

**Question 1**

Consider the initial-value problem, for  $t \geq 0$ ,

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0, \quad (y, y', y'', y''')(0) = (1, 0, 0, 0).$$

A student writes  $(s+1)^4 Y(s) = s^3$  and concludes that the initial displacement can be represented by a single term from the fourth derivative.

**Tasks**

1. Derive the correct transformed equation, displaying the initial terms contributed by every derivative. Identify the student's error.
2. Find  $Y(s)$  and invert it by organizing the numerator in powers of  $s + 1$ .
3. Verify the differential equation and all four initial values directly. Determine whether the response ever increases or changes sign.
4. Find a half-plane of convergence and the initial and final limits. Explain why a quadruple pole does not prevent decay, and sketch the response.

## Question 1 – Solution

**Strategy.** Collect every initial contribution before exploiting the repeated shifted pole.

**Step 1: Transform all derivatives.** The derivative transforms are  $s^4Y - s^3$ ,  $s^3Y - s^2$ ,  $s^2Y - s$ , and  $sY - 1$ . Thus

$$(s+1)^4Y = s^3 + 4s^2 + 6s + 4.$$

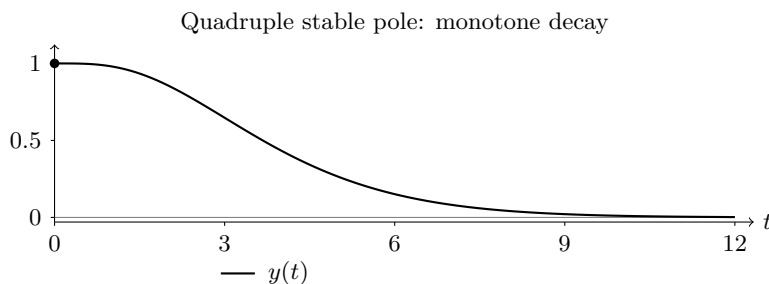
The student omitted the initial terms from  $4y''' + 6y'' + 4y'$ .

**Step 2: Use shifted powers.** Writing  $z = s + 1$  makes the numerator  $z^3 + z^2 + z + 1$ . Therefore

$$Y = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}, \quad \boxed{y(t) = e^{-t} \left( 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \right)}.$$

**Step 3: Check the state and shape.** For  $y = e^{-t}p(t)$ ,  $(D+1)^4y = e^{-t}p^{(4)} = 0$ . Moreover  $y' = -t^3e^{-t}/6$ , so  $y'(0) = y''(0) = y'''(0) = 0$  and  $y(0) = 1$ . Every term in  $p$  is positive on  $t \geq 0$ ; hence  $y > 0$  and  $y' < 0$  for  $t > 0$ .

**Step 4: Check convergence and limits.** The transform converges absolutely for  $\operatorname{Re} s > -1$ . Also  $\lim_{s \rightarrow \infty} sY = 1$  and  $\lim_{s \rightarrow 0} sY = 0$ . The final-value theorem applies because all poles of  $sY$  lie at  $-1$ . Polynomial growth from multiplicity is dominated by  $e^{-t}$ , so  $y(t) \rightarrow 0$ .



**Question 2**

A sixth-order initial-value problem is

$$(D^2 + 1)^3 y = 0, \quad y^{(j)}(0) = 0 \quad (0 \leq j \leq 4), \quad y^{(5)}(0) = 1,$$

where  $D = d/dt$  and  $t \geq 0$ . To invert a repeated quadratic pole, introduce  $H_a(s) = 1/(s^2 + a^2)$  for  $a > 0$ .

**Tasks**

1. Expand the differential operator and transform the problem. Explain why only one initial term survives.
2. Starting with  $\mathcal{L}^{-1}H_a = \sin(at)/a$ , use differentiation with respect to  $a$  twice to invert  $(s^2 + a^2)^{-3}$ . Then set  $a = 1$ .
3. Verify the equation and initial state without relying on the transform calculation. Identify the first nonzero Taylor term.
4. Describe the pole multiplicities and a convergence half-plane. Decide whether the solution is bounded and whether the final-value theorem can be used, giving a decisive argument for each claim.

## Question 2 – Solution

**Strategy.** Parameter differentiation resolves repeated quadratic poles while a Taylor check tests the high-order initial data.

**Step 1: Transform the sixth-order problem.** The equation is  $y^{(6)} + 3y^{(4)} + 3y'' + y = 0$ . Only  $\mathcal{L}(y^{(6)}) = s^6 Y - 1$  has a nonzero initial subtraction, so

$$Y(s) = \frac{1}{(s^2 + 1)^3}.$$

**Step 2: Differentiate a transform family.** Since  $\partial_a(s^2 + a^2)^{-m} = -2ma(s^2 + a^2)^{-m-1}$ ,

$$h_2(t) = \frac{\sin(at) - at \cos(at)}{2a^3}, \quad h_3(t) = -\frac{1}{4a} \partial_a h_2(t) = \frac{3 \sin(at) - 3at \cos(at) - a^2 t^2 \sin(at)}{8a^5}.$$

Differentiation under the integral is valid for  $a$  in a compact positive interval and  $\operatorname{Re} s > 0$ , since the derivatives have polynomial growth. Hence

$$y(t) = \frac{3 \sin t - 3t \cos t - t^2 \sin t}{8}.$$

**Step 3: Verify independently.** Each term is the real or imaginary part of  $e^{it}$  times a polynomial of degree at most two;  $(D - i)^3$  annihilates these complex modes, and the conjugate factor annihilates their conjugates. Thus  $(D^2 + 1)^3 y = 0$ . Expanding the displayed trigonometric expression gives

$$y(t) = \frac{t^5}{120} - \frac{t^7}{1680} + O(t^9).$$

This proves the five zero initial derivatives and  $y^{(5)}(0) = 1$ .

**Step 4: Distinguish oscillation from boundedness.** There are triple poles at  $s = \pm i$ ;  $\operatorname{Re} s > 0$  gives absolute convergence. Along  $t_n = \pi/2 + 2\pi n$ ,  $y(t_n) = (3 - t_n^2)/8 \rightarrow -\infty$ , so the response is unbounded. The poles of  $sY$  on the imaginary axis invalidate the final-value theorem. The formal value  $\lim_{s \rightarrow 0} sY = 0$  is not a time limit.

**Question 3**

Let  $u(t - a)$  denote the unit step at  $a$ . Solve the delayed-ramp problem

$$(D + 2)^3 y = u(t - 1)(t - 1), \quad (y, y', y'')(0) = (0, 0, 0), \quad t \geq 0.$$

The value assigned to  $u(0)$  does not affect an ordinary Laplace transform.

**Tasks**

1. Transform the initial-value problem, taking care to shift the entire ramp.
2. Obtain a closed form for the response on each side of  $t = 1$ . Display a partial-fraction decomposition for the unshifted transform.
3. Check the equation on the two open time intervals and determine exactly which derivative first jumps at  $t = 1$ . State the global smoothness of  $y$ .
4. Prove the response is positive after switching, find its large-time linear asymptote, and sketch it together with that asymptote on  $t \geq 1$ .

## Question 3 – Solution

**Strategy.** Solve a zero-state ramp response, then translate time and check the switching jet.

**Step 1: Transform the shifted input.** All initial terms vanish. The second shifting theorem gives

$$Y(s) = \frac{e^{-s}}{s^2(s+2)^3}, \quad \operatorname{Re} s > 0.$$

**Step 2: Invert before translating.** The unshifted fraction decomposes as

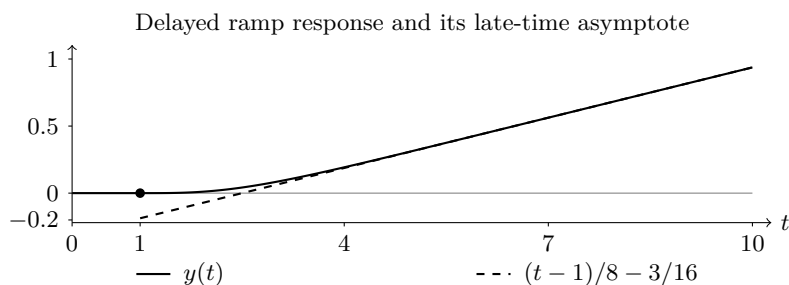
$$\frac{1}{s^2(s+2)^3} = -\frac{3}{16s} + \frac{1}{8s^2} + \frac{3}{16(s+2)} + \frac{1}{4(s+2)^2} + \frac{1}{4(s+2)^3}.$$

Thus, with  $r = t - 1$ ,

$$g(r) = \frac{r}{8} - \frac{3}{16} + e^{-2r} \left( \frac{3}{16} + \frac{r}{4} + \frac{r^2}{8} \right), \quad y(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ g(t-1), & t > 1. \end{cases}$$

**Step 3: Verify forcing and regularity.** The exponential term is annihilated by  $(D+2)^3$ ; applying it to  $r/8 - 3/16$  gives  $r$ . Also  $g(r) = r^4/24 + O(r^5)$  as  $r \downarrow 0$ . Consequently  $y, y', y'', y'''$  match zero at switching, while  $y^{(4)}(1+) - y^{(4)}(1-) = 1$ . The solution is  $C^3$  but not  $C^4$ ; the original equation also holds at  $t = 1$ .

**Step 4: Establish shape and asymptote.** Convolution gives  $g(r) = \int_0^r (r-v)v^2 e^{-2v}/2 dv > 0$  for  $r > 0$ . The exponential correction tends to zero, so the asymptote is  $y = (t-1)/8 - 3/16$ , approached from above. It is drawn only for  $t \geq 1$ .



**Question 4**

Consider the causal, distributional equation

$$y^{(4)} + 2y''' + y'' = 3\delta(t-1) + \delta'(t-1).$$

The response is zero for  $t < 0$ , with no impulse at the origin. Here  $\delta'$  is the distributional derivative of  $\delta$ , and  $[v]_1 = v(1+) - v(1-)$  denotes a jump.

**Tasks**

1. Find the transformed response, using the zero prehistory and both delayed distribution transforms.
2. First invert  $H(s) = 1/[s^2(s+1)^2]$ , then use differentiation to obtain a real-time formula for  $y$ .
3. Derive all four jumps  $[y]_1, [y']_1, [y'']_1, [y''']_1$  by matching the coefficients of distributional derivatives. Verify them from your formula.
4. Determine the motion before the impulse, the eventual linear asymptote and the smoothness at switching. Explain why assigning an isolated value at  $t = 1$  cannot remove the derivative jump. Sketch the response and its asymptote.

## Question 4 – Solution

**Strategy.** Separate the impulse kernel from the input derivative, then match distributions rather than imposing false continuity conditions.

**Step 1: Transform the distributions.** For positive delay,  $\mathcal{L}\delta(t-1) = e^{-s}$  and  $\mathcal{L}\delta'(t-1) = se^{-s}$ . Zero prehistory therefore gives

$$Y = e^{-s}(s+3)H(s), \quad H(s) = \frac{1}{s^2(s+1)^2}, \quad \operatorname{Re} s > 0.$$

**Step 2: Invert the kernel.** Since  $H = -2/s + 1/s^2 + 2/(s+1) + 1/(s+1)^2$ ,  $h(r) = r - 2 + (r+2)e^{-r}$  and  $h(0) = 0$ . Thus  $sH$  transforms  $h'$ , without an additional origin impulse. For  $r = t - 1$ ,

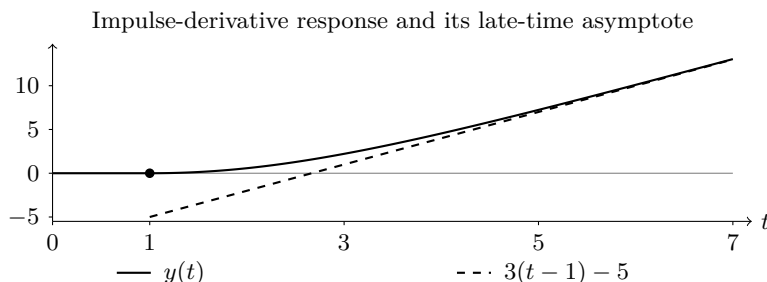
$$y(t) = u(t-1)k(r), \quad k(r) = 3r - 5 + (2r+5)e^{-r}.$$

**Step 3: Match every singular coefficient.** Put  $J_j = [y^{(j)}]_1$ . The coefficients of  $\delta''', \delta'', \delta', \delta$  in the left side are, respectively,

$$J_0, \quad J_1 + 2J_0, \quad J_2 + 2J_1 + J_0, \quad J_3 + 2J_2 + J_1.$$

Matching  $0, 0, 1, 3$  gives  $(J_0, J_1, J_2, J_3) = (0, 0, 1, 1)$ . Directly,  $(k, k', k'', k''')(0) = (0, 0, 1, 1)$ ; away from  $r = 0$ ,  $D^2(D+1)^2k = 0$ , completing the distributional verification.

**Step 4: Interpret the motion.** The response vanishes before  $t = 1$  and approaches  $3(t-1) - 5$  afterward. It is  $C^1$ , but its one-sided second derivatives differ. Changing a function at one point cannot change those one-sided limits or its distribution.



**Question 5**

An unstable homogeneous mode may or may not appear in a forced response. For real parameters  $a, b$ , consider

$$y''' + y'' - 2y' = e^{-t}, \quad y(0) = 0, \quad y'(0) = a, \quad y''(0) = b, \quad t \geq 0.$$

**Tasks**

1. Derive  $Y(s)$  and identify all candidate poles before cancellation.
2. Find the necessary and sufficient relation between  $a, b$  for a bounded response. Give a complete inverse transform that proves both necessity and sufficiency.
3. Among the bounded responses, find the unique initial pair for which  $y(t) \rightarrow 0$ . Compute that solution and verify its equation and initial data.
4. An experiment perturbs only  $b$  from this special pair to  $b + \varepsilon$ . Find the exact response error. Explain how an arbitrarily small nonzero error can invalidate boundedness, and state precisely when a final-value calculation is justified.

## Question 5 – Solution

**Strategy.** Inspect the residue of the positive pole before using a final-value theorem.

**Step 1: Keep the initial-data numerator.** Transformation yields

$$P(s)Y = a(s+1) + b + \frac{1}{s+1}, \quad P(s) = s(s-1)(s+2).$$

The candidate poles are  $1, 0, -1, -2$ ;  $\operatorname{Re} s > 1$  is a common convergence half-plane before any cancellation.

**Step 2: Classify bounded responses.** Residues give

$$y(t) = -\frac{a+b+1}{2} + \frac{2a+b+1/2}{3}e^t + \frac{1}{2}e^{-t} + \frac{-a+b-1}{6}e^{-2t}.$$

Only  $e^t$  is unbounded, and no other mode can cancel it asymptotically. Therefore

$$\boxed{y \text{ is bounded} \iff b = -2a - \frac{1}{2}.$$

For this line of data,  $\lim y = (a - 1/2)/2$ .

**Step 3: Remove the constant mode as well.** Zero limit requires  $a = 1/2$ ,  $b = -3/2$ , giving

$$\boxed{y_*(t) = \frac{1}{2}(e^{-t} - e^{-2t}).}$$

Its first three initial values are  $(0, 1/2, -3/2)$ . Since  $P(-1) = 2$  and  $P(-2) = 0$ ,  $P(D)y_* = e^{-t}$ , directly checking the equation. Its transform converges for  $\operatorname{Re} s > -1$ .

**Step 4: Quantify sensitivity and theorem use.** A change  $b \mapsto b + \varepsilon$  gives

$$\boxed{y - y_* = \varepsilon \left( -\frac{1}{2} + \frac{e^t}{3} + \frac{e^{-2t}}{6} \right).$$

This error has initial state  $(0, 0, \varepsilon)$  and is asymptotic to  $\varepsilon e^t/3$ . Every nonzero perturbation is eventually unbounded. Only after the positive pole cancels do all poles of  $sY$  lie strictly left of the imaginary axis; then  $\lim y = \lim_{s \rightarrow 0} sY$  is valid. For unstable data, the finite algebraic value at  $s = 0$  does not represent a time limit.

**Question 6**

A variable coefficient makes the transform satisfy a differential equation. Seek a real-analytic solution at  $t = 0$  of

$$ty''' + 3y'' + ty' + y = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

for  $t \geq 0$ , with a Laplace transform. A third initial value is not specified.

**Tasks**

1. Determine the only compatible value of  $y''(0)$ . Explain why the usual three-free-data theorem for a regular third-order equation does not apply at the origin.
2. Use  $\mathcal{L}\{tf(t)\} = -F'(s)$  to derive and solve an equation for  $Y(s)$  on real  $s > 0$ . Fix the integration constant from the behavior as  $s \rightarrow \infty$ .
3. Invert the result by writing it as an integral of elementary cosine transforms. Extend the answer to  $t = 0$  and verify the original differential equation.
4. Derive the power-series coefficient recurrence and prove uniqueness among analytic solutions with the prescribed data. Check the first three terms of the large- $s$  expansion of  $Y$  against the initial derivatives.

## Question 6 – Solution

**Strategy.** Use the singular-endpoint constraint before transforming; the transform's boundary condition replaces its integration constant.

**Step 1: Enforce compatibility.** At  $t = 0$ , continuity of the equation gives  $3y''(0) + 1 = 0$ , so  $y''(0) = -1/3$ . The leading coefficient  $t$  vanishes at the origin; normalizing by it would produce singular coefficients there.

**Step 2: Transform carefully.** Writing  $c = y''(0)$ , the four transforms are

$$-\frac{d}{ds}(s^3Y - s^2 - c), \quad 3(s^2Y - s), \quad -\frac{d}{ds}(sY - 1), \quad Y.$$

Their sum is  $-(s^3 + s)Y' - s = 0$ . Thus  $Y' = -1/(1 + s^2)$  for  $s > 0$ . Because the transform of a continuous exponential-order function tends to zero as  $s \rightarrow \infty$ , the constant is fixed:

$$Y(s) = \int_s^\infty \frac{dv}{1 + v^2} = \arctan(1/s), \quad s > 0.$$

**Step 3: Invert an integral representation.** Since  $Y(s) = \int_0^1 s/(s^2 + u^2) du$ , absolute convergence for  $\operatorname{Re} s > 0$  permits interchange of integrals. Therefore

$$y(t) = \int_0^1 \cos(ut) du = \frac{\sin t}{t} \quad (t > 0), \quad y(0) = 1.$$

For  $t > 0$ , this function satisfies  $ty'' + 2y' + ty = 0$ . Differentiating gives the required third-order equation; its analytic extension satisfies the equation at zero too.

**Step 4: Prove analytic uniqueness.** For  $y = \sum_{n \geq 0} c_n t^n$ , coefficient matching gives

$$(n+1)((n+2)(n+3)c_{n+2} + c_n) = 0 \quad (n \geq 0).$$

The seeds  $c_0 = 1, c_1 = 0$  uniquely give  $c_{2k} = (-1)^k/(2k+1)!$  and  $c_{2k+1} = 0$ , an everywhere convergent series. Ordinary uniqueness on  $t > 0$  extends local uniqueness to the whole positive axis. Finally  $Y = s^{-1} - (3s^3)^{-1} + (5s^5)^{-1} + O(s^{-7})$ , agreeing with  $y(0) = 1$ ,  $y''(0) = -1/3$ , and  $y^{(4)}(0) = 1/5$  in  $Y \sim \sum y^{(j)}(0)/s^{j+1}$ .

**Question 7**

A third-order equation has an exponentially fading memory:

$$(D + 1)^3 y(t) = \int_0^t e^{-(t-v)} y(v) dv + e^{-t}, \quad (y, y', y'')(0) = (0, 0, 0).$$

All functions are defined for  $t \geq 0$ .

**Tasks**

1. Transform the convolution equation and solve algebraically for  $Y(s)$ . Factor the resulting fourth-degree denominator.
2. Invert the transform in a real form, displaying a useful decomposition in  $z = s + 1$ .
3. Derive an equivalent ordinary fourth-order initial-value problem. Explain why differentiation requires an additional initial condition, and prove that your fourth-order solution really satisfies the original memory equation.
4. Determine the long-time limit and justify any final-value theorem used. Explain how the feedback changes the zero-state response from what it would be if the memory integral were omitted.

## Question 7 – Solution

**Strategy.** Convert memory into a rational factor, then guard against the extra solutions that differentiation can introduce.

**Step 1: Transform and factor.** The convolution theorem gives

$$(s+1)^3 Y = \frac{Y}{s+1} + \frac{1}{s+1}, \quad \boxed{Y = \frac{1}{(s+1)^4 - 1}}.$$

The denominator is  $s(s+2)((s+1)^2+1)$ , with roots  $0, -2, -1 \pm i$ . A sufficient convergence half-plane is  $\operatorname{Re} s > 0$ .

**Step 2: Invert real quadratic factors.** For  $z = s+1$ ,

$$\frac{1}{z^4 - 1} = \frac{1}{2} \left( \frac{1}{z^2 - 1} - \frac{1}{z^2 + 1} \right), \quad \boxed{y(t) = \frac{1}{2} e^{-t} (\sinh t - \sin t)}.$$

Equivalently,  $y = (1 - e^{-2t})/4 - e^{-t} \sin t/2$ .

**Step 3: Prove equivalence, including the lost condition.** If  $m(t) = \int_0^t e^{-(t-v)} y(v) dv$ , then  $(D+1)m = y$  and  $m(0) = 0$ . Applying  $D+1$  to the original equation gives

$$((D+1)^4 - 1)y = 0, \quad (y, y', y'', y''')(0) = (0, 0, 0, 1).$$

The fourth value follows from the original equation at zero. The displayed solution has  $y = t^3/6 + O(t^4)$  and lies in the four characteristic modes, so it satisfies this fourth-order IVP. Conversely its residual  $R = (D+1)^3 y - m - e^{-t}$  obeys  $R' + R = 0$  and  $R(0) = 0$ . Hence  $R \equiv 0$ , proving equivalence rather than just necessity.

**Step 4: Interpret the surviving mode.** The explicit formula gives  $\boxed{\lim_{t \rightarrow \infty} y(t) = 1/4}$ . This also equals  $\lim_{s \rightarrow 0} sY$ ; after cancellation, the poles of  $sY$  are  $-2, -1 \pm i$ , all strictly stable. Without memory, the transform is  $1/(s+1)^4$  and the response is  $t^3 e^{-t}/6 \rightarrow 0$ . Feedback introduces a zero characteristic root and thus a nonzero constant component, although the external forcing itself decays.

**Question 8**

A stable third-order filter receives identical unit impulses at times  $T, 2T, 3T, \dots$ , where  $T > 0$ :

$$(D + 1)^3 y = \sum_{n=1}^{\infty} \delta(t - nT).$$

Assume zero prehistory and zero initial state. Interpret the equation in the sense of distributions; there is no impulse at  $t = 0$ .

**Tasks**

1. Find  $Y(s)$ , stating a convergence half-plane. Give the exact finite-sum response at any finite time.
2. For  $t = NT + x$  with  $0 \leq x < T$ , find the limiting periodic profile as  $N \rightarrow \infty$ . Sum it explicitly using  $q = e^{-T}$ .
3. Determine the continuity and derivative jumps across each impulse, including the join in the limiting profile. Compute the mean of the limiting profile over a period.
4. Decide whether  $y(t)$  has a pointwise limit as  $t \rightarrow \infty$ . Explain the failure or validity of a final-value argument, and sketch the transient and limiting profile for  $T = 2$ .

## Question 8 – Solution

**Strategy.** Sum delayed impulse kernels and distinguish a periodic limiting profile from a constant final value.

**Step 1: Sum the transformed impulses.** For  $\operatorname{Re} s > 0$ ,

$$Y = \frac{e^{-sT}}{(1 - e^{-sT})(s + 1)^3}, \quad y(t) = \sum_{1 \leq n \leq \lfloor t/T \rfloor} h(t - nT), \quad h(v) = \frac{1}{2}v^2e^{-v}.$$

The sum is empty before  $T$ ; including the current impulse adds  $h(0) = 0$ .

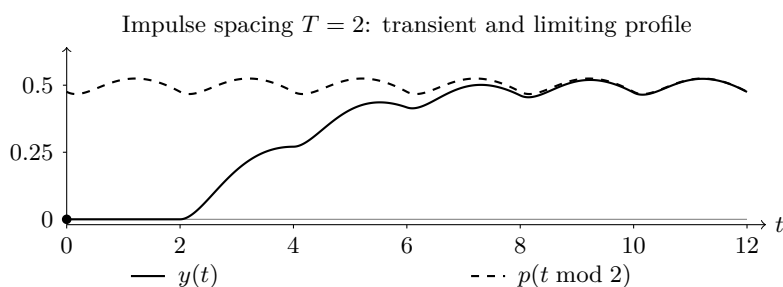
**Step 2: Sum the limiting profile.** At  $t = NT + x$  the sum is  $\sum_{k=0}^{N-1} h(x + kT)$ . Geometric-series identities give

$$p(x) = \frac{e^{-x}}{2} \left[ \frac{x^2}{1 - q} + \frac{2xTq}{(1 - q)^2} + \frac{T^2q(1 + q)}{(1 - q)^3} \right], \quad q = e^{-T}.$$

The omitted tail is uniformly  $O(N^2q^N)$  for  $0 \leq x \leq T$  and fixed  $T$ , so  $y(NT + x) \rightarrow p(x)$  uniformly on a period.

**Step 3: Check joins and mean.** Since  $(h, h', h'')(0) = (0, 0, 1)$ ,  $y, y'$  are continuous and  $y''$  jumps upward by one. Reindexing the convergent profile series gives the same periodic joins:  $p(0) = p(T-)$ ,  $p'(0) = p'(T-)$ ,  $p''(0) - p''(T-) = 1$ . Also  $\int_0^T p(x) dx = \int_0^\infty h(v) dv = 1$ , so its mean is  $1/T$ .

**Step 4: Reject a constant final value.** The profile is nonconstant: on each open gap  $(D+1)^3p = 0$ , so a constant profile would be zero, contradicting its integral. Distinct phases therefore have distinct subsequential limits. Poles at  $s = 2\pi ik/T$ ,  $k \neq 0$ , remain in  $sY$ ; the formal limit  $\lim_{s \rightarrow 0} sY = 1/T$  is a mean, not a final value.



**Question 9**

An experiment supplies the exact transform

$$Y(s) = \frac{s^4 + 3s^3 + 7s^2 + 11s + 18}{(s+3)(s^2+1)(s^2+4)}$$

of a solution to

$$y^{(4)} + 5y'' + 4y = Ae^{-3t}, \quad t \geq 0.$$

The forcing amplitude  $A$  and all four initial values are unknown real numbers.

**Tasks**

1. Recover  $A$  and  $(y, y', y'', y''')(0)$  by polynomial division of  $(s^2+1)(s^2+4)Y$ . Prove that this recovery is unique.
2. Invert  $Y$  completely using real sine, cosine and exponential terms.
3. Verify the recovered forcing and all four initial values directly from your answer, and state an absolute-convergence half-plane.
4. Find the monic constant-coefficient homogeneous differential operator of least order that annihilates this particular response. Explain why its order differs from that of the forced equation, and prove minimality.

## Question 9 – Solution

**Strategy.** Separate the polynomial encoding of initial data from the proper rational forcing term.

**Step 1: Recover data uniquely.** Write  $P(s) = s^4 + 5s^2 + 4$ . Polynomial division gives

$$P(s)Y = s^3 + 7s - 10 + \frac{48}{s+3}.$$

For initial values  $c_j = y^{(j)}(0)$ , transformation of the equation gives

$$PY = c_0s^3 + c_1s^2 + (c_2 + 5c_0)s + (c_3 + 5c_1) + \frac{A}{s+3}.$$

Thus  $\boxed{A = 48, \quad (c_0, c_1, c_2, c_3) = (1, 0, 2, -10)}$ . Polynomial division has a unique polynomial part and remainder; the displayed triangular relation then determines every  $c_j$  uniquely.

**Step 2: Resolve the real modes.** Real partial fractions yield

$$Y = \frac{(2/5)s + 22/15}{s^2 + 1} + \frac{(3/13)s - 14/39}{s^2 + 4} + \frac{24/65}{s + 3},$$

so

$$\boxed{y = \frac{2}{5} \cos t + \frac{22}{15} \sin t + \frac{3}{13} \cos(2t) - \frac{7}{39} \sin(2t) + \frac{24}{65} e^{-3t}.$$

**Step 3: Check the recovered experiment.** The oscillatory terms are annihilated by  $P(D)$  and  $P(-3) = 130$ , so the exponential contributes  $130(24/65)e^{-3t} = 48e^{-3t}$ . At zero the four derivatives are

$$\begin{aligned} y(0) &= 2/5 + 3/13 + 24/65 = 1, \\ y'(0) &= 22/15 - 14/39 - 72/65 = 0, \\ y''(0) &= -2/5 - 12/13 + 216/65 = 2, \\ y'''(0) &= -22/15 + 56/39 - 648/65 = -10. \end{aligned}$$

Absolute convergence holds for  $\operatorname{Re} s > 0$ .

**Step 4: Prove the minimum homogeneous order.** All five complex modes at  $i, -i, 2i, -2i, -3$  have nonzero coefficients. Their linear independence forces every annihilating polynomial to vanish at all five distinct roots. Hence the least monic operator is  $\boxed{(D+3)(D^2+1)(D^2+4)}$ , of order five. The fourth-order equation leaves the exponential as a nonzero forcing; annihilating that forcing adds the fifth factor.

**Question 10**

For  $\varepsilon \geq 0$ , let  $h_\varepsilon$  be the causal impulse response of

$$(D+1)^2(D+1+\varepsilon)h_\varepsilon = \delta(t),$$

with zero prehistory. The impulse is at the origin; use the causal transform convention  $\mathcal{L}\delta(t) = 1$ . Ordinary formulas below describe  $t \geq 0$ .

**Tasks**

1. Find the transform and invert it for  $\varepsilon > 0$  by partial fractions in  $z = s + 1$ .
2. Derive a convolution formula with a nonnegative integrand. Use it to obtain  $h_0$  and prove  $h_\varepsilon \rightarrow h_0$  without subtracting divergent coefficients.
3. Prove two-sided bounds for  $h_\varepsilon(t)$  and an explicit bound of the form  $\sup_{t \geq 0} |h_\varepsilon(t) - h_0(t)| \leq C\varepsilon$  with a numerical constant  $C$ .
4. Explain the change in pole multiplicities and verify the four right-hand initial derivatives required by a unit impulse. Discuss reliable evaluation for small  $\varepsilon t$ , and sketch the kernels for  $\varepsilon = 0, 1/2, 2$ .

## Question 10 – Solution

**Strategy.** Use partial fractions for a closed form and convolution for a stable coalescing-pole limit.

**Step 1: Invert distinct repeated poles.** For  $\varepsilon > 0$ ,  $H_\varepsilon = 1/[z^2(z + \varepsilon)^2]$  and

$$H_\varepsilon = \frac{1}{\varepsilon^2} \left( \frac{1}{z^2} + \frac{1}{(z + \varepsilon)^2} \right) + \frac{2}{\varepsilon^3} \left( \frac{1}{z + \varepsilon} - \frac{1}{z} \right).$$

Hence  $h_\varepsilon = e^{-t} \{t(1 + e^{-\varepsilon t})/\varepsilon^2 - 2(1 - e^{-\varepsilon t})/\varepsilon^3\}$ .

**Step 2: Take the limit through convolution.** Convolving  $te^{-t}$  with  $te^{-(1+\varepsilon)t}$  gives

$$h_\varepsilon(t) = e^{-t} \int_0^t v(t-v)e^{-\varepsilon v} dv, \quad h_0(t) = \frac{t^3 e^{-t}}{6}.$$

For fixed  $t$ , dominated convergence justifies the limit, including  $t = 0$ .

**Step 3: Prove global error bounds.** Bounding the integrand and using  $1 - e^{-\varepsilon v} \leq \varepsilon v$  yields

$$e^{-(1+\varepsilon)t} \frac{t^3}{6} \leq h_\varepsilon(t) \leq e^{-t} \frac{t^3}{6}, \quad 0 \leq h_0 - h_\varepsilon \leq \frac{\varepsilon t^4 e^{-t}}{12}.$$

The maximum of  $t^4 e^{-t}$  is  $256e^{-4}$  at  $t = 4$ . Thus  $\sup_{t \geq 0} |h_\varepsilon - h_0| \leq 64\varepsilon/(3e^4)$ .

**Step 4: Check poles, impulses and evaluation.** Two double poles merge into a quadruple pole at  $-1$ ; absolute convergence holds for  $\operatorname{Re} s > -1$ . Convolution gives  $h_\varepsilon = t^3/6 + O(t^4)$ , so its right-hand jet is  $(0, 0, 0, 1)$ , producing exactly a unit  $\delta$ . For small  $\varepsilon t$ , use the integral or its exponential series; the closed form subtracts large nearly equal terms and can lose significant digits.

