

Question 1

Consider the variable-coefficient initial-value problem on the real line,

$$(1 + t^2)y''' - 2ty'' + 2y' = 0, \quad (y, y', y'')(0) = (1, 0, 2).$$

A state representation must preserve the scalar equation's leading coefficient and all three initial data.

Tasks

1. Set $X = (y, y', y'')^T$ and derive $X' = A(t)X$. State the initial vector and the largest real interval on which the conversion is regular.
2. Find the solution by testing a polynomial of degree at most two. Verify every component of the system and explain why this trajectory is unique.
3. Let $\Phi' = A\Phi$, $\Phi(0) = I$. Determine $\det \Phi(t)$ without finding the full matrix. Interpret the result as a change in three-dimensional state volume.
4. Decide whether the increasing volume factor forces every individual trajectory to grow. Give an explicit counterexample or proof, and sketch all three components of the prescribed trajectory on $[-2, 2]$.

Question 1 – Solution

Strategy. Normalize first, and distinguish the evolution of a family of states from that of one trajectory.

Step 1: Preserve the derivative chain. Since $1 + t^2 > 0$ for every real t ,

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2/(1+t^2) & 2t/(1+t^2) \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

The conversion is regular on $\mathbb{R}$, and its first two rows enforce $X_2 = X_1'$, $X_3 = X_1''$, so it is reversible.

Step 2: Verify the complete state. The initial data select $y = 1 + t^2$ among quadratic polynomials. Its derivatives are $2t, 2, 0$, and the scalar residual is $-4t + 4t = 0$. Thus

$$X(t) = (1 + t^2, 2t, 2)^T.$$

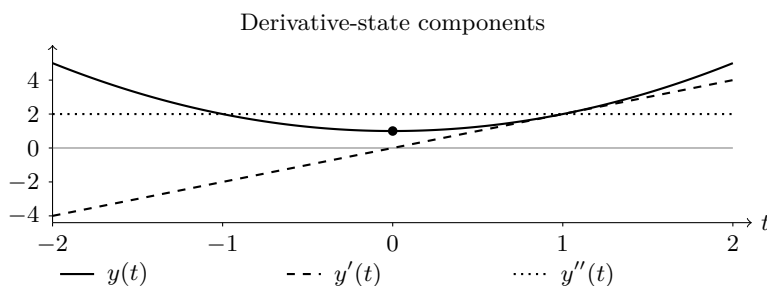
The system's last row gives $(-4t + 4t)/(1 + t^2) = 0 = X_3'$; the other rows and initial vector also agree. Continuity of $A(t)$ gives uniqueness on $\mathbb{R}$.

Step 3: Compute the state-volume factor. Liouville's formula gives

$$\det \Phi(t) = \exp\left(\int_0^t \frac{2v}{1+v^2} dv\right) = 1 + t^2.$$

An initial parallelepiped of states has its oriented volume multiplied by this positive factor. The factor increases with $|t|$, not with t on the entire line.

Step 4: Test the individual-growth claim. For any constant c , $X = (c, 0, 0)^T$ is a constant solution. Hence expansion of state volume does not force every trajectory to grow. The prescribed components are shown below; their different behaviors also caution against that inference.



Question 2

A scalar measurement of a three-state system is

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}, \quad y = X_1 + 2X_2.$$

The measured initial derivatives are $y(0) = 1$, $y'(0) = 0$, $y''(0) = -12$. All coefficients are constant and $t \in \mathbb{R}$.

Tasks

1. Form the matrix O such that $(y, y', y'')^T = OX$. Determine whether the three measured derivatives uniquely determine the state.
2. Derive a scalar third-order equation for y and explicit reconstruction formulas for all three state components in terms of y, y', y'' .
3. Recover $X(0)$ and find the measured response $y(t)$ explicitly.
4. Prove that every solution of your scalar equation reconstructs a solution of the original system. Verify the recovered trajectory and explain why differentiating the measurement has lost no state information.

Question 2 – Solution

Strategy. Invert the map from state coordinates to the measurement's derivative coordinates.

Step 1: Test the measurement map. With $c = (1, 2, 0)$, the rows of O are c, cA, cA^2 . Hence

$$O = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -12 & -22 & -11 \end{pmatrix}, \quad \det O = -15 \neq 0.$$

The derivative triple determines the state uniquely.

Step 2: Convert and reconstruct. The identity $A^3 + 6A^2 + 11A + 6I = 0$ gives

$$y''' + 6y'' + 11y' + 6y = 0.$$

Solving $OX = (y, y', y'')^T$ gives

$$\begin{aligned} X_1 &= (-33y - 22y' - 4y'')/15, \\ X_2 &= (24y + 11y' + 2y'')/15, \\ X_3 &= (-12y + 2y' - y'')/15. \end{aligned}$$

Step 3: Solve the recovered IVP. The measured data give $X(0) = (1, 0, 0)^T$. The scalar roots are $-1, -2, -3$; imposing the measured derivative triple yields

$$y = -3e^{-t} + 9e^{-2t} - 5e^{-3t}.$$

The state is $X = (f, f', f'')^T$, where $f = 3e^{-t} - 3e^{-2t} + e^{-3t}$. In particular, $f + 2f' = y$ and $(f, f', f'')(0) = (1, 0, 0)$.

Step 4: Prove full equivalence. If $Z = (y, y', y'')^T$, then $Z' = BZ$ with last row of B equal to $(-6, -11, -6)$. Direct multiplication gives $OA = BO$, so for any scalar solution, $X = O^{-1}Z$ satisfies $X' = O^{-1}BZ = AX$. Its measurement is y by the first row of O . Conversely every state solution produces this scalar equation. The recovered f satisfies $f''' + 6f'' + 11f' + 6f = 0$, checking the original last row directly. Invertibility of O proves that no hidden state was discarded.

Question 3

Only one coordinate of the following system is observed:

$$X' = AX, \quad A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad y = X_1, \quad t \geq 0.$$

A proposed scalar model is obtained by replacing the characteristic polynomial of A with a differential operator.

Tasks

1. Derive the lowest-order monic constant-coefficient homogeneous scalar equation satisfied by every possible measurement y . Prove that this order is minimal.
2. For a prescribed measurement, reconstruct all compatible states. Determine exactly which initial states are bounded on $[0, \infty)$.
3. Write the third-order scalar equation suggested by the characteristic polynomial. Identify its extra scalar mode, and find the initial constraint that excludes that mode.
4. Decide whether an arbitrarily long, exact record of y can certify boundedness of the full state. Exhibit two states with identical measurements and different boundedness behavior.

Question 3 – Solution

Strategy. Separate the measured Jordan block from the invisible unstable coordinate.

Step 1: Find the true measurement equation. From $y' = -y + X_2$ and $X_2' = -X_2$,

$$y'' + 2y' + y = 0, \quad y = (a + bt)e^{-t}.$$

Both e^{-t} and te^{-t} occur as measurements and are independent. Any constant-coefficient annihilator must have a double root at -1 , so order two is minimal.

Step 2: Recover the visible and hidden states. Every compatible state is

$$X(t) = ((a + bt)e^{-t}, be^{-t}, ce^{2t})^T, \quad c \in \mathbb{R}.$$

The first two coordinates always decay. Therefore the full state is bounded if and only if $c = X_3(0) = 0$. The measurement does not determine c .

Step 3: Remove the spurious scalar mode. The characteristic polynomial gives

$$(D - 2)(D + 1)^2 y = 0, \quad \text{equivalently} \quad y''' - 3y' - 2y = 0.$$

Its general solution adds de^{2t} to the actual measurement family. Put $r = (D + 1)^2 y$. Then $r' = 2r$ and $r = 9de^{2t}$, so the precise exclusion condition is

$$y''(0) + 2y'(0) + y(0) = 0.$$

The characteristic-polynomial equation is necessary, but without this constraint it is not sufficient for a scalar function to be a measurement.

Step 4: Test what observation can certify. The initial states $(1, 0, 0)^T$ and $(1, 0, 1)^T$ both produce $y = e^{-t}$ forever. Their full trajectories are $(e^{-t}, 0, 0)^T$ and $(e^{-t}, 0, e^{2t})^T$; one is bounded and the other is not. Thus even an exact infinite measurement record cannot certify boundedness without additional information about $X_3(0)$.

Question 4

A smooth system on the entire real line is

$$x' = z, \quad z' = tw, \quad w' = 0.$$

Eliminating z, w appears to produce a third-order scalar equation for x . The elimination must be examined at $t = 0$.

Tasks

1. Eliminate z, w for $t \neq 0$. Explain why the resulting scalar equation is singular at the origin although the original system is regular there.
2. Solve the original system for arbitrary data $(x, z, w)(0) = (a, b, c)$. Determine which derivatives of x at zero encode these three constants.
3. Classify all $C^3(\mathbb{R})$ solutions of the scalar equation. Prove that each reconstructs a unique smooth system solution, explicitly treating the origin.
4. Decide whether $x(0), x'(0), x''(0)$ alone determine a unique scalar solution. Give a one-parameter counterexample if needed, and sketch three members sharing the zero derivative triple on $[-1, 1]$.

Question 4 – Solution

Strategy. Retain the state coordinate that division by t conceals at the singular point.

Step 1: Eliminate on the regular subintervals. Since $x'' = tw$, for $t \neq 0$ we have $w = x''/t$. Differentiating gives

$$tx''' - x'' = 0.$$

The leading coefficient vanishes at zero. This singularity came from the chosen scalar coordinate; the original system matrix is continuous everywhere.

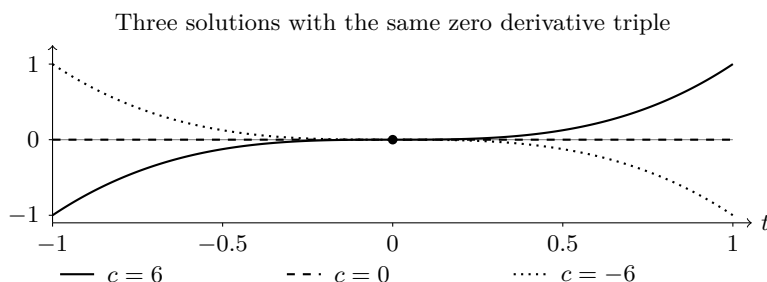
Step 2: Solve before dividing. Integrating successively gives

$$w = c, \quad z = b + ct^2/2, \quad x = a + bt + ct^3/6.$$

Thus $(x, x', x'')(0) = (a, b, 0)$, while $x'''(0) = c$. The missing state coordinate is encoded in the third derivative, not in the usual second derivative.

Step 3: Check global equivalence. On either side of zero, $(x''/t)' = 0$, so $x = a_{\pm} + b_{\pm}t + c_{\pm}t^3/6$. Continuity of x, x' matches $a_{\pm}, b_{\pm}$, and continuity of x''' matches $c_{\pm}$. All C^3 solutions therefore have the single global polynomial form above. Reconstruct $z = x'$ and $w = x''/t$ off zero, setting $w(0) = x'''(0)$. These are smooth and satisfy all three system equations. At zero, the scalar equation itself requires $x''(0) = 0$.

Step 4: Explain the apparent nonuniqueness. The family $x = ct^3/6$ has $(x, x', x'')(0) = (0, 0, 0)$ for every c . It corresponds to distinct system data $(0, 0, c)$, so it does not contradict uniqueness for the regular system. The plotted members use $c = -6, 0, 6$.



Question 5

Two coupled displacements satisfy

$$x'' + 2x - z = 0, \quad z'' + 2z - x = 0,$$

with $(x, x', z, z')(0) = (1, 0, 0, 0)$. Time is measured in units that make the coefficients dimensionless.

Tasks

1. Write a four-state first-order system using the order (x, x', z, z') . Eliminate z to obtain a fourth-order scalar equation for x and translate all four initial values.
2. Introduce $u = x + z$, $v = x - z$. Solve the resulting independent equations and recover x, z .
3. Prove that every solution of the scalar fourth-order equation reconstructs a unique solution of the coupled equations. Verify the prescribed solution and its initial data.
4. Derive a positive conserved energy, express it in u, v , and use it to prove boundedness. Sketch both displacements for $0 \leq t \leq 12$ and explain how two different frequencies appear.

Question 5 – Solution

Strategy. Compare the derivative-state, scalar-elimination and normal-mode descriptions of the same four-dimensional motion.

Step 1: Convert and transfer the initial state. For $X = (x, x', z, z')^T$,

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} X.$$

Since $z = x'' + 2x$, substitution into the second equation gives

$$x^{(4)} + 4x'' + 3x = 0, \quad (x, x', x'', x''')(0) = (1, 0, -2, 0).$$

Step 2: Solve the normal modes. Adding and subtracting give $u'' + u = 0$, $v'' + 3v = 0$, with initial displacements one and velocities zero. Thus

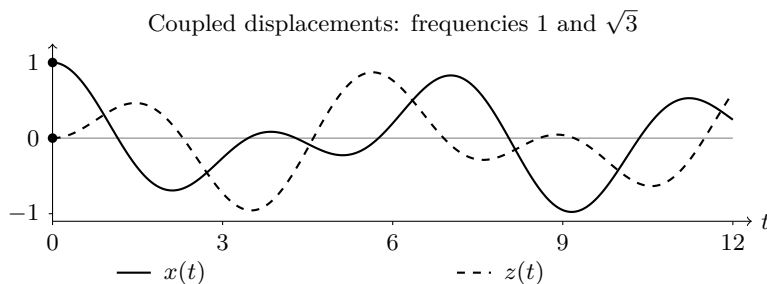
$$x = \frac{1}{2}(\cos t + \cos(\sqrt{3}t)), \quad z = \frac{1}{2}(\cos t - \cos(\sqrt{3}t)).$$

Step 3: Prove reversibility. For any scalar solution, define $z = x'' + 2x$. The first coupled equation is then an identity, and the second has residual $x^{(4)} + 4x'' + 3x = 0$. This formula also proves uniqueness of z . The displayed cosines satisfy the scalar factors $(D^2 + 1)(D^2 + 3)$ and give $z(0) = z'(0) = 0$ as required.

Step 4: Check energy and boundedness. Multiplying the equations by x', z' and adding proves conservation of

$$E = \frac{1}{2}(x'^2 + z'^2) + x^2 + z^2 - xz = \frac{1}{4}(u'^2 + u'^2 + v'^2 + 3v'^2) = 1.$$

Since $x^2 + z^2 - xz \geq (x^2 + z^2)/2$, both displacements are bounded; the explicit formulas sharpen this to $|x|, |z| \leq 1$. Each displacement mixes the frequencies 1 and $\sqrt{3}$.



Question 6

Let $D = d/dt$. Ordered differential factors act from right to left in

$$(D + 1)(D - t)(D - 1)y = 0, \quad (y, y', y'')(0) = (0, 0, 1).$$

Here $(D - t)f = f' - tf$. Multiplication by t does not commute with differentiation.

Tasks

1. Define $q_1 = y$, $q_2 = (D - 1)y$, $q_3 = (D - t)q_2$. Derive the first-order system for q and its initial vector.
2. Expand the scalar third-order equation and show that the change between (y, y', y'') and q is invertible for every real t .
3. Solve the triangular system by successive integrating factors. Give an explicit nested-integral formula for y and verify the initial data and equation through the state relations.
4. Compare the given operator with $(D - t)(D + 1)(D - 1)$. Compute their difference, and exhibit a function that solves the reordered equation but not the original one.

Question 6 – Solution

Strategy. Keep the factors in their stated order and regard intermediate derivatives as state coordinates.

Step 1: Form the ordered cascade. The definitions give

$$\boxed{q'_1 = q_1 + q_2, \quad q'_2 = tq_2 + q_3, \quad q'_3 = -q_3, \quad q(0) = (0, 0, 1)^T.}$$

In particular, $q_3 = y'' - (t+1)y' + ty$; using only y'' as this coordinate would lose the variable-coefficient terms.

Step 2: Expand and invert the coordinate map. Differentiating the expression for q_3 and adding q_3 yields

$$\boxed{y''' - ty'' - 2y' + (t+1)y = 0.}$$

The coordinate matrix is

$$q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ t & -(t+1) & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, \quad \det = 1.$$

Its inverse relations are $y = q_1$, $y' = q_1 + q_2$ and $y'' = q_1 + (t+1)q_2 + q_3$, so the conversion is globally reversible.

Step 3: Integrate in triangular order. Successive integrating factors give

$$q_3 = e^{-t}, \quad q_2 = e^{t^2/2} \int_0^t e^{-v-v^2/2} dv,$$

$$\boxed{y(t) = e^t \int_0^t e^{-u+u^2/2} \left(\int_0^u e^{-v-v^2/2} dv \right) du.}$$

These oriented integrals apply to every real t . Their derivatives give the three cascade equations. At zero the state is $(0, 0, 1)$, and the inverse relations give $(y, y', y'') = (0, 0, 1)$. Finally $q'_3 + q_3 = 0$ verifies the original ordered product directly.

Step 4: Quantify the order error. The commutator $(D+1)(D-t) - (D-t)(D+1)$ is multiplication by -1 . Thus the original operator minus the reordered one equals $-(D-1)$. The reordered operator is $D^3 - tD^2 - D + t$, whereas the original has the extra $-D + 1$. For $y = e^{-t}$ the reordered product vanishes, but the original gives $2e^{-t} \neq 0$. Reordering these factors changes the solution space.

Question 7

A controlled third derivative moves a state from rest to rest in unit time:

$$y''' = u(t), \quad (y, y', y'')(0) = (0, 0, 0), \quad (y, y', y'')(1) = (1, 0, 0).$$

For square-integrable controls u on $[0, 1]$, interpret the derivative chain through its integral solution. The effort is $J(u) = \int_0^1 u(t)^2 dt$.

Tasks

1. Write the first-order state system and its transition matrix. Derive the three terminal constraints as integrals of u .
2. Find the unique quadratic control satisfying the terminal conditions. Integrate it to obtain the displacement.
3. Prove this quadratic is the unique minimum-effort control among all square-integrable admissible controls, up to equality almost everywhere. Compute the minimum effort.
4. Verify both endpoint states and explain why the minimization proof covers controls that are not polynomials. Sketch the displacement and $u/60$, explicitly labeling the scaling.

Question 7 – Solution

Strategy. Express the terminal state as three moments, then use orthogonality to prove optimality over the full control class.

Step 1: Derive the terminal moments. For $X = (y, y', y'')^T$, $X' = NX + e_3u$, where N has ones just above the diagonal and zeros elsewhere. Since $N^3 = 0$,

$$e^{Nr} = \begin{pmatrix} 1 & r & r^2/2 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}, \quad X(1) = \int_0^1 \begin{pmatrix} (1-t)^2/2 \\ 1-t \\ 1 \end{pmatrix} u(t) dt.$$

The required three moments are $(1, 0, 0)^T$.

Step 2: Solve within the quadratic space. For $u = a + bt + ct^2$, the constraints are

$$a/6 + b/24 + c/60 = 1, \quad a/2 + b/6 + c/12 = 0, \quad a + b/2 + c/3 = 0.$$

Their solution is

$$u_* = 60 - 360t + 360t^2, \quad y_* = 10t^3 - 15t^4 + 6t^5.$$

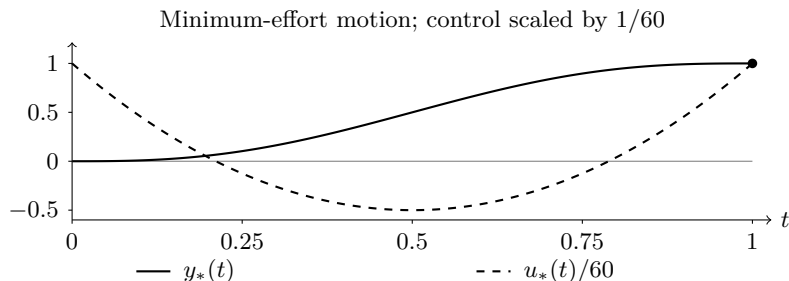
The moment kernels span all quadratics. A quadratic in the homogeneous null space is orthogonal to itself and hence zero, proving uniqueness here.

Step 3: Prove global minimum effort. For any admissible $u = u_* + v$, all three terminal moments of v vanish. Because u_* is in their span, $\int_0^1 u_* v = 0$. Therefore

$$J(u) = J(u_*) + \int_0^1 v^2 dt \geq J(u_*) = 720.$$

Equality holds exactly when $v = 0$ almost everywhere. This argument only uses square integrability and moment constraints, not polynomial form for v .

Step 4: Verify the endpoint motion. We have $y'_* = 30t^2(1-t)^2$ and $y''_* = 60t(1-t)(1-2t)$, so the two endpoint states are exactly $(0, 0, 0)$ and $(1, 0, 0)$; differentiating again gives u_* . The graph uses $u_*/60$ so the control and displacement are both readable.



Question 8

The functions $1, t, e^t$ form a scalar basis for $y''' - y'' = 0$. For the derivative state $X = (y, y', y'')^T$, let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

A fundamental matrix need not equal the identity at the chosen base time.

Tasks

1. Form the fundamental matrix $F(t)$ from the given scalar basis and its derivatives. Check $F' = AF$ and compute its determinant.
2. Derive the transition matrix $\Phi(t, s) = F(t)F(s)^{-1}$ explicitly. Verify its normalization, composition law and determinant.
3. Use this transition matrix to solve $y''' - y'' = 1$ with $(y, y', y'')(1) = (0, 0, 0)$ for every real t . Treat the integral orientation when $t < 1$.
4. Explain why using $F(t)$ directly as the transition from time zero gives wrong initial data in general. Prove that replacing F by FC , for any constant invertible C , leaves the true transition matrix unchanged.

Question 8 – Solution

Strategy. Normalize a scalar-derived fundamental matrix at the actual base time before transporting states or forcing.

Step 1: Lift the scalar basis. The derivative columns give

$$F(t) = \begin{pmatrix} 1 & t & e^t \\ 0 & 1 & e^t \\ 0 & 0 & e^t \end{pmatrix}, \quad \det F = e^t \neq 0.$$

Differentiating the rows gives $F' = AF$, including the last row because $(e^t)' = e^t$. Thus F is fundamental on the entire real line.

Step 2: Normalize and compose. With $r = t - s$, direct multiplication gives

$$\Phi(t, s) = \begin{pmatrix} 1 & r & e^r - 1 - r \\ 0 & 1 & e^r - 1 \\ 0 & 0 & e^r \end{pmatrix}.$$

It equals I at $t = s$, and $\det \Phi = e^{t-s}$. The identity $F(t)F(s)^{-1}F(s)F(a)^{-1} = F(t)F(a)^{-1}$ proves $\Phi(t, s)\Phi(s, a) = \Phi(t, a)$ for all real base times.

Step 3: Transport the forcing in both directions. The zero initial vector at time one gives

$$X(t) = \int_1^t \Phi(t, v)e_3 dv, \quad y(t) = e^{t-1} - 1 - (t-1) - \frac{1}{2}(t-1)^2.$$

Indeed its first component integrates $e^{t-v} - 1 - (t-v)$. For $t < 1$ the integral is oriented: it is minus the integral from t to 1. Writing $r = t - 1$, we obtain $y' = e^r - 1 - r$, $y'' = e^r - 1$, $y''' = e^r$. These give the zero state at $t = 1$ and residual $y''' - y'' = 1$ on all of $\mathbb{R}$.

Step 4: Separate basis from transition. Here $F(0) \neq I$. For example, $F(t)e_3 = (e^t, e^t, e^t)^T$ starts at $(1, 1, 1)^T$, not at e_3 . The correct expression is $F(t)F(0)^{-1}e_3$. For any constant invertible C ,

$$(F(t)C)(F(s)C)^{-1} = F(t)CC^{-1}F(s)^{-1} = \Phi(t, s).$$

A basis change alters coordinates of solution columns, but not the transport of a specified physical state.

Question 9

A fourth-order oscillator has a repeated quadratic factor:

$$y^{(4)} + 2y'' + y = 0, \quad t \geq 0.$$

Introduce the state

$$W = (y, y', y'' + y, y''' + y')^T.$$

All four scalar initial derivatives are arbitrary real numbers.

Tasks

1. Derive the system for W and show that this coordinate change is invertible. Identify the autonomous oscillator inside the system.
2. Write the full solution in terms of $a = y(0)$, $b = y'(0)$, $A = y''(0) + y(0)$ and $B = y'''(0) + y'(0)$.
3. Give necessary and sufficient conditions on the four initial derivatives for boundedness. Prove necessity even when the secular terms oscillate and have infinitely many zeros.
4. Find a conserved quadratic expression involving only the last two state coordinates. Decide whether the full four-state system can have a conserved positive-definite quadratic form, and justify your answer.

Question 9 – Solution

Strategy. Expose an internal oscillator that resonantly forces the displacement oscillator.

Step 1: Reveal the oscillator cascade. The original equation gives

$$W' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} W.$$

The coordinate map is triangular with determinant one; its inverse is $(y, y', y'', y''') = (W_1, W_2, W_3 - W_1, W_4 - W_2)$. The last pair satisfies $W_3' = W_4$, $W_4' = -W_3$ independently.

Step 2: Solve the resonant forcing. We have $W_3 = A \cos t + B \sin t$ and $y'' + y = W_3$. A resonant particular solution is $(A/2)t \sin t - (B/2)t \cos t$. Enforcing the first two data gives

$$y = a \cos t + (b + B/2) \sin t + (A/2)t \sin t - (B/2)t \cos t.$$

It has $y''(0) = A - a$ and $y'''(0) = B - b$, checking all four data. Applying $D^2 + 1$ twice verifies the fourth-order equation.

Step 3: Classify bounded data. If $A = B = 0$, only bounded sine and cosine remain. If $(A, B) \neq (0, 0)$, choose a phase θ with $A \sin \theta - B \cos \theta = \sqrt{A^2 + B^2}$. Along $t_n = \theta + 2\pi n \geq 0$, the secular part grows as $t_n \sqrt{A^2 + B^2}/2$, while the other terms remain bounded. Consequently

$$y \text{ bounded} \iff y''(0) = -y(0), \quad y'''(0) = -y'(0).$$

The whole state is bounded exactly on this same two-dimensional subspace.

Step 4: Distinguish two kinds of conserved energy. The internal oscillator has

$$E_{\text{int}} = \frac{1}{2}(W_3^2 + W_4^2) = \frac{1}{2}(A^2 + B^2), \quad E'_{\text{int}} = W_3 W_4 - W_4 W_3 = 0.$$

This is not positive definite on the full state space. If a conserved positive-definite form $W^T P W$ existed, its smallest eigenvalue would bound $\|W(t)\|^2$ by a constant for every initial state. The unbounded solutions just constructed contradict that conclusion. Purely imaginary characteristic roots with repeated factors therefore do not imply boundedness of all trajectories.

Question 10

For the stable scalar equation $(D + 1)^3 y = 0$, introduce scaled state coordinates

$$q_1 = y, \quad q_2 = (y' + y)/4, \quad q_3 = (y'' + 2y' + y)/16.$$

Let $\|\cdot\|_2$ be the Euclidean norm and consider $t \geq 0$.

Tasks

1. Derive the constant system for q and the inverse map to (y, y', y'') . Explain why all characteristic roots are still -1 .
2. Solve the system for $q(0) = (0, 0, 1)^T$. Show explicitly that its Euclidean norm can exceed its initial value even though every solution tends to zero.
3. Set $w = (q_1, 4q_2, 16q_3)^T$. Prove a quadratic norm estimate that is strictly decreasing for nonzero states, using

$$2w_2(w_1 + w_3) \leq \sqrt{2}\|w\|_2^2.$$

Obtain an exponential bound for $\|q(t)\|_2$ valid for every initial state.

4. Explain why transient growth is consistent with asymptotic stability. For the prescribed initial state, sketch $\|q(t)\|_2$ and $\|w(t)\|_2/16$, labeling the normalization, and distinguish the two quantities.

Question 10 – Solution

Strategy. Compare Euclidean growth in scaled coordinates with decay in a suitable weighted norm.

Step 1: Derive the scaled cascade. With N having ones just above the diagonal,

$$q' = (-I + 4N)q, \quad y = q_1, \quad y' = 4q_2 - q_1, \quad y'' = 16q_3 - 8q_2 + q_1.$$

The coordinate map is invertible, so this is a similarity transformation of the scalar companion system. Its triangular matrix has three eigenvalues -1 .

Step 2: Compute an actual transient. Since $N^3 = 0$, $e^{(-I+4N)t} = e^{-t}(I + 4tN + 8t^2N^2)$. Therefore

$$q = e^{-t}(8t^2, 4t, 1)^T, \quad \|q(t)\|_2 = e^{-t}\sqrt{64t^4 + 16t^2 + 1}.$$

The initial norm is one, but $\|q(1)\|_2 = 9/e > 1$. Every state still tends to zero because its components are polynomials times e^{-t} .

Step 3: Prove a uniform weighted estimate. The new state satisfies $w' = (-I + N)w$, so

$$\frac{d}{dt}\|w\|_2^2 = -2\|w\|_2^2 + 2w_2(w_1 + w_3) \leq -(2 - \sqrt{2})\|w\|_2^2.$$

The stated inequality follows from Cauchy–Schwarz and $2\sqrt{2}|w_2|\sqrt{w_1^2 + w_3^2} \leq \sqrt{2}\|w\|_2^2$. With $\alpha = 1 - 1/\sqrt{2} > 0$, integration and $\|q\|_2 \leq \|w\|_2 \leq 16\|q\|_2$ give

$$\|q(t)\|_2 \leq 16e^{-\alpha t}\|q(0)\|_2.$$

Step 4: Interpret coordinate-dependent growth. Asymptotic stability allows a finite transient; it does not require every norm to decrease at every instant. For this trajectory, $\|w\|_2/16 = e^{-t}\sqrt{t^4/4 + t^2 + 1}$ decreases strictly, whereas $\|q\|_2$ has a pronounced transient. They measure the same state with different weights.

