: Check the displacement from rest to rest. For $a = 1, b = 0$,

$$q = -12, \quad y = 3x^2 - 2x^3.$$

Its third derivative is -12 , its endpoint values are zero and one, and $y' = 6x(1 - x)$ vanishes at both ends. Since this derivative is positive inside, the displacement is strictly increasing on $(0, 1)$.

Step 4: Interpret the extra measurement. For $q = 6, a = 1$, compatibility requires $b - 2 = 1$, so $b = 3$ and $y = x^3$. Any other prescribed right slope is inconsistent with those fixed data. The left conditions already remove two of the three integration constants; with fixed q , the remaining constant cannot fit arbitrary two right data. When q is unknown, it supplies one more adjustable quantity, which the fourth measurement identifies. Counting conditions is useful only with this distinction.

Question 7

Consider a singular-endpoint problem on $0 < x \leq 1$:

$$(x^2 y')' = 0, \quad y(1) = 1.$$

An additional condition at zero is to be imposed through a limit or regularity requirement. The differential equation is initially required only for $x > 0$.

Tasks

1. Find every solution on $(0, 1]$ satisfying the right boundary condition. Identify why zero is not a regular endpoint of the normalized equation.
2. Select the solutions that remain bounded as $x \downarrow 0$, and separately those with finite weighted energy $\int_0^1 x^2 (y')^2 dx$. Determine whether these two selections agree.
3. Replace boundedness by $\lim_{x \downarrow 0} xy(x) = k$, with k prescribed. Find the unique solution for every real k and its limiting weighted derivative $\lim_{x \downarrow 0} x^2 y'(x)$.
4. Decide whether a continuous solution can satisfy $y(0) = 0$ and $y(1) = 1$. Explain why two stated endpoint values do not justify invoking the usual regular boundary-value theory at this singular endpoint.

Question 7 – Solution

Strategy. Solve on the open interval first, then apply each proposed endpoint condition to the possible singular term.

Step 1: Find the family on the actual domain. Integration gives $x^2 y' = C$ and $y = A - C/x$. Incorporating $y(1) = 1$, write

$$y = 1 - k + k/x, \quad k \in \mathbb{R}.$$

For $x > 0$ the normalized equation is $y'' + (2/x)y' = 0$. Its coefficient $2/x$ is singular at zero; the original leading coefficient also vanishes there.

Step 2: Test boundedness and energy. Boundedness at zero forces $k = 0$, giving $y \equiv 1$. Since $y' = -k/x^2$,

$$\int_0^1 x^2 (y')^2 dx = \lim_{t \downarrow 0} k^2 \left(\frac{1}{t} - 1 \right).$$

This is infinite for $k \neq 0$ and zero for $k = 0$. Thus boundedness and finite weighted energy select the same single solution in this family.

Step 3: Prescribe a weighted endpoint value. For the displayed family,

$$\lim_{x \downarrow 0} xy(x) = k, \quad \lim_{x \downarrow 0} x^2 y'(x) = -k.$$

Therefore each real prescribed weighted limit selects exactly one solution, including unbounded solutions when $k \neq 0$. A weighted boundary condition is not the same as an ordinary finite value $y(0)$.

Step 4: Test the proposed continuous boundary data. A continuous extension is bounded, so it must be $y \equiv 1$ and must have $y(0) = 1$. Consequently no continuous solution satisfies $y(0) = 0, y(1) = 1$. One cannot assign an unrelated value at zero and retain continuity. Regular endpoint theory requires hypotheses on the normalized coefficients that fail here; the explicit family and its limits decide this problem instead.

Question 8

A linear differential equation is paired with a nonlinear boundary condition:

$$y'' = 0, \quad y(0) = 0, \quad y'(1) = y(1)^2 + \lambda,$$

where $\lambda \in \mathbb{R}$.

Tasks

1. Reduce the boundary-value problem to an algebraic equation and classify the number of real solutions for every λ .
2. At $\lambda = 0$, test whether the average of the two solutions is also a solution. Explain which part of the full problem prevents superposition.
3. Set $\lambda = 1/4 - \varepsilon$ with $\varepsilon > 0$. Measure the uniform distances of the two solutions from the unique critical solution at $\lambda = 1/4$. Decide whether a local Lipschitz estimate in the parameter can hold there.
4. Plot the admissible slopes $c = y'(0)$ against λ , clearly indicating the critical point and where no real solution exists. Explain why uniqueness at a single parameter need not imply stable dependence on nearby parameters.

Question 8 – Solution

Strategy. Solve the differential equation completely, then retain the nonlinear structure of its endpoint constraint.

Step 1: Classify the slope equation. Every solution of $y'' = 0, y(0) = 0$ is $y = cx$. Its remaining boundary condition is $c = c^2 + \lambda$, so

$$c_{\pm} = \frac{1 \pm \sqrt{1 - 4\lambda}}{2}.$$

There are two real solutions for $\lambda < 1/4$, one for $\lambda = 1/4$, and none for $\lambda > 1/4$. The critical solution is $y_* = x/2$.

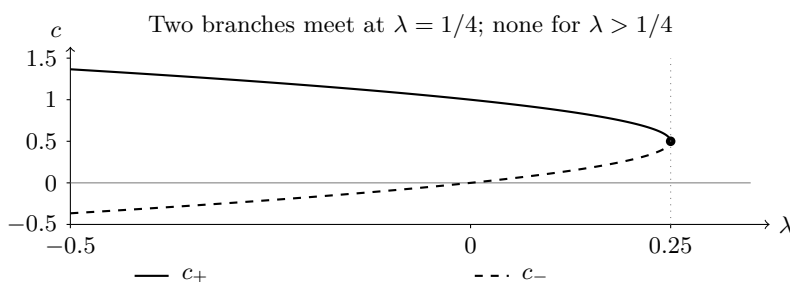
Step 2: Check the superposition claim. At $\lambda = 0$, the solutions are 0 and x . Their average $x/2$ has $y'(1) = 1/2$ but $y(1)^2 = 1/4$, so it fails the boundary condition. The differential equation is linear, but the square in the boundary operator makes the full boundary-value problem nonlinear.

Step 3: Quantify critical sensitivity. At $\lambda = 1/4 - \varepsilon$, the slopes are $1/2 \pm \sqrt{\varepsilon}$. Hence

$$\max_{[0,1]} |y_{\pm} - y_*| = \sqrt{\varepsilon}.$$

A Lipschitz bound with fixed constant K would require $\sqrt{\varepsilon} \leq K\varepsilon$ for all sufficiently small positive ε , which is impossible. Both branches approach y_* continuously from this side, but not at a rate bounded linearly by the parameter error.

Step 4: Read the complete branch picture. The branches meet at $(\lambda, c) = (1/4, 1/2)$ and have no real continuation for larger λ . For $\lambda < 1/4$, $dc_{\pm}/d\lambda = \mp 1/\sqrt{1 - 4\lambda}$, whose magnitude diverges at the join. Critical uniqueness therefore does not guarantee a locally unique family or a Lipschitz-stable response.



Question 9

Let $L[y] = -y'' + q(x)y$ on $[0, 1]$, where $q \geq 0$ is continuous. All solutions below satisfy $y(0) = y(1) = 0$. A useful comparison statement is: if $L[w] \geq 0$ and $w(0), w(1) \geq 0$, then $w \geq 0$ on the interval.

Tasks

1. Prove that $L[y] = f$ has exactly one solution for every continuous f . Use an energy identity for uniqueness and a family of initial-value solutions for existence.
2. Prove the stated comparison result. To handle the case where q can vanish, apply the interior-minimum test first to $w + \eta[1 + x(1 - x)]$ with $\eta > 0$, then let $\eta \downarrow 0$.
3. If an approximation v has zero endpoint values and $|L[v] - f| \leq \epsilon$, prove a pointwise error bound using $\phi = x(1 - x)/2$. Give a sharp constant for the uniform error estimate over all such problems.
4. Apply the result to $q = x^2$, $f = 1$, $v = \phi$. Compute the maximum residual exactly and give rigorous upper and lower bounds for the unknown exact solution, without claiming the approximation solves the equation.

Question 9 – Solution

Strategy. Combine an energy identity, a strict comparison argument and a simple barrier to certify the boundary-value error.

Step 1: Establish existence and uniqueness. A homogeneous difference w with zero endpoint values satisfies $0 = \int_0^1 wL[w]dx = \int_0^1 ((w')^2 + qw^2)dx$, hence $w' = 0$ and $w = 0$. For existence, let p solve $L[p] = f$ with $p(0) = p'(0) = 0$, and let h solve $L[h] = 0$ with $h(0) = 0, h'(0) = 1$. Regular IVP theory supplies both. If $h(1) = 0$, uniqueness just proved would force $h = 0$, a contradiction. Thus $y = p - p(1)h/h(1)$ is the required solution.

Step 2: Make comparison strict before taking a limit. Put $\psi = 1 + x(1 - x) > 0$. Then $L[\psi] = 2 + q\psi > 0$, so $L[w + \eta\psi] > 0$ and its endpoint values are positive. At a nonpositive interior minimum, its second derivative is nonnegative and $q \geq 0$ would give $L[w + \eta\psi] \leq 0$, a contradiction. Thus $w + \eta\psi > 0$. Letting $\eta \downarrow 0$ proves $w \geq 0$.

Step 3: Bound the error with a barrier. Since $L[\phi] = 1 + q\phi \geq 1$, for $e = y - v$ we have $L[\epsilon\phi \pm e] \geq 0$, with zero endpoints. Comparison yields

$$|y(x) - v(x)| \leq \epsilon\phi(x), \quad \max_{[0,1]} |y - v| \leq \epsilon/8.$$

The constant $1/8$ is sharp: take $q = 0, v = 0, f = \epsilon$, for which $y = \epsilon\phi$ attains equality at $x = 1/2$.

Step 4: Certify the proposed approximation. Here $L[\phi] - 1 = x^2\phi = x^3(1 - x)/2$. Its derivative is $x^2(3 - 4x)/2$, so its maximum is $\epsilon = 27/512$ at $x = 3/4$. Also $L[v - y] \geq 0$, hence $y \leq v$. Combining this sign with the error bound gives

$$\left(1 - \frac{27}{512}\right)\phi \leq y \leq \phi, \quad \max_{[0,1]} |y - \phi| \leq \frac{27}{4096}.$$

The residual is not zero; these are certified bounds, not an exact formula for y .

Question 10

A positive point load acts at an unknown location $a \in (0, 1)$:

$$-y'' = F\delta(x - a), \quad y(0) = y(1) = 0, \quad F > 0.$$

Seek a continuous, piecewise linear displacement. The distributional equation requires $y'(a+) - y'(a-) = -F$; no jump in y is allowed.

Tasks

1. Derive the solution on each side of a using the endpoint values, continuity and derivative jump. Verify the distributional equation.
2. Find the peak height and the total area $\int_0^1 y(x) dx$. For fixed F , determine which load location maximizes the height.
3. Suppose only the peak height and total area are measured, while both F and a are unknown. Decide whether these measurements determine the load uniquely, and characterize all compatible loads for a positive measured height H .
4. Instead measure the endpoint slopes $m_L = y'(0)$ and $m_R = y'(1)$. Recover F, a and state their admissibility conditions. Solve the case $m_L = 2, m_R = -1$, and sketch it alongside the reflected load at $1 - a$ with the same F .

Question 10 – Solution

Strategy. Use the boundary conditions to determine two lines, then distinguish independent measurements from redundant ones.

Step 1: Match the two linear pieces. Write $y = mx$ to the left and $y = n(1-x)$ to the right. Continuity requires $ma = n(1-a)$, and the slope jump gives $-n - m = -F$. Thus

$$y(x) = \begin{cases} F(1-a)x, & 0 \leq x \leq a, \\ Fa(1-x), & a \leq x \leq 1. \end{cases}$$

Both endpoint values vanish. The continuous join produces no δ' term; the derivative jump $-F$ gives exactly $-y'' = F\delta(x-a)$, with zero ordinary second derivative elsewhere.

Step 2: Compute the peak and area. The left slope is positive and the right slope negative, so the peak is at a :

$$H = Fa(1-a), \quad \int_0^1 y \, dx = H/2.$$

The area is that of a triangle with base one and height H . For fixed F , $a(1-a) = 1/4 - (a - 1/2)^2$ is maximized at $a = 1/2$, giving height $F/4$.

Step 3: Test whether the measurements are independent. If the measured area differs from $H/2$, no load fits the model. If it equals $H/2$, every $a \in (0, 1)$ is compatible with $\boxed{F = H/[a(1-a)]}$. The two measurements supply only one independent number, so there are infinitely many compatible location-strength pairs.

Step 4: Recover the load from endpoint slopes. Since $m_L = F(1-a)$ and $m_R = -Fa$,

$$F = m_L - m_R, \quad a = \frac{-m_R}{m_L - m_R}.$$

The precise admissibility conditions are $m_L > 0$ and $m_R < 0$. For $(m_L, m_R) = (2, -1)$, $F = 3$, $a = 1/3$, with height $2/3$ and area $1/3$. The reflected load has the same height and area but different endpoint slopes.

