

Question 1

Consider the entire-coefficient problem

$$y^{(4)} = x^2 y'' + 2xy', \quad (y, y', y'', y''')(0) = (0, 1, 0, 0).$$

Write $y = \sum_{n \geq 0} a_n x^n$ using ordinary power-series coefficients.

Tasks

1. Derive the recurrence, including its exceptional first index. Identify all potentially nonzero coefficient classes.
2. Find the terms through degree thirteen and verify the initial data and the recurrence relations determining those terms.
3. Prove that the solution series is entire. For the degree-nine truncation p_9 , give a rigorous uniform error bound below 10^{-6} on $[-1, 1]$.
4. Determine the sign of $y - p_9$ on either side of zero. Plot the error on $[0, 1]$ together with a pointwise certificate, explicitly scaling the vertical values by 10^6 .

Question 1 – Solution

Strategy. Follow the surviving coefficient chain and bound its tail by a geometric majorant.

Step 1: Match every coefficient. The coefficient of x^n on the right is $n(n+1)a_n$, so

$$a_{n+4} = \frac{n a_n}{(n+4)(n+3)(n+2)} \quad (n \geq 0).$$

In particular $a_4 = 0$, not a freely chosen coefficient. The initial coefficients are $(a_0, a_1, a_2, a_3) = (0, 1, 0, 0)$; only indices $1 \pmod{4}$ survive.

Step 2: Compute the first terms. Successive use of $n = 1, 5, 9$ gives

$$y = x + \frac{x^5}{60} + \frac{x^9}{6048} + \frac{x^{13}}{1153152} + \cdots$$

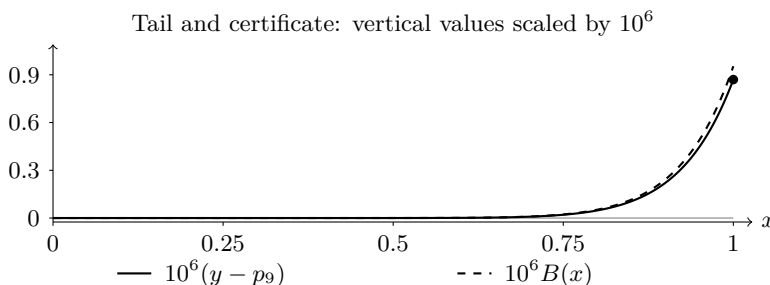
The missing coefficients are zero. The first four derivatives at zero match the data, and substitution into the recurrence verifies the displayed chain.

Step 3: Prove convergence and certify the tail. For fixed x , successive nonzero terms have ratio in magnitude $|x|^4 n / [(n+4)(n+3)(n+2)] \rightarrow 0$, so the series is entire and may be differentiated termwise. For $n \geq 9$, the coefficient ratio is at most $1/(n+2)^2 < 1/100$. Consequently, for $|x| \leq 1$,

$$|y - p_9| \leq \frac{100|x|^{13}}{99 \cdot 1153152} \leq B(x) := \frac{11|x|^{13}}{10 \cdot 1153152} \leq \frac{11}{10 \cdot 1153152} < 10^{-6}.$$

This bound concerns the full infinite tail, not just its first omitted term.

Step 4: Check the error's sign and graph. Every surviving coefficient is positive and every surviving degree is odd. Thus $y - p_9$ is positive for $x > 0$, negative for $x < 0$, and zero at zero. The figure compares $10^6(y - p_9)$ and $10^6 B(x)$ on $[0, 1]$.



Question 2

For the fourth-order problem

$$y^{(4)} + y'' + y = e^x, \quad y^{(j)}(0) = 0 \quad (0 \leq j \leq 3),$$

let $b_n = y^{(n)}(0)$ and $a_n = b_n/n!$. The ordinary generating function $B(z) = \sum_{n \geq 0} b_n z^n$ is not the solution's Taylor series.

Tasks

1. Derive a recurrence for b_n and prove that the derivative sequence is periodic. Give one full period.
2. Find $B(z)$ and its radius of convergence. Write the actual solution series using a_n and determine its radius separately.
3. Verify the differential equation directly from the solution series and explain why treating b_n as ordinary Taylor coefficients would give a wrong answer.
4. Determine $\lim_{x \rightarrow +\infty} e^{-x}y(x)$ without explicitly solving for all homogeneous constants. Justify why the homogeneous contribution disappears in this limit.

Question 2 – Solution

Strategy. Keep derivative values separate from coefficients divided by factorials.

Step 1: Derive periodic derivative data. Differentiating the equation n times at zero gives

$$b_{n+4} + b_{n+2} + b_n = 1 \quad (n \geq 0).$$

The first six values are $(0, 0, 0, 0, 1, 1)$. Subtracting this equation from its version at $n + 2$ gives $b_{n+6} = b_n$. Thus $b_n = 1$ precisely when $n \equiv 4, 5 \pmod{6}$, and otherwise $b_n = 0$.

Step 2: Compare the two generating functions. The ordinary generating function of the derivatives is

$$B(z) = \frac{z^4 + z^5}{1 - z^6}, \quad R_B = 1.$$

There is a genuine pole at $z = 1$. In contrast, the solution is

$$y(x) = \sum_{k=0}^{\infty} \left(\frac{x^{6k+4}}{(6k+4)!} + \frac{x^{6k+5}}{(6k+5)!} \right), \quad R_y = \infty.$$

Absolute convergence follows by comparison with $e^{|x|}$.

Step 3: Verify the equation and normalization. The coefficient of $x^n/n!$ in $y^{(4)} + y'' + y$ is $b_{n+4} + b_{n+2} + b_n = 1$, giving e^x . The first four derivative values vanish. Using b_n as ordinary coefficients would make the fourth derivative at zero 24, whereas the equation and initial values require $y^{(4)}(0) = 1$.

Step 4: Find the dominant exponential. A particular solution is $e^x/3$, since $1^4 + 1^2 + 1 = 3$. The characteristic polynomial factors as

$$r^4 + r^2 + 1 = (r^2 + r + 1)(r^2 - r + 1).$$

Its distinct roots have real parts $-1/2$ or $1/2$. Every homogeneous term is therefore $O(e^{x/2})$ as $x \rightarrow +\infty$, regardless of its constants. It follows that

$$\lim_{x \rightarrow +\infty} e^{-x} y(x) = 1/3.$$

Question 3

For a real parameter λ , investigate polynomial solutions of

$$y^{(4)} - xy' + \lambda y = 0.$$

A polynomial solution is required to be nonzero; its degree is not prescribed.

Tasks

1. Derive the ordinary coefficient recurrence and use the highest-degree term to restrict λ .
2. Prove that your restriction is sufficient. For each allowed parameter, determine the dimension of the polynomial solution space and give a monic formula.
3. Find the monic solution when $\lambda = 8$ and verify it directly in the differential equation.
4. Explain why the other three coefficient chains cannot supply additional independent polynomials for the same parameter. Decide what remains of the polynomial classification when λ is negative or nonintegral.

Question 3 – Solution

Strategy. Combine a highest-degree argument with the exact termination mechanism in the recurrence.

Step 1: Identify the only possible parameters. Coefficient matching gives

$$a_{n+4} = \frac{(n-\lambda)a_n}{(n+4)(n+3)(n+2)(n+1)}.$$

If a nonzero polynomial has degree m , its leading term in $-xy' + \lambda y$ is $(\lambda - m)a_mx^m$; $y^{(4)}$ has smaller degree. Hence $\lambda = m$ must be a nonnegative integer.

Step 2: Construct and count the terminating chain. For $\lambda = m$, normalize $a_m = 1$. Descending the recurrence gives

$$p_m(x) = \sum_{j=0}^{\lfloor m/4 \rfloor} \frac{(-1)^j m!}{4^j j! (m-4j)!} x^{m-4j}.$$

At $n = m$ the next coefficient is zero and the chain terminates. The displayed finite sum satisfies every coefficient equation, so it is a solution. All its coefficients are fixed by the leading coefficient; thus the polynomial solution space, including zero, is one-dimensional.

Step 3: Check a concrete eighth-degree solution. For $m = 8$,

$$p_8 = x^8 - 420x^4 + 1260.$$

We have $p_8^{(4)} = 1680x^4 - 10080$ and $-xp_8' + 8p_8 = -1680x^4 + 10080$, so the residual is identically zero.

Step 4: Rule out hidden terminating chains. The recurrence preserves the index modulo four. A nonzero chain can terminate only when its numerator $n - m$ vanishes, which occurs solely in the class containing m . Any nonzero seed in another class continues forever and cannot be part of a polynomial. Equivalently, a second polynomial of lower degree would require that degree to equal m , a contradiction. For negative or nonintegral λ , only the zero polynomial solves the equation.

Question 4

Near $x = 0$, consider

$$(1 - x^2)y''' - 6xy'' - 6y' = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \quad y''(0) = 2a_2.$$

The leading coefficient vanishes at $x = \pm 1$, but this does not by itself prove that every solution is singular there.

Tasks

1. Derive the recurrence for the Taylor coefficients and sum the resulting series on $|x| < 1$.
2. Classify exactly which initial triples give analytic continuation through $x = 1$, through $x = -1$, and through both points.
3. Determine the Taylor radius at zero for every initial triple. Explain how an entire solution can coexist with singular coefficients in the normalized equation.
4. For the data $(a_0, a_1, a_2) = (0, 1, -1)$, find the continued value and first two derivatives at $x = 1$. Verify the original equation at that point, and give the maximal real interval on which this solution is analytic.

Question 4 – Solution

Strategy. Sum the two coefficient tails, then check cancellation of each possible pole separately.

Step 1: Derive and sum the recurrence. The coefficient equation simplifies to

$$(n+3)(n+2)(n+1)(a_{n+3} - a_{n+1}) = 0 \quad (n \geq 0).$$

Thus the odd tail repeats a_1 and the positive even tail repeats a_2 :

$$y = a_0 + \frac{a_1x + a_2x^2}{1 - x^2}, \quad |x| < 1.$$

As an independent check, the differential equation is exactly $((1 - x^2)y)''' = 0$.

Step 2: Classify pole cancellation. The numerator of the fractional term vanishes at 1 exactly when $a_1 + a_2 = 0$, and at -1 exactly when $-a_1 + a_2 = 0$. Because the denominator's zeros are simple, these are also sufficient for analytic continuation through the respective point. Continuation through both requires $a_1 = a_2 = 0$, yielding the constant solution.

Step 3: Determine every convergence radius. If $a_1 = a_2 = 0$, the Taylor series is constant and its radius is infinite. Otherwise at least one genuine pole remains at distance one, so the radius is exactly one, even if the other pole cancels. Singularities of the normalized coefficients constrain the general theory; they need not be singularities of a particular solution after cancellation.

Step 4: Verify passage through a singular point. For the stated data, $y = x/(1+x)$ after cancellation. Therefore

$$y(1) = \frac{1}{2}, \quad y'(1) = \frac{1}{4}, \quad y''(1) = -\frac{1}{4}.$$

At $x = 1$, the original equation reduces to $-6y'' - 6y' = 0$, which these values satisfy. The continued solution is analytic on $(-1, \infty)$, its maximal real analytic interval containing zero. Its Taylor series about zero still has radius one because the pole at -1 remains.

Question 5

A formal power series is an algebraic coefficient expansion without an assumed positive radius of convergence. Investigate

$$x^2 y''' - y'' = -x, \quad y(0) = y'(0) = y''(0) = 0.$$

For a possible one-sided smooth solution, you may use

$$z(x) = \int_0^\infty e^{-t} \frac{x}{1 - xt} dt, \quad x \leq 0.$$

Tasks

1. Put $z = y''$ and derive its formal coefficient recurrence. Find the first four nonzero terms of the formal series for y .
2. Prove that the formal series for y has radius zero. Deduce whether an analytic solution at zero can satisfy the equation and data.
3. Verify that the given integral defines a smooth function from the left at zero and satisfies $x^2 z' - z = -x$. Recover a one-sided smooth solution y with the required initial data.
4. Explain how this smooth solution can have the divergent formal series as its Taylor series. Identify the hypothesis that prevents application of the ordinary-point analytic existence theorem at zero.

Question 5 – Solution

Strategy. Separate formal coefficient solvability from convergence and from one-sided smooth existence.

Step 1: Compute the formal coefficients. For $z = \sum b_n x^n$, coefficient matching gives $b_0 = 0$, $b_1 = 1$, and $b_n = (n-1)b_{n-1}$ for $n \geq 2$. Thus $b_n = (n-1)!$ for $n \geq 1$. Two integrations with zero constants give

$$y \sim \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)(n+2)} x^{n+2} = \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{10} + \frac{x^6}{5} + \cdots.$$

Step 2: Prove divergence. The ratio of consecutive displayed coefficients is $n(n+1)/(n+3) \rightarrow \infty$. For every fixed nonzero x , the terms eventually fail to tend to zero. The radius is zero. Any analytic solution would have exactly these forced coefficients, so no analytic solution at zero exists.

Step 3: Construct and verify a smooth solution. For $x \leq 0$, $1 - xt \geq 1$. For $n \geq 1$,

$$\partial_x^n \frac{x}{1-xt} = \frac{n! t^{n-1}}{(1-xt)^{n+1}}.$$

An integrable majorant permits differentiation of every order up to the left endpoint, giving $z(0) = 0$ and $z^{(n)}(0-) = n!(n-1)!$. Also

$$x^2 z' - z = x \int_0^\infty \frac{d}{dt} \left(\frac{e^{-t}}{1-xt} \right) dt = -x.$$

Hence $y(x) = \int_0^x (x-v)z(v) dv$, with oriented integration, is smooth on $(-\infty, 0]$, satisfies the equation there and has the zero initial triple. The equality at zero follows by continuity.

Step 4: Resolve the apparent contradiction. Its derivatives give exactly the formal coefficients above, but smoothness does not require the Taylor series to converge or represent the function. The leading coefficient x^2 vanishes at zero, so this is not an ordinary point. The analytic existence theorem for a normalized regular equation does not apply.

Question 6

The series solution of $y''' + y' = 0$ with $(y, y', y'')(0) = (0, 1, 0)$ describes an oscillator derivative pair (y', y'') . A proposed rational approximation is

$$U(x) = \frac{1 - x^2/4}{1 + x^2/4}, \quad V(x) = -\frac{x}{1 + x^2/4}.$$

Its designer claims that correct initial values and exact conservation of $U^2 + V^2$ are enough to make it the true derivative pair.

Tasks

1. Derive the series recurrence and identify the exact solution. Give its terms through degree seven.
2. Expand U, V through the first coefficient at which each differs from the true pair. Check their initial values and quadratic invariant.
3. Test the necessary derivative relation $U' = V$. Compute its defect exactly and decide whether any function can have both $y' = U$ and $y'' = V$ on an interval containing zero.
4. Find the phase $\tau(x)$ for which $(U, V) = (\cos \tau, -\sin \tau)$. Explain the actual evolution law and plot the phase lag $x - \tau(x)$ beside its leading cubic approximation on $[0, 2]$.

Question 6 – Solution

Strategy. Test derivative compatibility in addition to an invariant and a few Taylor coefficients.

Step 1: Solve the series IVP. Coefficient matching gives $a_{n+3} = -a_{n+1}/[(n+3)(n+2)]$. The data $(a_0, a_1, a_2) = (0, 1, 0)$ give the entire series

$$y = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

The true pair is $(\cos x, -\sin x)$.

Step 2: Check local accuracy and the invariant. Near zero,

$$U = 1 - x^2/2 + x^4/8 + O(x^6), \quad V = -x + x^3/4 + O(x^5).$$

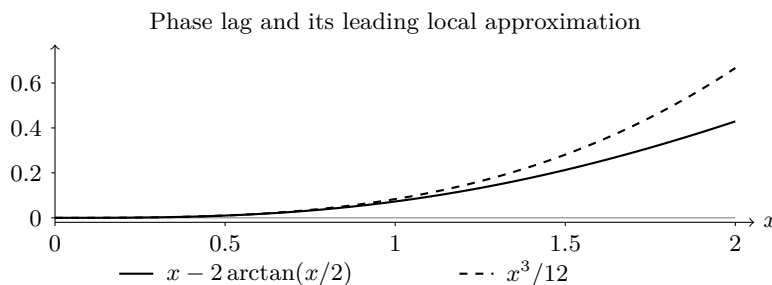
The correct coefficients are $x^4/24$ and $x^3/6$, respectively. Nevertheless $(U, V)(0) = (1, 0)$, and direct squaring gives $U^2 + V^2 = 1$ exactly.

Step 3: Expose the derivative defect. Writing $d = 1 + x^2/4$, we obtain

$$U' - V = \frac{x^3}{4d^2}.$$

This is nonzero for $x \neq 0$. If $y' = U$, differentiation forces $y'' = U'$, so no such function has both proposed derivatives on a nontrivial interval containing zero. Conservation alone does not enforce the differential equation.

Step 4: Identify the altered clock. The half-angle identities give $\tau = 2 \arctan(x/2)$ and $\tau' = 1/d$. Thus $U' = V/d$, $V' = -U/d$: the pair traverses the correct circle with a different speed. The phase lag is $x - \tau = x^3/12 + O(x^5)$ and is positive for $x > 0$ because its derivative is $x^2/(4 + x^2) > 0$.



Question 7

Use series to investigate the directional boundedness of solutions to

$$y''' - 2xy'' - 4y' = 0, \quad y(0) = a, \quad y'(0) = c, \quad y''(0) = 2b.$$

You may use $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$.

Tasks

1. Derive the coefficient recurrence and construct three entire basis functions normalized by the three initial coefficients.
2. Sum the even chain and express the odd chain as an integral involving e^{-t^2} . Verify the resulting general solution and its initial values.
3. Give necessary and sufficient conditions for boundedness as $x \rightarrow +\infty$, as $x \rightarrow -\infty$, and in both directions. Justify cancellation limits with an explicit Gaussian-tail bound.
4. For $a = b = -\sqrt{\pi}/2$, $c = 1$, determine the limit at $+\infty$ and the sign of the response for $x \geq 0$. Sketch that one-sided bounded response on $[0, 4]$.

Question 7 – Solution

Strategy. Build an entire basis from the recurrence, then examine its exponentially amplified coefficients at each end of the real line.

Step 1: Follow the even and odd chains. Matching x^n gives $a_{n+3} = 2a_{n+1}/(n+3)$ for $n \geq 0$. The independent seeds are $a_0 = a$, $a_1 = c$, $a_2 = b$. Ratio tests along each chain give infinite radius. A normalized basis is

$$1, \quad e^{x^2} - 1, \quad H(x) = x + \frac{2x^3}{3} + \frac{4x^5}{15} + \cdots.$$

Step 2: Identify and verify the odd basis. The function $H = e^{x^2} \int_0^x e^{-t^2} dt$ satisfies $H' = 2xH + 1$; differentiating twice proves the third-order equation and its initial triple $(0, 1, 0)$. The other two basis functions also satisfy the equation. Thus

$$y = a + b(e^{x^2} - 1) + ce^{x^2} \int_0^x e^{-t^2} dt.$$

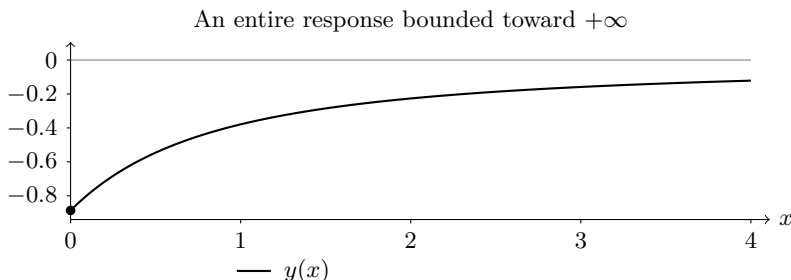
Its initial coefficients are exactly (a, c, b) in degrees zero, one and two.

Step 3: Classify both ends. Put $K = \sqrt{\pi}/2$. At $+\infty$, a nonzero $b + cK$ forces exponential growth. If $b + cK = 0$, the remaining tail tends to zero because, for $x > 0$,

$$0 \leq e^{x^2} \int_x^\infty e^{-t^2} dt \leq \frac{1}{2x}.$$

The inequality follows by replacing 1 with t/x in the integral. Therefore boundedness at $+\infty$ is equivalent to $b + cK = 0$, with limit $a - b$. Oddness of the integral gives the condition $b - cK = 0$ at $-\infty$, also with limit $a - b$. Both conditions hold exactly when $b = c = 0$, leaving constants.

Step 4: Check the selected response. Here $y = -e^{x^2} \int_x^\infty e^{-t^2} dt < 0$ for $x \geq 0$, and the tail estimate gives $y \rightarrow 0$ as $x \rightarrow +\infty$.



Question 8

Investigate the forced singular equation

$$xy''' - y'' = x, \quad x > 0.$$

One wishes to continue a solution to $x = 0$. Analytic continuation, twice continuous differentiability and three times continuous differentiability are different requirements here.

Tasks

1. Attempt an ordinary power series at zero. Identify the precise coefficient equation that obstructs a formal Taylor solution.
2. Solve the equation on $x > 0$ by first setting $z = y''$. Give the complete three-constant family.
3. Determine the highest integer k for which the solutions extend as C^k functions to $[0, \infty)$. Explain why no choice of the homogeneous constants removes the obstruction.
4. Select the constants giving $y = x^3 \log x/6$. Find its endpoint limit, its minimum on $(0, 1]$ and its value at one. Sketch it, showing zero as an excluded endpoint of the original domain.

Question 8 – Solution

Strategy. Let the failed coefficient equation identify a logarithmic term forced by resonance.

Step 1: Locate the formal obstruction. For $y = \sum a_n x^n$, matching x^n gives

$$(n+2)(n+1)(n-1)a_{n+2} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

At $n = 0$ this requires $a_2 = 0$, but at $n = 1$ it requires $0 = 1$. No formal ordinary power series, and hence no analytic solution, exists at zero.

Step 2: Solve the reduced equation. For $z = y''$, $(z/x)' = 1/x$, giving $z = x \log x + Cx$. Integrating twice yields

$$y = A + Bx + \frac{x^3 \log x}{6} + \left(\frac{C}{6} - \frac{5}{36}\right)x^3.$$

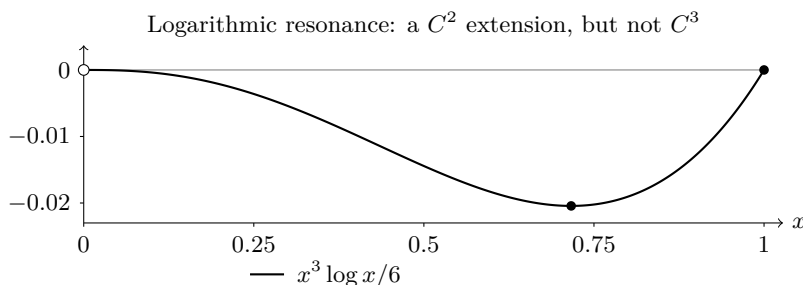
The three homogeneous modes are $1, x, x^3$; the forced logarithm is additional.

Step 3: Classify endpoint smoothness. As $x \downarrow 0$, $y \rightarrow A$, $y' \rightarrow B$ and $y'' = x \log x + Cx \rightarrow 0$. These limits give a C^2 extension. But $y''' = \log x + 1 + C \rightarrow -\infty$, so no C^3 extension exists. Constants can change finite terms, not cancel the coefficient of $\log x$. This is a formal inconsistency, unlike a solvable but divergent coefficient recurrence.

Step 4: Verify the illustrated member. Set $A = B = 0$, $C = 5/6$. Then $y' = x^2(3 \log x + 1)/6$, so

$$x_{\min} = e^{-1/3}, \quad y_{\min} = -\frac{1}{18e}, \quad y(1) = 0, \quad \lim_{x \downarrow 0} y(x) = 0.$$

The derivative changes from negative to positive at this unique minimum. The open marker records the original domain $x > 0$, even though a C^2 extension to zero is possible.



Question 9

A solution is known to satisfy a constant-coefficient homogeneous equation

$$y''' = Ay'' + By' + Cy, \quad (y, y', y'')(0) = (0, 0, 1),$$

where A, B, C are unknown real constants. Its measured Taylor expansion is

$$y(x) = \frac{x^2}{2} + \frac{x^3}{3} + \frac{7x^4}{24} + \frac{x^5}{5} + O(x^6).$$

Tasks

1. Convert the measured coefficients into derivatives and recover A, B, C uniquely. Show the triangular recovery equations.
2. Explain why degree five is the lowest Taylor degree that suffices in general. Predict the coefficient of x^6 from the recovered equation.
3. Derive the full coefficient recurrence and justify that it determines an entire solution with the given data. Rule out any lower-order monic constant-coefficient homogeneous equation for this same solution.
4. Explain why the stated model assumption is essential. Construct analytic functions with the same measured expansion through degree five that do not solve the recovered equation.

Question 9 – Solution

Strategy. Recover constant coefficients from the earliest derivatives at which they enter, while keeping the scope of identification explicit.

Step 1: Recover the constants. The measured derivatives are $b_2 = 1$, $b_3 = 2$, $b_4 = 7$, $b_5 = 24$, where $b_n = y^{(n)}(0)$. The equation and its first two derivatives at zero give

$$b_3 = A, \quad b_4 = A^2 + B, \quad b_5 = A^3 + 2AB + C.$$

Thus $\boxed{A = 2, B = 3, C = 4}$. The triangular equations prove uniqueness.

Step 2: Check the information threshold. Through degree four, only A, B are fixed; arbitrary C gives the same earlier derivatives. Degree five first reveals C , so it is necessary and sufficient in general. The next derivative is

$$b_6 = 2b_5 + 3b_4 + 4b_3 = 77, \quad \boxed{a_6 = 77/720}.$$

Step 3: Extend the series and check minimal order. For ordinary coefficients,

$$a_{n+3} = \frac{2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + 4a_n}{(n+3)(n+2)(n+1)} \quad (n \geq 0).$$

The three seeds determine every coefficient. Equivalently, its constant companion system has the everywhere-convergent matrix exponential, so the series is entire. A homogeneous regular equation of order one or two with $y(0) = y'(0) = 0$ would have only the zero solution by uniqueness, contradicting $y''(0) = 1$. Thus the least possible monic homogeneous order is three.

Step 4: Show the model assumption matters. Let y be the recovered solution and $y_\varepsilon = y + \varepsilon x^6$. For any nonzero ε , this is entire and has the same measured coefficients through degree five. Yet for $L = D^3 - 2D^2 - 3D - 4$,

$$Ly_\varepsilon = \varepsilon(120x^3 - 60x^4 - 18x^5 - 4x^6) \neq 0.$$

A finite Taylor record identifies the equation within the stipulated model class; it does not establish that an arbitrary analytic function belongs to that class.

Question 10

For $\varepsilon > 0$, consider the fourth-order initial-value problem

$$\varepsilon y^{(4)} + y'' = 0, \quad (y, y', y'', y''')(0) = (0, 0, 1, 0).$$

A Taylor approximation accurate for each fixed parameter need not be accurate uniformly as the parameter tends to zero.

Tasks

1. Derive the Taylor recurrence, sum the solution series, and determine its radius for each fixed $\varepsilon > 0$.
2. Prove uniform convergence of y_ε and y'_ε on the whole real line as $\varepsilon \downarrow 0$. Show that y''_ε has no limit at any fixed nonzero x .
3. At fixed $x \neq 0$, investigate the limit of every fixed even-degree Taylor truncation of degree at least two. Explain why taking a parameter limit and truncating the series give incompatible conclusions.
4. Use $\xi = x/\sqrt{\varepsilon}$ to obtain a parameter-independent profile. Give a uniform Taylor error bound for bounded ξ , and plot the profile and its degree-six polynomial on $0 \leq \xi \leq \pi$.

Question 10 – Solution

Strategy. Resolve the shrinking oscillation scale before judging a series truncation.

Step 1: Derive and sum the series. The recurrence is $a_{n+4} = -a_{n+2}/[\varepsilon(n+4)(n+3)]$, with $a_0 = a_1 = a_3 = 0$, $a_2 = 1/2$. Therefore

$$y_\varepsilon = \sum_{j=1}^{\infty} \frac{(-1)^{j+1} x^{2j}}{(2j)! \varepsilon^{j-1}} = \varepsilon (1 - \cos(x/\sqrt{\varepsilon})).$$

The radius is infinite for each fixed positive parameter. The formula verifies all four initial derivatives and the equation directly.

Step 2: Distinguish levels of derivative convergence. The bounds $|y_\varepsilon| \leq 2\varepsilon$ and $|y'_\varepsilon| \leq \sqrt{\varepsilon}$ prove uniform convergence to zero on $\mathbb{R}$. But $y''_\varepsilon = \cos(x/\sqrt{\varepsilon})$. For fixed $x \neq 0$, the sequences $\varepsilon_n = x^2/(2\pi n)^2$ and $\tilde{\varepsilon}_n = x^2/((2n+1)\pi)^2$ give values 1 and -1 . No pointwise limit exists there. The reduced equation $y'' = 0$ cannot retain the original second-derivative datum.

Step 3: Test fixed truncations. The degree-two polynomial is $x^2/2$, which fails to approach zero at fixed nonzero x . For any fixed degree $2m \geq 4$, its highest term is a nonzero constant times ε^{1-m} and dominates the lower powers as $\varepsilon \downarrow 0$. Thus the truncation diverges while the full sum tends to zero. The Taylor remainder is not uniform in this parameter at fixed x .

Step 4: Rescale and certify. With $y_\varepsilon/\varepsilon = 1 - \cos \xi$, Taylor's theorem gives, for $|\xi| \leq R$,

$$\left| 1 - \cos \xi - \sum_{j=1}^m \frac{(-1)^{j+1} \xi^{2j}}{(2j)!} \right| \leq \frac{R^{2m+2}}{(2m+2)!}.$$

Multiplying by ε bounds the original displacement error. For the graph, $m = 3$, $R = \pi$, so the profile error is at most $\pi^8/40320$.

