

Question 1

Consider the fourth-order IVP

$$(1 + x^2)y^{(4)} = 1, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Use the normalized homogeneous basis $1, x, x^2/2, x^3/6$.

Tasks

1. Normalize the equation and verify that the stated basis is fundamental. Write the four equations for the parameter derivatives in variation of parameters.
2. Solve those equations. Integrate each parameter from zero and combine the result into one definite integral for $y(x)$.
3. Verify the equation and all initial data by differentiating the combined integral. Explain exactly what goes wrong if the leading coefficient is not divided out.
4. Prove upper and lower bounds for $y(x)$ when $x \geq 0$ using the largest and smallest values of $1/(1 + t^2)$ on $[0, x]$. State when equality holds.

Question 1 – Solution

Strategy. Normalize first, then combine the four parameter integrals into one positive kernel.

Step 1: Form the normalized parameter system. Put $g(x) = 1/(1+x^2)$. The derivative matrix of the given basis is triangular with unit diagonal, so its Wronskian is 1. The parameter derivatives satisfy

$$\begin{pmatrix} 1 & x & x^2/2 & x^3/6 \\ 0 & 1 & x & x^2/2 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u'_0 \\ u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ g \end{pmatrix}.$$

The first three zero rows remove derivative terms introduced by varying the constants; the last row supplies the normalized forcing.

Step 2: Solve and combine. Back substitution gives $(u'_0, u'_1, u'_2, u'_3) = g(-x^3/6, x^2/2, -x, 1)$. Choose each $u_j(0) = 0$. In $y = \sum u_j f_j$, the combined numerator is $(x-t)^3$, yielding

$$y(x) = \frac{1}{6} \int_0^x \frac{(x-t)^3}{1+t^2} dt.$$

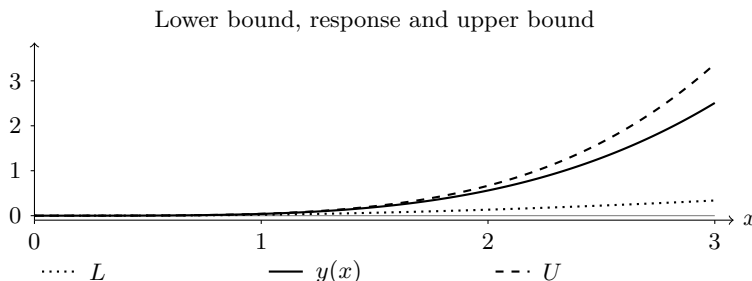
Oriented integration also defines the solution for negative x .

Step 3: Verify the initial state and residual. Successive derivatives have kernels $(x-t)^2/2$, $x-t$, and 1, all divided by $1+t^2$. The fourth derivative is $1/(1+x^2)$. The first four state components vanish at zero, and multiplication by $1+x^2$ gives the original forcing. If one incorrectly uses $g = 1$, the same construction gives $x^4/24$, whose original residual is $1+x^2$, not 1.

Step 4: Bound the positive integral. For $0 \leq t \leq x$, $1/(1+x^2) \leq 1/(1+t^2) \leq 1$. Hence

$$\frac{x^4}{24(1+x^2)} \leq y(x) \leq \frac{x^4}{24} \quad (x \geq 0).$$

Both inequalities are strict for $x > 0$, since their integrands differ on an interval of positive length where $(x-t)^3 > 0$. Both become equalities at $x = 0$. The figure labels the lower and upper bounds L and U , respectively.



Question 2

For $x \geq 0$, consider

$$y''' - y' = e^{-x^2}.$$

Use the homogeneous basis $1, e^x, e^{-x}$; definite integrals may be left unevaluated.

Tasks

1. Verify the basis and derive the three parameter derivatives by variation of parameters.
2. Find the solution y_0 with $y_0(0) = y_0'(0) = y_0''(0) = 0$, expressed as one integral. Verify its kernel and initial data.
3. Determine $\lim_{x \rightarrow \infty} e^{-x} y_0(x)$ and prove that it is positive. Justify convergence of every improper integral used.
4. Keep $y(0) = y'(0) = 0$ but choose $y''(0)$ so that the response is bounded on $[0, \infty)$. Prove uniqueness of this choice and determine the finite limit of the resulting solution.

Question 2 – Solution

Strategy. Use parameter integrals to isolate the growing homogeneous mode and the initial curvature needed to cancel it.

Step 1: Solve the derivative system. The functions solve $y''' - y' = 0$ and have Wronskian 2. With $g = e^{-x^2}$, the equations are

$$u'_0 + e^x u'_+ + e^{-x} u'_- = 0, \quad e^x u'_+ - e^{-x} u'_- = 0, \quad e^x u'_+ + e^{-x} u'_- = g.$$

Thus $u'_0 = -g$, $u'_+ = e^{-x}g/2$, and $u'_- = e^xg/2$.

Step 2: Construct the zero-data response. Integrating from zero and combining gives

$$y_0(x) = \int_0^x [\cosh(x-t) - 1] e^{-t^2} dt.$$

For $G(s) = \cosh s - 1$, we have $G(0) = G'(0) = 0$, $G''(0) = 1$, and $G''' - G' = 0$. Leibniz differentiation therefore gives $y_0''' - y_0' = g$ and the three zero initial data.

Step 3: Identify the growing coefficient. Define

$$A(x) = \int_0^x e^{-t^2-t} dt, \quad B(x) = \int_0^x e^{-t^2+t} dt, \quad C(x) = \int_0^x e^{-t^2} dt.$$

Their limits $A_\infty, B_\infty, C_\infty$ are finite: for $t \geq 2$, each integrand is at most $e^{-t^2/2} \leq e^{-t}$. All are positive. Since $y_0 = e^x A(x)/2 + e^{-x} B(x)/2 - C(x)$,

$$\lim_{x \rightarrow \infty} e^{-x} y_0(x) = A_\infty/2 > 0.$$

Step 4: Cancel growth without losing the first two data. Any other solution with the same value and slope is $y = y_0 + k(\cosh x - 1)$, where $k = y''(0)$. Boundedness requires $k = -A_\infty$, and no other choice cancels the growing coefficient. For this choice,

$$y = -\frac{1}{2}e^x(A_\infty - A(x)) + \frac{1}{2}e^{-x}(B(x) - A_\infty) + A_\infty - C(x).$$

The first term tends to zero: for $x > 0$, $e^x \int_x^\infty e^{-t^2-t} dt \leq e^{-x^2}$. The second also tends to zero. Therefore the choice is sufficient, and

$$y''(0) = -A_\infty, \quad \lim_{x \rightarrow \infty} y = A_\infty - C_\infty.$$

Question 3

On $x > 0$, consider the Euler-type equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x.$$

Use the proposed homogeneous basis x, x^2, x^3 and variation of parameters, rather than a guessed logarithmic trial.

Tasks

1. Verify the three homogeneous solutions and their Wronskian. Normalize the equation and write the parameter system with the correct right-hand side.
2. Solve for the parameter derivatives and integrate to obtain one particular solution. Write the complete general solution.
3. Impose $y(1) = y'(1) = y''(1) = 0$. Verify all three initial data and the forcing directly.
4. State the interval containing 1 on which this construction and the standard linear IVP theorem apply. Explain why evaluating the same parameter system at $x = 0$ is invalid, even if some expressions have finite limits there.

Question 3 – Solution

Strategy. Normalize the Euler operator before solving the derivative equations, and retain its positive-domain restriction.

Step 1: Verify the basis and normalization. For $y = x^m$, the left side is $(m-1)(m-2)(m-3)x^m$. Thus the proposed functions are homogeneous. Their Wronskian is $2x^3 \neq 0$ for $x > 0$. Division by x^3 gives normalized forcing $x \ln x$, so

$$\begin{pmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x \ln x \end{pmatrix}.$$

Step 2: Integrate the solved parameter derivatives. Elimination gives $u'_1 = x^2 \ln x / 2$, $u'_2 = -x \ln x$, $u'_3 = \ln x / 2$. Convenient antiderivatives are

$$u_1 = \frac{x^3 \ln x}{6} - \frac{x^3}{18}, \quad u_2 = -\frac{x^2 \ln x}{2} + \frac{x^2}{4}, \quad u_3 = \frac{x \ln x}{2} - \frac{x}{2}.$$

Combining $xu_1 + x^2u_2 + x^3u_3$ gives

$$y_p = x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right), \quad y = y_p + c_1x + c_2x^2 + c_3x^3.$$

Integration constants are already represented by the homogeneous terms.

Step 3: Fit and check the initial vector. The particular solution has vector $(-11/36, -19/18, -5/2)$ at 1. The three linear equations for its cancellation give

$$y = \frac{x}{18} - \frac{x^2}{4} + \frac{x^3}{2} + x^4 \left(\frac{\ln x}{6} - \frac{11}{36} \right).$$

The added terms have vector $(11/36, 19/18, 5/2)$, verifying the data. For a direct residual check, let $\theta = xD$. The operator is $(\theta-1)(\theta-2)(\theta-3)$; on $x^4(a \ln x + b)$ it gives $x^4(6a \ln x + 11a + 6b)$. With $a = 1/6$, $b = -11/36$, this is exactly $x^4 \ln x$.

Step 4: Respect the regular interval. The normalized coefficients, forcing and nonsingular derivative matrix are valid throughout $(0, \infty)$, the largest such interval containing 1. At zero the leading coefficient and Wronskian vanish and $\ln x$ is undefined. Finite limits of selected terms do not make that parameter system invertible or extend the theorem across zero.

Question 4

Let g be continuous on $\mathbb{R}$ and consider $y''' = g(x)$. Two homogeneous bases are

$$f = (1, x, x^2/2), \quad \tilde{f} = (1 + x, x + x^2/2, 1 + x^2/2).$$

Regard these as row vectors of functions and write $y_p = fu = \tilde{f}v$, with column vectors of parameters.

Tasks

1. Find the constant matrix M such that $\tilde{f} = fM$. Verify that both bases are fundamental.
2. Derive u' by variation of parameters, then obtain v' from the change of basis. Explain why the lower-limit-zero constructions produce exactly the same particular solution.
3. Replace the lower integration limit 0 by an arbitrary real b . Find the difference between the two resulting particular solutions and prove that it is homogeneous.
4. Explain why changing basis or choosing separate integration constants does not create more than three free constants in the general solution. Distinguish unique parameter coordinates in a chosen basis from the nonuniqueness of a particular solution.

Question 4 – Solution

Strategy. Track the constant coordinate transformation and isolate the effect of the integration base point.

Step 1: Identify the basis transformation. The coefficient columns give

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \det M = 2.$$

All six displayed functions are quadratic or lower and solve $y''' = 0$. The Wronskian of f is 1, and that of $\tilde{f}$ is $\det M = 2$, so both triples are fundamental.

Step 2: Transform the coordinates. The standard derivative system gives $u' = g(x)(x^2/2, -x, 1)^T$. Since the derivative matrices satisfy $\tilde{F} = FM$, their parameter equations imply $Mv' = u'$, hence

$$v' = g(x) \begin{pmatrix} x^2/4 - x/2 - 1/2 \\ -x^2/4 - x/2 + 1/2 \\ x^2/4 + x/2 + 1/2 \end{pmatrix}.$$

With $u(0) = v(0) = 0$, integration gives $u = Mv$ at every x . Thus $fu = fMv = \tilde{f}v$ exactly, not merely up to a homogeneous term. Both equal

$$Y_0(x) = \frac{1}{2} \int_0^x (x-t)^2 g(t) dt.$$

Step 3: Change the integration base point. The construction based at b gives $Y_b(x) = \frac{1}{2} \int_b^x (x-t)^2 g(t) dt$. Therefore

$$Y_b - Y_0 = -\frac{x^2}{2} \int_0^b g(t) dt + x \int_0^b t g(t) dt - \frac{1}{2} \int_0^b t^2 g(t) dt.$$

The three integrals are constants with respect to x , even when $b < 0$. The difference is a quadratic polynomial and hence homogeneous.

Step 4: Count genuine solution freedom. Once a basis and lower limit are fixed, the invertible derivative matrix uniquely fixes the parameter derivatives, and the specified parameter values fix their integrals. Allowing arbitrary integration constants adds a constant linear combination of the three homogeneous basis functions. Adding a separate homogeneous triple afterward repeats that same freedom rather than creating six independent constants. Particular solutions need not coincide, but their differences are homogeneous; the full solution space of $y''' = g$ is one particular solution plus the three-dimensional homogeneous space.

Question 5

On the positive half-line, consider

$$y''' = \frac{1}{x}, \quad y(1) = y'(1) = y''(1) = 0.$$

Use the homogeneous basis $1, x, x^2/2$ and integrate the variation parameters from 1.

Tasks

1. Derive the parameter derivatives and compute the complete IVP solution.
2. Verify the original equation and all initial data. Find the one-sided limits of y, y', y'' as $x \rightarrow 0^+$.
3. Determine whether the solution admits a C^1 or C^2 extension to the endpoint 0. Explain why a divergent parameter does not necessarily force y itself to diverge.
4. Test the attempted lower-limit-zero formula $\frac{1}{2} \int_0^x (x-t)^2/t dt$. Decide whether it converges for fixed $x > 0$, and explain why finite endpoint values of y, y' do not justify using zero as a regular IVP base point.

Question 5 – Solution

Strategy. Separate parameter divergence, solution limits and the regularity needed for initial data.

Step 1: Integrate from the regular point. The triangular parameter system for $y''' = g$ gives $(u'_0, u'_1, u'_2) = (x^2 g/2, -xg, g)$. Here $g = 1/x$, so

$$u_0 = (x^2 - 1)/4, \quad u_1 = 1 - x, \quad u_2 = \ln x.$$

Combining $u_0 + xu_1 + x^2 u_2/2$ yields

$$y = \frac{x^2 \ln x}{2} - \frac{3x^2}{4} + x - \frac{1}{4} \quad (x > 0).$$

Step 2: Check derivatives and endpoint behavior. Direct differentiation gives

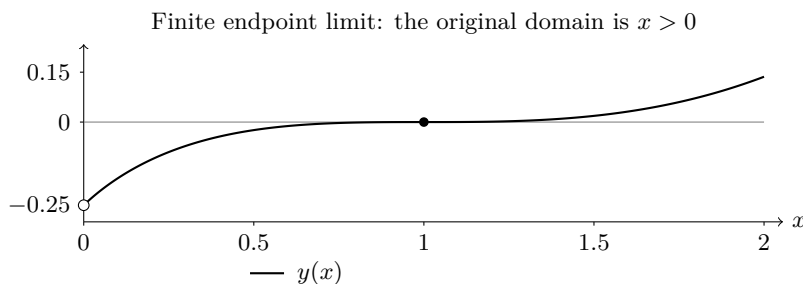
$$y' = x \ln x - x + 1, \quad y'' = \ln x, \quad y''' = 1/x.$$

These formulas and $\ln 1 = 0$ verify all three data at 1. Since $x \ln x$ and $x^2 \ln x$ tend to zero at the positive endpoint,

$$\lim_{x \rightarrow 0^+} y = -1/4, \quad \lim_{x \rightarrow 0^+} y' = 1, \quad \lim_{x \rightarrow 0^+} y'' = -\infty.$$

Step 3: Classify the extension. Defining $y(0) = -1/4$ gives right derivative 1, because $(y(x) + 1/4)/x \rightarrow 1$. This derivative is continuous from the right, so the solution extends as a C^1 function to $[0, \infty)$; it may also be continued to the left with matching value and slope. No C^2 extension is possible because y'' diverges. Although $u_2 = \ln x$ diverges, it is multiplied by $x^2/2$ in y , whose contribution tends to zero.

Step 4: Reject the singular base point. For fixed $x > 0$ and $0 < t < x/2$, $(x - t)^2/(2t) \geq x^2/(8t)$. The proposed improper integral therefore diverges to $+\infty$. Its lower limit cannot simply be moved to zero. The equation is singular there, and even the solution's second derivative lacks finite initial data. Two finite endpoint limits do not supply a regular third-order initial state.



Question 6

For continuous f on $[0, \infty)$, consider

$$(D + 1)^4 y = f(x), \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Use the factorial-normalized homogeneous basis $e^{-x}, xe^{-x}, x^2e^{-x}/2, x^3e^{-x}/6$.

Tasks

1. Derive the variation-parameter equations or an equivalent triangular system, and combine their integrals into one response kernel.
2. Verify that the kernel gives the original equation and all four zero initial data for every continuous forcing.
3. If $|f(x)| \leq M$ for $M \geq 0$, derive an explicit bound on $|y(x)|$ for each $x \geq 0$. Prove its sharpness and deduce an all-time bound.
4. Apply the same reasoning to the difference of two responses with identical initial data and forcing discrepancy at most ε . Explain why bounded forcing need not produce a response tending to zero.

Question 6 – Solution

Strategy. Use the repeated-root basis to produce a positive kernel whose integral is explicit.

Step 1: Solve the shifted parameter system. Write $y = e^{-x}v$. The equation becomes $v^{(4)} = e^x f(x)$. The original basis becomes $1, x, x^2/2, x^3/6$ for v , with parameter derivatives

$$(u'_0, u'_1, u'_2, u'_3) = e^x f(x)(-x^3/6, x^2/2, -x, 1).$$

The homogeneous basis is fundamental: its Wronskian is $e^{-4x} \neq 0$. Integrating from zero gives

$$y(x) = \int_0^x G(x-t)f(t) dt, \quad G(s) = \frac{s^3 e^{-s}}{6}.$$

Step 2: Verify the response kernel. The function G satisfies $(D+1)^4 G = 0$ for $s > 0$, and $G(0) = G'(0) = G''(0) = 0$, $G'''(0) = 1$. Repeated Leibniz differentiation therefore adds no endpoint terms until the fourth derivative, where it adds exactly $f(x)$. Combining derivatives in $(D+1)^4 y$ cancels the integral terms. All four initial data vanish because the relevant integrals have zero length.

Step 3: Integrate the positive kernel sharply. For $s \geq 0$, $G(s) \geq 0$. Repeated integration by parts yields

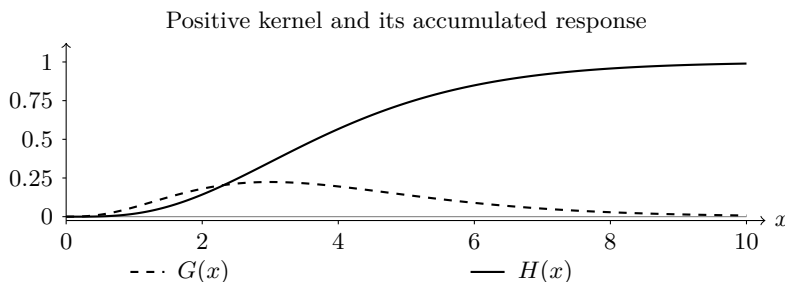
$$\int_0^x G(s) ds = 1 - e^{-x} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} \right) = H(x).$$

Consequently,

$$|y(x)| \leq MH(x) \leq M \quad (x \geq 0).$$

The forcing $f \equiv M$ gives $y = MH$ and attains the pointwise bound. Also $H(x) \rightarrow 1$, so no smaller all-time constant than M works for all admissible forcings when $M > 0$. The case $M = 0$ gives the zero solution.

Step 4: Bound input error without claiming decay. For identical initial data, the response difference has zero initial vector and forcing difference bounded by ε . Thus $|y_1 - y_2| \leq \varepsilon H(x) \leq \varepsilon$. But $f \equiv 1$ gives $y = H(x) \rightarrow 1$, not zero. A positive integrable kernel controls size; it does not force the output to vanish for a persistent input.



Question 7

For the variable-coefficient IVP

$$y''' - xy'' = 1, \quad y(0) = y'(0) = y''(0) = 0,$$

define $F(x) = \int_0^x (x-t)e^{t^2/2} dt$ and use the homogeneous basis $1, x, F(x)$.

Tasks

1. Verify the basis and its Wronskian, then derive the three variation-parameter derivatives.
2. Combine the lower-limit-zero parameter integrals into $y(x) = \int_0^x K(x,t) dt$. Write K both using F, F' and as a definite integral for $x \geq t$.
3. Differentiate to verify the IVP. Prove that y is strictly convex and $y(x) > x^3/6$ for every $x > 0$.
4. Determine whether $K(x,t)$ can depend only on $x-t$. Use a derivative of the kernel to give a precise obstruction to treating this variable-coefficient response as a translation-invariant convolution.

Question 7 – Solution

Strategy. Exploit the triangular derivative matrix while retaining both kernel variables.

Step 1: Solve the parameter equations. We have $F(0) = F'(0) = 0$, $F'' = e^{x^2/2}$ and $F''' = xF''$. Thus $1, x, F$ solve the homogeneous equation and have Wronskian $e^{x^2/2} > 0$. The triangular system gives

$$u_2' = e^{-x^2/2}, \quad u_1' = -F' e^{-x^2/2}, \quad u_0' = (xF' - F)e^{-x^2/2}.$$

Step 2: Combine the parameter integrals. Multiplying by $1, x, F(x)$ and integrating from zero yields

$$K(x, t) = e^{-t^2/2} [F(x) - F(t) - (x - t)F'(t)].$$

Taylor's integral identity, obtained by integrating F'' twice, gives for $x \geq t$

$$K(x, t) = e^{-t^2/2} \int_t^x (x - s) e^{s^2/2} ds, \quad y(x) = \int_0^x K(x, t) dt.$$

The factor involving t is essential; it comes from the inverse derivative matrix evaluated at the integration variable.

Step 3: Verify the equation and inequalities. At the diagonal, $K(t, t) = K_x(t, t) = 0$, while

$$K_{xx}(x, t) = e^{(x^2-t^2)/2}, \quad y''(x) = e^{x^2/2} \int_0^x e^{-t^2/2} dt.$$

Hence $y''' = xy'' + 1$ and all three initial data vanish. For $x > 0$, $y'' > 0$, proving strict convexity. Also $y''' = 1 + xy'' > 1$ there. Integrating this strict inequality three times from the zero initial vector gives $y(x) > x^3/6$ for every $x > 0$.

Step 4: Test translation invariance. If $K(x, t) = H(x - t)$, then $K_{xx} = H''(x - t)$ would have the same value at all pairs with the same difference. But for any fixed $h > 0$,

$$K_{xx}(t + h, t) = e^{th+h^2/2},$$

which varies with t . Therefore this kernel is not a function of $x - t$ alone. The variable coefficient distinguishes the absolute source time from the elapsed time.

Question 8

Let g be continuous and nonnegative on $[0, \infty)$, and consider

$$y''' + y' = g(x), \quad y(0) = y'(0) = y''(0) = 0.$$

Use the homogeneous basis $1, \cos x, \sin x$.

Tasks

1. Derive the variation parameters and combine them into a single response integral.
2. Prove that $y(x) \geq 0$ for every $x \geq 0$. Give a sufficient condition for strict positivity at a specified $x > 0$ and justify it analytically.
3. For the continuous pulse $g(x) = \sin^2 x$ on $[0, \pi]$ and $g(x) = 0$ for $x \geq \pi$, compute $y(x)$ for all $x \geq \pi$ using the integrals of g , $g \cos x$ and $g \sin x$.
4. Determine the exact minimum and maximum of this post-pulse response. Prove that a nonnegative forcing need not make y monotone, even though the solution stays nonnegative.

Question 8 – Solution

Strategy. Distinguish a nonnegative response kernel from the sign of its derivative.

Step 1: Compute the response kernel. The derivative matrix of $1, \cos x, \sin x$ has Wronskian 1. Its parameter equations give

$$u'_0 = g, \quad u'_c = -g \cos x, \quad u'_s = -g \sin x.$$

Integrating from zero and combining by the cosine difference identity yields

$$y(x) = \int_0^x [1 - \cos(x-t)]g(t) dt.$$

The kernel and its first derivative vanish at zero, its second derivative is 1 there, and its third plus first derivative vanishes. These facts verify the equation and zero data.

Step 2: Prove positivity without a monotonicity assumption. Both factors in the integrand are nonnegative. If $g(t_0) > 0$ for some $0 \leq t_0 < x$, continuity gives an interval of positive forcing inside $[0, x)$. The zeros of $1 - \cos(x-t)$ are isolated, so the integral is strictly positive. In particular, a continuous nonnegative forcing that is not identically zero on $[0, x)$ suffices.

Step 3: Evaluate the post-pulse moments. For the stated pulse,

$$\int_0^\pi \sin^2 t dt = \frac{\pi}{2}, \quad \int_0^\pi \sin^2 t \cos t dt = 0, \quad \int_0^\pi \sin^3 t dt = \frac{4}{3}.$$

The first follows from the double-angle identity, the second from the derivative of $\sin^3 t/3$, and the third by substituting $u = \cos t$. Thus

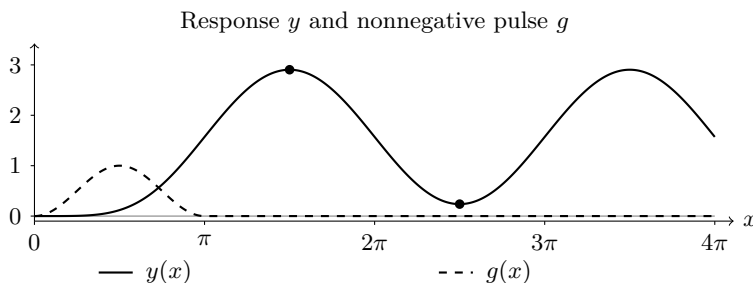
$$y(x) = \frac{\pi}{2} - \frac{4}{3} \sin x \quad (x \geq \pi).$$

For plotting the initial interval, evaluating the same kernel gives $y = x/2 - 2 \sin x/3 + \sin 2x/12$ on $[0, \pi]$; its state matches the displayed branch at π .

Step 4: Separate positivity from monotonicity. The post-pulse range is

$$\frac{\pi}{2} - \frac{4}{3} \leq y(x) \leq \frac{\pi}{2} + \frac{4}{3}.$$

Both bounds are attained infinitely often; the lower bound is positive. Yet $y' = -4 \cos x/3$ changes sign, and $y'(2\pi) = -4/3 < 0$. Thus the nonnegative input gives a nonnegative response that continues oscillating after the pulse. Positivity of $1 - \cos s$ does not imply positivity of its derivative $\sin s$.



Question 9

On $x > 0$, consider

$$x^4 y^{(4)} = 1, \quad y(1) = y'(1) = y''(1) = y'''(1) = 0.$$

A proposed response kernel for $x \geq t > 0$ is

$$\hat{K}(x, t) = \frac{(x-t)^3}{6x^4}.$$

The placement of the leading-coefficient factor must be checked, not guessed.

Tasks

1. Normalize the equation and derive the correct kernel from variation of parameters using $1, x, x^2/2, x^3/6$.
2. For fixed t , verify the correct kernel's homogeneous equation when $x > t$ and its first four diagonal data, through the third x -derivative.
3. Integrate the correct kernel from 1 to x to find the IVP solution in elementary form. Verify its initial data and forcing.
4. Show that $\hat{K}$ has the same diagonal data but is still wrong. Compute the response it would predict and evaluate $x^4 \hat{y}^{(4)}$ at $x = 2$. Explain which additional kernel requirement fails.

Question 9 – Solution

Strategy. Evaluate the normalized forcing at the integration variable and check the kernel away from its diagonal.

Step 1: Derive the correctly normalized kernel. The equation is $y^{(4)} = x^{-4}$. For the given basis, variation of parameters gives $u' = x^{-4}(-x^3/6, x^2/2, -x, 1)^T$. Integrating from 1 and combining yields

$$K(x, t) = \frac{(x-t)^3}{6t^4}, \quad y(x) = \int_1^x K(x, t) dt.$$

The denominator is evaluated at the source variable t . The same polynomial expression and oriented integral apply for $0 < x < 1$.

Step 2: Verify both kernel requirements. For fixed $t > 0$, K is a cubic polynomial in x , so $x^4 K_{xxxx} = 0$ for $x > t$. Its diagonal data are

$$K(t, t) = K_x(t, t) = K_{xx}(t, t) = 0, \quad K_{xxx}(t, t) = 1/t^4.$$

Leibniz differentiation gives $y^{(4)}(x) = 1/x^4$, because the fourth kernel derivative is zero and the third diagonal derivative supplies exactly that term.

Step 3: Evaluate and check the elementary response. Expanding $(x-t)^3/t^4$ and integrating its four powers of t gives

$$y = \frac{x^3}{18} - \frac{x^2}{4} + \frac{x}{2} - \frac{\ln x}{6} - \frac{11}{36}.$$

Its derivatives are $y' = x^2/6 - x/2 + 1/2 - 1/(6x)$, $y'' = x/3 - 1/2 + 1/(6x^2)$, and $y''' = 1/3 - 1/(3x^3)$. These and y vanish at 1; one further derivative gives $y^{(4)} = x^{-4}$.

Step 4: Expose the misleading diagonal agreement. The factor $(x-t)^3$ ensures that $\hat{K}$ has the same three zero diagonal values and $\hat{K}_{xxx}(t, t) = 1/t^4$. But its proposed response is

$$\hat{y} = \frac{1}{6x^4} \int_1^x (x-t)^3 dt = \frac{(x-1)^4}{24x^4}.$$

It satisfies the zero initial data, yet

$$x^4 \hat{y}^{(4)} = -\frac{4}{x} + \frac{30}{x^2} - \frac{60}{x^3} + \frac{35}{x^4}, \quad x^4 \hat{y}^{(4)} \Big|_{x=2} = \boxed{3/16 \neq 1}.$$

The missing requirement is the homogeneous equation in x away from $x = t$. Correct diagonal data alone do not make a valid response kernel.

Question 10

For the zero-data IVP

$$y''' + y' = e^x, \quad y(0) = y'(0) = y''(0) = 0,$$

a proposed approximation on $[0, 1]$ is $q(x) = x^3/6 + x^4/24$.

Tasks

1. Compute the residual $r = q''' + q' - e^x$ and check the approximation's initial data. Express the residual using the remainder after the cubic Taylor polynomial of e^x .
2. Use variation of parameters to represent the error $E = y - q$ by a definite integral with zero initial data. Prove whether the approximation lies above or below the true solution on $(0, 1]$.
3. Use Taylor's remainder theorem and $1 - \cos s \leq s^2/2$ to prove $0 \leq E(x) \leq ex^7/5040$ for $0 \leq x \leq 1$. Decide whether an absolute-error tolerance of 10^{-3} is certified throughout this interval.
4. Independently verify $Y(x) = (e^x + \cos x - \sin x)/2 - 1$ as the exact IVP solution. Compare its error at $x = 1$ with the certified upper bound, and explain why direct agreement at finitely many sample points is weaker than the integral certificate.

Question 10 – Solution

Strategy. Treat the residual as an error forcing and bound its full response, not just sampled values.

Step 1: Compute the forcing error. We have $q''' = 1 + x$ and $q' = x^2/2 + x^3/6$, so

$$r = 1 + x + x^2/2 + x^3/6 - e^x = -R_4(x),$$

where $R_4 = e^x - (1 + x + x^2/2 + x^3/6)$. The values $q(0), q'(0), q''(0)$ are zero. Thus $E = y - q$ has a zero initial vector and satisfies $E''' + E' = R_4$.

Step 2: Derive the positive error integral. For the basis $1, \cos x, \sin x$, the variation derivatives for this error equation are $(R_4, -R_4 \cos x, -R_4 \sin x)$. Integrating from zero gives

$$E(x) = \int_0^x [1 - \cos(x-t)] R_4(t) dt.$$

For $0 < t < x \leq 1$, both factors are strictly positive. Hence $E(x) > 0$ on $(0, 1]$: the approximation underestimates the true solution.

Step 3: Certify a uniform error bound. Taylor's theorem gives $0 \leq R_4(t) \leq et^4/24$ on $[0, 1]$. Therefore

$$0 \leq E(x) \leq \frac{e}{48} \int_0^x (x-t)^2 t^4 dt = \boxed{\frac{ex^7}{5040}},$$

where the polynomial integral equals $x^7/105$. The maximum certified error is $e/5040 \approx 0.00053934 < 10^{-3}$, so the tolerance holds throughout $[0, 1]$.

Step 4: Compare with an exact independent check. For $Y = (e^x + \cos x - \sin x)/2 - 1$, direct differentiation gives $Y''' + Y' = e^x$ and its initial vector is zero. Uniqueness therefore implies $Y = y$. At 1, $q(1) = 5/24$, so the actual error is

$$E(1) = \frac{e + \cos 1 - \sin 1}{2} - \frac{29}{24}.$$

Numerically, $E(1) \approx 0.00022324$, positive and smaller than the certified bound. Exact evaluation or sample agreement is useful corroboration, but finitely many values cannot exclude larger errors between them. The integral inequality proves the bound for every point of the interval.

