

Question 1

Write $D = d/dx$ and let

$$L = D^2(D - 1)^3(D^2 + 4)^2.$$

Consider on $\mathbb{R}$ the equation

$$Ly = 3x^2 + (x + 1)e^x + x \cos 2x + e^{-x} \sin x.$$

This question concerns designing and justifying the coefficient trial, not carrying out a large elimination.

Tasks

1. State the order and complete real homogeneous solution, with all root multiplicities.
2. Write the standard undetermined-coefficient trial for a particular solution, including all polynomial degrees, both trigonometric partners, and every resonance factor. Count its unknown coefficients.
3. Explain why a trial containing only cosine terms for $x \cos 2x$, or only sine terms for $e^{-x} \sin x$, is not the appropriate full trial. Explain why one common resonance factor for all four forcing terms is inappropriate.
4. Prove that substitution determines exactly one member of your trial space. Use the effect of L on each polynomial-exponential family and its intersection with the homogeneous solution space. You need not compute the eleven coefficients.

Question 1 – Solution

Strategy. Treat each distinct exponential frequency separately and remove only its own homogeneous overlap.

Step 1: List the nine homogeneous modes. The roots are 0 twice, 1 three times, and $\pm 2i$ twice each. Thus L has order nine and

$$y_h = c_0 + c_1x + e^x(c_2 + c_3x + c_4x^2) + (c_5 + c_6x)\cos 2x + (c_7 + c_8x)\sin 2x.$$

Step 2: Build the trial block by block. A standard trial is

$$\begin{aligned} y_p = & x^2(a_2x^2 + a_1x + a_0) + x^3e^x(b_1x + b_0) \\ & + x^2((d_1x + d_0)\cos 2x + (e_1x + e_0)\sin 2x) \\ & + e^{-x}(f\cos x + g\sin x). \end{aligned}$$

There are $3 + 2 + 4 + 2 = \boxed{11}$ unknowns. The resonance powers are respectively 2, 3, 2, 0, because $-1 \pm i$ are not roots of L .

Step 3: Keep the derivative families complete. Differentiation mixes sine and cosine, and differentiating polynomial factors also generates lower powers of x . Therefore the full paired families must be allowed; coefficients that ultimately vanish should emerge from the equations. Resonance is tied to the particular exponent, not to the equation's total order. A common factor would either retain homogeneous overlap in some blocks or unnecessarily alter others.

Step 4: Establish solvability and uniqueness in the trial. For a root λ of multiplicity m , write the characteristic polynomial as $(r - \lambda)^m R(r)$ with $R(\lambda) \neq 0$. The shift identity gives

$$L(e^{\lambda x} x^m q) = e^{\lambda x} D^m R(D + \lambda)(x^m q).$$

For $\deg q \leq d$, the resulting polynomial has degree at most d ; its top coefficient is the top coefficient of q times $R(\lambda)(m + d)!/d!$, which is nonzero when that coefficient is nonzero. Successive coefficient matching is therefore invertible. Real and imaginary parts give the same conclusion for each paired block.

Equivalently, the eleven-dimensional trial maps into the eleven-dimensional forcing family and contains no nonzero homogeneous member: each block starts beyond its root's allowed homogeneous degree. Its kernel is zero, so this map is bijective. Hence the stated forcing determines exactly one trial member, although other particular solutions can differ by y_h .

Question 2

Consider the resonant third-order equation

$$(D - 1)^3 y = e^x(x^2 + 1)$$

on $\mathbb{R}$.

Tasks

1. Explain why $e^x(Ax^2 + Bx + C)$ cannot be a particular-solution trial. Give a complete resonance-corrected trial.
2. Use $y = e^x v$ to determine a particular solution by coefficient matching and write the full real general solution.
3. Solve the IVP $y(0) = y'(0) = y''(0) = 0$. Verify all three initial data and the original forcing.
4. For an arbitrary solution, calculate $\lim_{x \rightarrow \infty} e^{-x} y(x)/x^5$. Prove that no choice of homogeneous constants makes a solution bounded on $[0, \infty)$.

Question 2 – Solution

Strategy. Shift the triple root to zero, where the coefficient calculation becomes polynomial differentiation.

Step 1: Remove the homogeneous overlap. The characteristic root 1 has multiplicity three. Every proposed term $e^x(Ax^2 + Bx + C)$ is already homogeneous and gives zero under $(D-1)^3$. The corrected trial is

$$y_p = x^3 e^x (Ax^2 + Bx + C).$$

Step 2: Solve the shifted coefficient equations. Since $(D-1)(e^x v) = e^x v'$, the equation becomes $v''' = x^2 + 1$. For $v_p = Ax^5 + Bx^4 + Cx^3$, matching gives

$$60Ax^2 + 24Bx + 6C = x^2 + 1, \quad A = \frac{1}{60}, \quad B = 0, \quad C = \frac{1}{6}.$$

The complete solution is consequently

$$y = e^x \left(c_0 + c_1 x + c_2 x^2 + \frac{x^3}{6} + \frac{x^5}{60} \right).$$

Step 3: Impose and verify the initial data. With $v = e^{-x}y$, the initial values are

$$v(0) = y(0) = 0, \quad v'(0) = y'(0) - y(0) = 0, \quad v''(0) = y''(0) - 2y'(0) + y(0) = 0.$$

Thus $c_0 = c_1 = c_2 = 0$. The polynomial $v = x^3/6 + x^5/60$ and its first two derivatives vanish at zero, so the corresponding initial data for $y = e^x v$ vanish as well. Its third derivative is $v''' = 1 + x^2$, and the shift identity verifies the forcing exactly.

Step 4: Identify unavoidable resonant growth. For every choice of c_0, c_1, c_2 ,

$$\lim_{x \rightarrow \infty} \frac{e^{-x} y(x)}{x^5} = \frac{1}{60}.$$

The lower-degree terms disappear after division by x^5 . Hence $y(x)$ is eventually positive and asymptotic to $e^x x^5/60$, so it is unbounded. Homogeneous terms have polynomial degree at most two after removing e^x and cannot cancel the forced degree-five term.

Question 3

Consider the initially resting response of a fourth-order equation:

$$(D^2 + 1)^2 y = \cos x, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

Tasks

1. Identify the resonance multiplicity and choose a full real particular-solution trial. Explain why multiplying the ordinary sine-cosine trial by only x still fails.
2. Determine a particular solution by direct substitution, then add the complete homogeneous family.
3. Impose the four initial data and verify the resulting solution explicitly.
4. Evaluate the IVP solution at $x = 2\pi n$ and prove quadratic growth along this sequence. Explain why no homogeneous correction can make any solution bounded on $[0, \infty)$.

Question 3 – Solution

Strategy. Account for the double conjugate roots before imposing the initial state.

Step 1: Use the full resonance multiplicity. The roots $\pm i$ both have multiplicity two. The functions $\cos x, \sin x, x \cos x, x \sin x$ are all homogeneous, so an x -weighted trial still gives zero. Use $y_p = x^2(A \cos x + B \sin x)$.

Step 2: Match the forcing directly. A first application gives $(D^2 + 1)(x^2 \cos x) = 2 \cos x - 4x \sin x$, and a second gives $-8 \cos x$. Likewise, $(D^2 + 1)^2(x^2 \sin x) = -8 \sin x$. Thus $A = -1/8$, $B = 0$, and

$$y = -\frac{x^2 \cos x}{8} + (a + bx) \cos x + (c + dx) \sin x.$$

Step 3: Recover the initially resting solution. The particular term has initial vector $(0, 0, -1/4, 0)$. The homogeneous term has vector $(a, b + c, -a + 2d, -3b - c)$. Setting their sum to zero gives $a = b = c = 0, d = 1/8$, hence

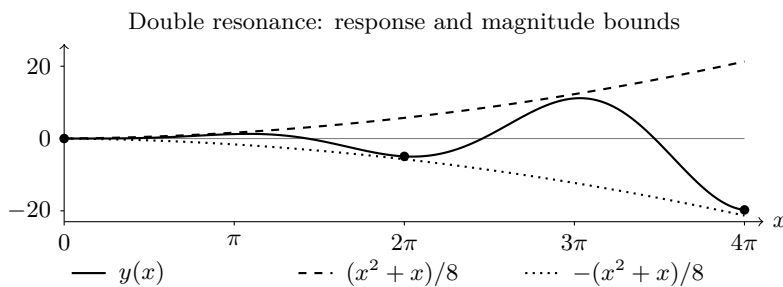
$$y = \frac{x \sin x - x^2 \cos x}{8}.$$

The two numerator terms cancel to order two: $x \sin x - x^2 \cos x = x^4/3 + O(x^6)$ near zero. Thus y and its first three derivatives vanish there. The term $x \sin x/8$ is homogeneous, while the already checked particular term produces $\cos x$.

Step 4: Separate quadratic forcing from linear freedom. At $x = 2\pi n$,

$$y(2\pi n) = -\frac{(2\pi n)^2}{8}.$$

This proves unbounded quadratic growth in magnitude along that sequence. Every homogeneous term is $O(x)$ on the positive half-line, so no correction can cancel these quadratic values. The figure shows the response and the valid envelopes $\pm(x^2 + x)/8$.



Question 4

On $\mathbb{R}$, consider

$$y^{(4)} - y = 6x^2 + 8e^{-x} + 10 \cos x.$$

Its three forcing terms require different coefficient calculations.

Tasks

1. Factor the characteristic polynomial, write the full real homogeneous solution, and select a separate particular-solution trial for each forcing term.
2. Determine all coefficients and assemble one particular solution.
3. Verify each forcing contribution directly using $L = D^4 - 1$, rather than relying only on the trial-selection rule. Write the complete general solution.
4. Decide whether any solution is bounded on $[0, \infty)$. Prove your answer even when homogeneous constants are chosen to cancel all removable growing terms.

Question 4 – Solution

Strategy. Use linearity to separate polynomial forcing from two distinct resonant frequencies.

Step 1: Identify roots and trials. The characteristic roots are $1, -1, i, -i$, all simple, so

$$y_h = Ae^x + Be^{-x} + C \cos x + D \sin x.$$

Use $ax^2 + bx + c$ for $6x^2$, kxe^{-x} for $8e^{-x}$, and $x(m \cos x + n \sin x)$ for $10 \cos x$. Only the last two blocks are resonant.

Step 2: Compute the coefficients. The fourth derivative of a quadratic is zero, giving $a = -6, b = c = 0$. For a simple root λ of $P(r) = r^4 - 1$, the shift identity gives $P(D)(xe^{\lambda x}) = P'(\lambda)e^{\lambda x}$. Since $P'(-1) = -4$, we obtain $k = -2$.

For the trigonometric terms, $L(x \cos x) = 4 \sin x$ and $L(x \sin x) = -4 \cos x$. Therefore $m = 0, n = -5/2$, and

$$y_p = -6x^2 - 2xe^{-x} - \frac{5}{2}x \sin x.$$

Step 3: Verify all three residuals. Direct differentiation gives

$$L(-6x^2) = 6x^2, \quad D^4(xe^{-x}) = e^{-x}(x - 4),$$

$$D^4(x \sin x) = x \sin x - 4 \cos x.$$

Thus the second and third terms produce $8e^{-x}$ and $10 \cos x$ exactly. The full solution is $y = y_h + y_p$; no homogeneous constants are determined without additional data.

Step 4: Rule out bounded solutions. If $A \neq 0$, the Ae^x term dominates every other term, so boundedness is impossible. If $A = 0$, division by x^2 gives

$$\lim_{x \rightarrow \infty} \frac{y(x)}{x^2} = -6,$$

because the remaining oscillatory growth is only $O(x)$ and the other exponential terms decay. Hence even after removing the positive exponential, quadratic forcing remains. No solution is bounded on the positive half-line.

Question 5

Consider

$$(D + 1)^2(D - 2)y = e^{-x}.$$

An annihilator can simplify the forcing while introducing extra solutions that must later be rejected.

Tasks

1. Apply a minimal constant-coefficient annihilator of the forcing and write the full solution of the resulting fourth-order homogeneous equation.
2. Substitute that enlarged family into the original third-order equation. Find the constraint that removes the extra freedom and write the actual general solution.
3. Find the solution with $y(0) = y'(0) = y''(0) = 0$. Determine the coefficient of its growing exponential and its limit after multiplication by e^{-2x} .
4. Instead require $y(0) = y'(0) = 0$ and decay as $x \rightarrow +\infty$. Find this solution and the initial value $y''(0)$ it requires. Explain why decaying forcing does not guarantee that the zero-data response decays.

Question 5 – Solution

Strategy. Retain the original equation as a constraint after annihilating its right-hand side.

Step 1: Solve the enlarged equation. The forcing is annihilated by $D + 1$. Applying it gives $(D + 1)^3(D - 2)y = 0$, whose general solution is

$$y = e^{-x}(A + Bx + Cx^2) + Ee^{2x}.$$

This family has four constants, although the original equation has order three.

Step 2: Restore the original forcing. Under $y = e^{-x}v$, the original operator becomes $e^{-x}D^2(D - 3)v$. For $v = A + Bx + Cx^2$ its value is $-6Ce^{-x}$; the Ee^{2x} term is homogeneous. Thus $C = -1/6$, leaving

$$y = e^{-x}(A + Bx - x^2/6) + Ee^{2x}.$$

Annihilation alone only ensures that the original residual is a multiple of e^{-x} , not that its coefficient equals one.

Step 3: Fit the zero initial vector. The three equations are $A + E = 0$, $B - A + 2E = 0$, and $A - 2B - 1/3 + 4E = 0$. They yield $E = 1/27$, $A = -1/27$, $B = -1/9$, so

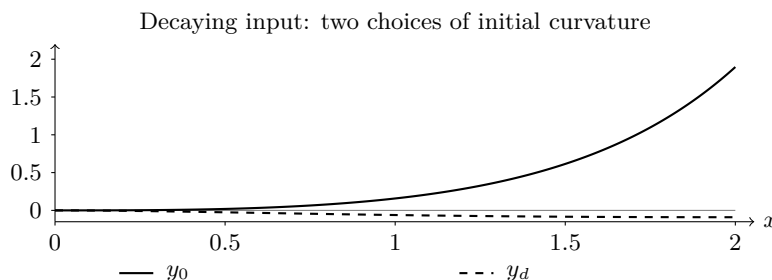
$$y_0 = \frac{e^{2x}}{27} - e^{-x} \left(\frac{1}{27} + \frac{x}{9} + \frac{x^2}{6} \right), \quad \lim_{x \rightarrow \infty} e^{-2x} y_0 = \frac{1}{27}.$$

The displayed coefficient equations verify the initial data, and Step 2 verifies the forcing.

Step 4: Replace curvature data by a future condition. Decay requires $E = 0$; with $y(0) = y'(0) = 0$, this gives $A = B = 0$. Hence

$$y_d = -x^2 e^{-x}/6, \quad y_d''(0) = -1/3.$$

This is the unique decaying response with those first two data. The zero-data response must add a growing homogeneous mode to cancel the particular solution's initial curvature. Decay of the input alone does not suppress that mode.



Question 6

Let $\omega \geq 0$ and consider the zero-data response

$$(D^2 + 1)(D^2 + 4)y_\omega = \cos(\omega x), \quad y_\omega(0) = y'_\omega(0) = y''_\omega(0) = y'''_\omega(0) = 0.$$

Tasks

1. For $\omega \neq 1, 2$, find a particular solution and then the full zero-data response by adding homogeneous modes.
2. Solve the IVP separately at each resonant frequency $\omega = 1$ and $\omega = 2$, using resonance-corrected trials.
3. Rewrite the nonresonant answer using differences of cosines and prove that it tends to the resonant answer as $\omega \rightarrow 1$ for every fixed x .
4. Classify which zero-data responses are bounded on $[0, \infty)$. Explain why the fixed- x limit in the previous task does not provide a bound uniform over all $x \geq 0$ as $\omega \rightarrow 1$.

Question 6 – Solution

Strategy. Enforce the initial data before taking a resonant limit; separate divergent terms can cancel.

Step 1: Solve away from resonance. Set $K = [(1 - \omega^2)(4 - \omega^2)]^{-1}$. The trial $K \cos \omega x$ has the required residual. Odd initial derivatives vanish; adding $A \cos x + C \cos 2x$ and matching the value and second derivative gives

$$y_\omega = K \cos \omega x - \frac{\cos x}{3(1 - \omega^2)} + \frac{\cos 2x}{3(4 - \omega^2)}, \quad \omega \neq 1, 2.$$

Indeed $A + C = -K$ and $A + 4C = -K\omega^2$, which verify the remaining two data.

Step 2: Solve at the two simple resonances. For $L = (D^2 + 1)(D^2 + 4)$, direct differentiation gives $L(x \sin x) = 6 \cos x$ and $L(x \sin 2x) = -12 \cos 2x$. Adding homogeneous cosine terms to enforce zero value and curvature yields

$$y_1 = \frac{x \sin x}{6} + \frac{\cos 2x - \cos x}{9}, \quad y_2 = -\frac{x \sin 2x}{12} + \frac{\cos x - \cos 2x}{9}.$$

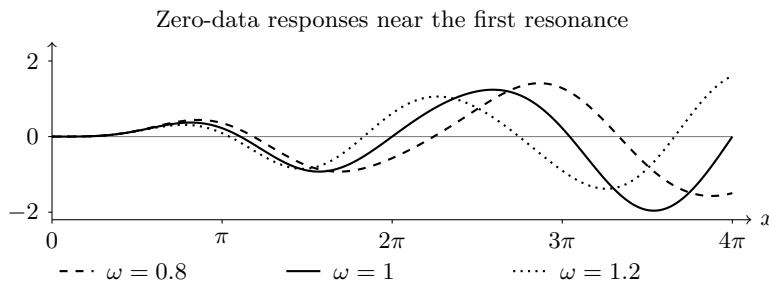
Both are even and vanish at zero. Their second derivatives at zero vanish by cancellation of $1/3$ with $-1/3$ in each case, so all four initial data hold.

Step 3: Take the limit of the combined response. Partial fractions give

$$y_\omega = \frac{1}{3} \left(\frac{\cos \omega x - \cos x}{1 - \omega^2} - \frac{\cos \omega x - \cos 2x}{4 - \omega^2} \right).$$

For fixed x , the first ratio tends to $x \sin x/2$ by differentiating numerator and denominator with respect to ω . The second tends to $(\cos x - \cos 2x)/3$. The resulting limit is exactly y_1 .

Step 4: Distinguish the two limit questions. Every fixed nonresonant response is a bounded sum of cosines. At resonance, the linear sine term has unbounded values along its phase maxima, and the added cosines remain bounded. Thus boundedness holds exactly when $\omega \notin \{1, 2\}$. A common bound for all $x \geq 0$ and all nonresonant ω sufficiently near 1 would pass to the fixed- x limit and bound y_1 everywhere, a contradiction.



Question 7

For real parameters a, b, c , consider

$$y^{(4)} + y'' = a + b \cos x + c \cos 2x.$$

A forcing with zero average over a period need not admit a periodic response.

Tasks

1. Use undetermined coefficients to find a particular solution for all a, b, c , and write the full real general solution.
2. Give necessary and sufficient conditions on the parameters and homogeneous constants for boundedness on $[0, \infty)$.
3. Determine exactly when the equation admits a 2π -periodic solution. Verify the necessary restrictions independently by integrating the equation against 1 and $\cos x$ over one period.
4. Give a forcing in this family with zero average but no periodic solution, and another nonzero forcing that does admit periodic solutions. Explain the different obstructions caused by a nonzero mean and by a resonant harmonic.

Question 7 – Solution

Strategy. Track the zero-frequency resonance and the oscillatory resonance separately.

Step 1: Calculate the resonant and nonresonant pieces. The characteristic polynomial is $r^2(r^2 + 1)$. Direct substitution gives

$$L(x^2/2) = 1, \quad L(x \sin x) = -2 \cos x, \quad L(\cos 2x) = 12 \cos 2x.$$

Consequently,

$$y = \frac{ax^2}{2} - \frac{b}{2}x \sin x + \frac{c}{12} \cos 2x + k_0 + k_1x + A \cos x + B \sin x.$$

The factors x^2 and x reflect the double root at zero and simple roots at $\pm i$, respectively.

Step 2: Classify boundedness exactly. If $a \neq 0$, division by x^2 gives a nonzero limit, so boundedness fails. Once $a = 0$, a bounded solution would satisfy

$$0 = \lim_{x \rightarrow \infty} \frac{y(x)}{x}, \quad \frac{y(x)}{x} = k_1 - \frac{b}{2} \sin x + o(1).$$

Along sequences with sine equal to 1 and -1 , this requires both $k_1 - b/2 = 0$ and $k_1 + b/2 = 0$. Hence

$$\text{bounded on } [0, \infty) \iff a = b = k_1 = 0.$$

These conditions suffice because all remaining terms are bounded.

Step 3: Check periodic compatibility independently. With $a = b = 0$, choosing $k_1 = 0$ gives a 2π -periodic solution for every c . Conversely, for a 2π -periodic C^4 solution, integration gives

$$\int_0^{2\pi} (y^{(4)} + y'') dx = 0 = 2\pi a.$$

Integration by parts transfers the even derivatives onto $\cos x$ without boundary terms. Since $(D^4 + D^2) \cos x = 0$, orthogonality then gives $0 = b\pi$. Thus $a = b = 0$ is necessary and sufficient for existence of a 2π -periodic response.

Step 4: Explain why zero mean is insufficient. The forcing $\cos x$ has zero average but produces the unavoidable term $-x \sin x/2$, so no solution is periodic. In contrast, $\cos 2x$ admits $y = \cos 2x/12$. A nonzero mean excites quadratic drift through the double zero root; a first harmonic excites linear oscillatory growth. Removing only the mean addresses just one of these two resonances.

Question 8

Consider the initially resting response

$$(D^2 + 1)(D^2 + 9)y = \cos^3 x, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

A nonlinear expression in the independent variable can conceal several resonant harmonics while the differential equation remains linear in y .

Tasks

1. Expand the forcing into elementary harmonics. Identify all resonances and select a complete real particular-solution trial.
2. Determine a particular solution by coefficient matching, and write the full homogeneous correction.
3. Impose the initial data and verify the resulting response, including the forcing and the initial curvature.
4. Evaluate the IVP solution at $x_n = \pi/2 + 2\pi n$, $n = 0, 1, \dots$, to prove unboundedness. Derive an explicit envelope for its magnitude valid for every $x \geq 0$.

Question 8 – Solution

Strategy. Expand the trigonometric product before choosing resonance factors.

Step 1: Expose both forcing frequencies. The identity $\cos^3 x = (3 \cos x + \cos 3x)/4$ reveals simple resonance at both $\pm i$ and $\pm 3i$. A full trial is

$$x(A \cos x + B \sin x + C \cos 3x + D \sin 3x).$$

There is no nonlinear dependence on the unknown y ; the right-hand side is a known function of x .

Step 2: Match each harmonic. For $L = (D^2 + 1)(D^2 + 9)$, differentiation gives $L(x \sin x) = 16 \cos x$ and $L(x \sin 3x) = -48 \cos 3x$. The cosine-weighted partners generate sine forcing and therefore have zero coefficients. Thus

$$y_p = \frac{3x \sin x}{64} - \frac{x \sin 3x}{192}, \quad y_h = a \cos x + b \sin x + c \cos 3x + d \sin 3x.$$

The residual is $3 \cos x/4 + \cos 3x/4 = \cos^3 x$.

Step 3: Correct the initial curvature. The particular term is even, has value zero, and has $y_p''(0) = 3/32 - 1/32 = 1/16$. The zero data require $a + c = 0$, $-a - 9c = -1/16$, and $b + 3d = -b - 27d = 0$. Hence

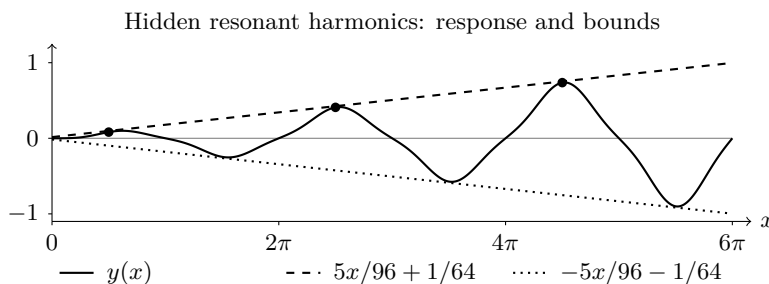
$$y = \frac{x(9 \sin x - \sin 3x)}{192} + \frac{\cos 3x - \cos x}{128}.$$

Here $a = -1/128, c = 1/128, b = d = 0$. The cosine correction has second derivative $-1/16$ at zero and cancels the particular curvature; evenness verifies the odd initial derivatives.

Step 4: Prove growth and a global envelope. At $x_n = \pi/2 + 2\pi n$, the sine values are 1, -1 and both cosines vanish, so $y(x_n) = 5x_n/96 \rightarrow \infty$. Also $9 \sin x - \sin 3x = 6 \sin x + 4 \sin^3 x$, whose absolute value is at most 10. Therefore

$$|y(x)| \leq \frac{5x}{96} + \frac{1}{64} \quad (x \geq 0).$$

The displayed envelopes are bounds, not assertions that equality holds at every point.



Question 9

For the operator $L = (D + 1)^3$, consider three separate forcing functions

$$f_1(x) = x^2 e^{-x}, \quad f_2(x) = \sin^2 x, \quad f_3(x) = e^{x^2}.$$

The standard finite undetermined-coefficient method uses polynomial-exponential-trigonometric families closed under differentiation, with resonance corrections when required.

Tasks

1. Decide which forcings admit this standard finite method. Give complete real trials for each admissible forcing, first simplifying trigonometric products.
2. Compute and verify one particular solution for each admissible forcing. State the common homogeneous family to be added in both cases.
3. Prove that the derivatives of e^{x^2} span an infinite-dimensional space: write $D^k e^{x^2} = P_k(x) e^{x^2}$ and determine the degree and leading coefficient of P_k .
4. Explain why this rules out the standard finite trial method for f_3 , while not ruling out existence or uniqueness of the IVP with any prescribed $y(0), y'(0), y''(0)$. Do not use variation of parameters to solve it here.

Question 9 – Solution

Strategy. Test closure under differentiation, then distinguish a method's scope from the equation's solvability.

Step 1: Select the admissible trial families. The root -1 has multiplicity three. For f_1 , use $y_{p,1} = x^3 e^{-x}(Ax^2 + Bx + C)$. For $f_2 = (1 - \cos 2x)/2$, none of the frequencies $0, \pm 2i$ is a characteristic root, so use $y_{p,2} = a + b \cos 2x + c \sin 2x$. Both have finite derivative families. The third forcing does not, as shown below.

Step 2: Match and check the admissible cases. For $y = e^{-x}v$, the equation becomes $v''' = x^2$ in the first case. Thus

$$y_{p,1} = x^5 e^{-x} / 60.$$

Its shifted third derivative is exactly x^2 . In the second case,

$$L(1) = 1, \quad L(\cos 2x) = -11 \cos 2x + 2 \sin 2x, \quad L(\sin 2x) = -2 \cos 2x - 11 \sin 2x.$$

The coefficient equations are $a = 1/2$, $-11b - 2c = -1/2$, $2b - 11c = 0$. They give

$$y_{p,2} = \frac{1}{2} + \frac{11}{250} \cos 2x + \frac{1}{125} \sin 2x.$$

Each complete solution adds $e^{-x}(c_0 + c_1 x + c_2 x^2)$.

Step 3: Prove infinite derivative dimension. Start with $P_0 = 1$. Differentiation gives $P_{k+1} = P'_k + 2xP_k$. If P_k has degree k and leading coefficient 2^k , the second term has degree $k+1$ and leading coefficient 2^{k+1} , whereas the derivative has lower degree. Induction proves these claims for every k . Polynomials of distinct degrees are linearly independent; multiplying by the never-zero e^{x^2} preserves independence. Hence no nonzero constant-coefficient operator annihilates f_3 .

Step 4: Interpret the obstruction correctly. Every standard finite trial, even after multiplication by a resonance power, lies in a finite-dimensional differentiation-invariant family. Applying L keeps it in that family. It therefore cannot produce e^{x^2} , whose derivative family is infinite-dimensional. This is a limitation of the specified method. The normalized equation has constant coefficients and smooth forcing on $\mathbb{R}$, so the linear IVP theorem still gives a unique global solution for any three initial data.

Question 10

A forcing pulse acts on a third-order equation for $x \geq 0$:

$$(D+1)^3 y = \begin{cases} e^{-x} p(x), & 0 < x < 1, \\ 0, & x > 1, \end{cases} \quad p(x) = 120x^3 + ax^2 + bx + c.$$

Initially $y(0) = y'(0) = y''(0) = 0$. Require y, y', y'' to be continuous at $x = 1$; the equation is imposed on the two open intervals, not at the switch itself.

Tasks

1. Set $y = e^{-x}v$. Write the shifted equations and identify the conditions at $x = 1$ equivalent to having $y(x) = 0$ for every $x \geq 1$.
2. Use a polynomial undetermined-coefficient solution on $[0, 1]$ to find the unique a, b, c producing this complete cancellation. Give y on both intervals.
3. Prove that no nonzero polynomial pulse of degree at most two can achieve the same cancellation under the same zero initial data.
4. Verify all initial and joining conditions and the forcing. Determine whether the resulting solution is C^3 across $x = 1$, and locate the minimum of the scaled response $e^x y$ on $[0, 1]$.

Question 10 – Solution

Strategy. Convert the pulse into a polynomial third derivative and enforce triple zeros at both ends.

Step 1: Translate the initial and terminal states. The shift gives $v''' = p$ on $(0, 1)$ and $v''' = 0$ on $(1, \infty)$. Multiplication by the smooth, nonzero e^{-x} preserves matching of the first three state components. Thus $v(0) = v'(0) = v''(0) = 0$, and complete cancellation requires $v(1) = v'(1) = v''(1) = 0$. These terminal conditions also suffice: the post-pulse quadratic solution is then zero.

Step 2: Determine the polynomial exactly. The zero-data polynomial solution of $v''' = p$ has degree six and leading coefficient 1, since $D^3x^6 = 120x^3$. The three zero conditions at each endpoint require divisibility by $x^3(x-1)^3$. Equal degrees and leading coefficients therefore force

$$v = x^3(x-1)^3 = x^6 - 3x^5 + 3x^4 - x^3.$$

Differentiating three times gives $p = 120x^3 - 180x^2 + 72x - 6$. Hence

$$(a, b, c) = (-180, 72, -6), \quad y(x) = \begin{cases} e^{-x}x^3(x-1)^3, & 0 \leq x \leq 1, \\ 0, & x \geq 1. \end{cases}$$

The formulas agree at the shared endpoint and uniquely determine the pulse.

Step 3: Prove the degree is necessary. For a polynomial p of degree at most two, the zero-data solution v has degree at most five. The terminal requirements would still make it divisible by the degree-six polynomial $x^3(x-1)^3$. It must therefore be zero, and so must $p = v'''$. No nonzero lower-degree polynomial pulse works.

Step 4: Check the join and the scaled shape. The triple endpoint zeros verify all six state conditions. The computed third derivative verifies the forcing before the switch; the zero branch verifies it afterward. At 1, the left third derivative of y is $e^{-1}v'''(1) = 6/e$, while the right third derivative is zero. Thus y is C^2 but not C^3 across the switch, exactly as allowed.

On $[0, 1]$, $e^xy = -[x(1-x)]^3$. Since $x(1-x)$ has maximum $1/4$ at $x = 1/2$, the scaled response has its unique minimum $-1/64$ there.

