

Question 1

Write $D = d/dx$. Consider the constant-coefficient homogeneous equation

$$D^2(D-1)(D+2)((D+1)^2+4)y=0$$

on $\mathbb{R}$. Long-term behavior depends on both the locations and the multiplicities of its characteristic roots.

Tasks

1. List every characteristic root with its multiplicity, state the order, and write the complete real general solution.
2. Characterize exactly which solutions are bounded on $[0, \infty)$. Give the dimension of that subspace.
3. Characterize exactly which solutions tend to zero as $x \rightarrow +\infty$. Give a bounded nonconstant solution that does not tend to zero.
4. Determine every solution bounded on the whole real line. Justify why terms cannot cancel to create additional bounded solutions.

Question 1 – Solution

Strategy. Separate exponential growth, polynomial drift, decay and oscillation before imposing boundedness.

Step 1: Read all multiplicities. The roots are 0 (twice), 1, -2 , and $-1 \pm 2i$. Their multiplicities total six, so a real fundamental family gives

$$y = c_0 + c_1 x + Ae^x + Be^{-2x} + e^{-x}(C \cos 2x + S \sin 2x).$$

The six constants are independent. In particular, the double zero root requires both 1 and x .

Step 2: Remove forward growth. If $A \neq 0$, the e^x term dominates as $x \rightarrow +\infty$. Once $A = 0$, the remaining exponential terms decay, so boundedness also requires $c_1 = 0$. Conversely those two conditions suffice:

$$\text{bounded on } [0, \infty) \iff A = c_1 = 0.$$

This is a four-dimensional subspace, with free constants c_0, B, C, S .

Step 3: Distinguish a limit from zero. Every forward-bounded solution tends to c_0 . Thus the decaying subspace has $A = c_1 = c_0 = 0$ and dimension three. For example, $1 + e^{-2x}$ is bounded and nonconstant on $[0, \infty)$ but tends to 1.

Step 4: Impose the backward restriction. Start with a forward-bounded solution. Multiplication by e^{2x} and passage to $x \rightarrow -\infty$ shows $B = 0$, since bounded y would give a zero limit. If $(C, S) \neq (0, 0)$, write $C \cos 2x + S \sin 2x = R \cos(2x - \phi)$ with $R > 0$. Along a sequence tending to $-\infty$ at cosine maxima, Re^{-x} is unbounded. Hence $C = S = 0$ as well, leaving

$$y(x) = c_0 \quad \text{as the entire-line bounded family.}$$

Different growth rates and the explicit oscillatory subsequence rule out hidden cancellation.

Question 2

Consider the fourth-order initial-value problem

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 2, \quad y'''(0) = -6.$$

All characteristic roots can lie strictly in the left half-plane while a particular response first grows in magnitude.

Tasks

1. Factor the characteristic polynomial and write a complete real general solution.
2. Set $y = e^{-x}v$. Derive the equation and all four initial data for v , then solve the IVP without a four-by-four coefficient elimination.
3. Find every critical point of the IVP solution on $[0, \infty)$ and its exact global maximum there. Determine its limit at infinity.
4. Prove that every solution of the differential equation tends to zero as $x \rightarrow +\infty$. Explain why this does not require each solution's magnitude to decrease from its initial value.

Question 2 – Solution

Strategy. Remove the repeated exponential factor, then analyze the resulting polynomial times exponential.

Step 1: Include the entire repeated-root chain. The polynomial is $(r + 1)^4$, so

$$y = e^{-x}(c_0 + c_1x + c_2x^2 + c_3x^3).$$

Four distinct powers are needed for a root of multiplicity four.

Step 2: Transfer the initial derivatives. Since $(D + 1)(e^{-x}v) = e^{-x}v'$, four applications give $v^{(4)} = 0$. Also $v = e^x y$, so the product rule gives

$$v(0) = 0, \quad v'(0) = y'(0) + y(0) = 0, \quad v''(0) = y''(0) + 2y'(0) + y(0) = 2,$$

$$v'''(0) = y'''(0) + 3y''(0) + 3y'(0) + y(0) = 0.$$

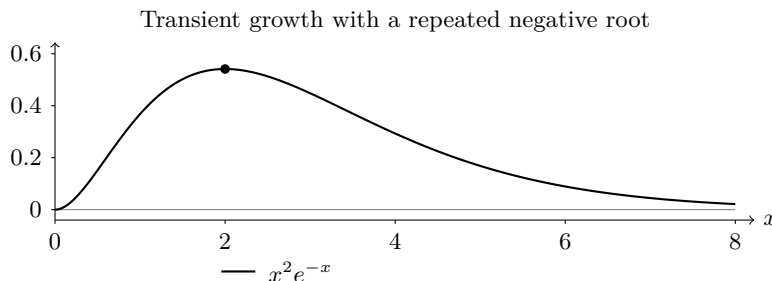
Therefore $v = x^2$ and $\boxed{y = x^2 e^{-x}}$. Its initial value and first three derivatives are 0, 0, 2, -6 as required; the shift identity verifies the equation.

Step 3: Locate the transient maximum. Here $y' = e^{-x}x(2 - x)$. It vanishes at 0 and 2, is positive on $(0, 2)$ and negative on $(2, \infty)$. Thus

$$\boxed{\max_{x \geq 0} y(x) = 4e^{-2} \text{ at } x = 2, \quad \lim_{x \rightarrow \infty} y(x) = 0.}$$

The endpoint 0 is a zero of the derivative and a minimum on the half-line.

Step 4: Separate eventual decay from monotonicity. Each $x^k e^{-x}$, $0 \leq k \leq 3$, tends to zero; hence every linear combination does too. Decay at infinity is an asymptotic statement. This IVP solution starts at zero and increases before decreasing, so neither its value nor its magnitude must decrease throughout the half-line.



Question 3

Consider

$$y^{(4)} + 8y'' + 16y = 0, \quad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 8.$$

A repeated imaginary root differs substantially from a simple imaginary root.

Tasks

1. Find the characteristic roots and write the full real general solution, taking account of multiplicities.
2. Determine the unique IVP solution and verify its four initial data.
3. Evaluate the solution and its first derivative at $x_n = n\pi/2$, $n = 1, 2, \dots$. Use these values to prove unboundedness on $[0, \infty)$.
4. Characterize all solutions of the equation that are bounded on $[0, \infty)$. Prove that no nonzero solution tends to zero at infinity.

Question 3 – Solution

Strategy. Retain the polynomial factors attached to repeated conjugate roots and test growth at suitable phases.

Step 1: Build the real basis. The polynomial is $(r^2 + 4)^2$, with roots $\pm 2i$, each twice. Thus

$$y = (A + Bx) \cos 2x + (C + Dx) \sin 2x.$$

Omitting the factors $x \cos 2x$ and $x \sin 2x$ would lose half the solution space.

Step 2: Fit the initial vector. At zero the four derivatives are

$$A, \quad B + 2C, \quad -4A + 4D, \quad -12B - 8C.$$

The data give $A = D = 0$, $B = -1$ and $C = 1/2$, hence

$$y = \frac{1}{2} \sin 2x - x \cos 2x.$$

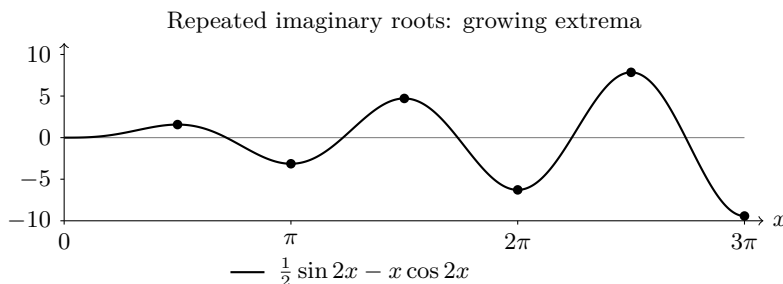
Indeed $y' = 2x \sin 2x$, $y'' = 2 \sin 2x + 4x \cos 2x$, and $y''' = 8 \cos 2x - 8x \sin 2x$, verifying all four data. Each basis term satisfies the factored equation.

Step 3: Exhibit growing extrema. For $x_n = n\pi/2$, we have

$$y'(x_n) = 0, \quad y(x_n) = (-1)^{n+1} \frac{n\pi}{2}, \quad y''(x_n) = 4x_n(-1)^n \neq 0.$$

These are alternating strict extrema with unbounded magnitude. In particular, purely imaginary roots alone do not imply boundedness when they are repeated.

Step 4: Classify the bounded family. If $(B, D) \neq (0, 0)$, choose a sequence $x \rightarrow \infty$ on which $B \cos 2x + D \sin 2x = \sqrt{B^2 + D^2} > 0$. The linear-in- x part grows, whereas $A \cos 2x + C \sin 2x$ is bounded. Therefore boundedness is equivalent to $B = D = 0$, a two-dimensional subspace. A nonzero member of that subspace attains its positive amplitude infinitely often, so it cannot tend to zero. An unbounded member cannot tend to zero either.



Question 4

An observed signal is known exactly on $\mathbb{R}$:

$$f(x) = e^{2x}(1 + x^2) + x \sin 3x.$$

Seek a real, monic, constant-coefficient homogeneous differential equation of the smallest possible order having this signal as a solution. Here monic means that the coefficient of the highest derivative is 1.

Tasks

1. Find a factored characteristic polynomial that annihilates f , carefully determining the multiplicity of every root.
2. Prove that no nonzero constant-coefficient differential operator of smaller order can annihilate f . State which independence fact prevents cancellation between its different exponential modes.
3. Write the complete real general solution of the resulting minimal equation and explain why its monic characteristic polynomial is unique.
4. Must this same equation also admit xe^{2x} and $\cos 3x$ as solutions? Explain why the two displayed summands of f do not mean that a second-order equation suffices. Restrict the conclusion to constant coefficients.

Question 4 – Solution

Strategy. Read the highest polynomial degree at each distinct exponential frequency, including complex conjugates.

Step 1: Find sufficient root multiplicities. The shift identity $P(D)(e^{\lambda x}q) = e^{\lambda x}P(D + \lambda)q$ shows that a quadratic polynomial multiplying e^{2x} is killed by $(D - 2)^3$. Also

$$x \sin 3x = \frac{x}{2i}(e^{3ix} - e^{-3ix}),$$

so each of $\pm 3i$ requires multiplicity two. A sufficient real monic polynomial is

$$P(r) = (r - 2)^3(r^2 + 9)^2, \quad \deg P = 7.$$

Step 2: Prove minimality, not just sufficiency. Suppose a polynomial Q has a root of multiplicity m at λ ; write $Q(r) = (r - \lambda)^m R(r)$ with $R(\lambda) \neq 0$. Applied to $e^{\lambda x}q(x)$, the polynomial $R(D + \lambda)q$ retains the degree of q because its leading coefficient is multiplied by $R(\lambda)$. Thus D^m kills it exactly when $m > \deg q$.

Exponential-polynomial modes with distinct complex exponents are linearly independent on $\mathbb{R}$; this is the generalized characteristic-root independence theorem. Therefore the contributions at $2, 3i, -3i$ cannot cancel each other. Every annihilating Q must have multiplicities at least $3, 2, 2$, so must be divisible by P and have degree at least seven.

Step 3: Recover the entire solution space. The real general solution is

$$y = e^{2x}(a_0 + a_1x + a_2x^2) + (b_0 + b_1x) \cos 3x + (c_0 + c_1x) \sin 3x.$$

There are seven free real constants. A monic degree-seven annihilator divisible by the monic degree-seven P must equal P , proving uniqueness at minimal order.

Step 4: Interpret the forced companion modes. Both xe^{2x} and $\cos 3x$ belong to this solution space. The two written summands package several repeated-root modes: their number is not the differential order. The lower bound concerns nonzero constant-coefficient operators only; it does not assert a minimum order among variable-coefficient equations.

Question 5

For the fourth-order equation

$$y^{(4)} - 5y'' + 4y = 0,$$

write the initial derivatives as $(y(0), y'(0), y''(0), y'''(0)) = (a, b, c, d)$. For each simple characteristic root λ , define the polynomial

$$\Pi_\lambda(r) = \prod_{\mu \neq \lambda} \frac{r - \mu}{\lambda - \mu},$$

where the product runs over the other characteristic roots.

Tasks

1. Find the roots and prove that $\Pi_\lambda(D)y$ isolates the single mode $A_\lambda e^{\lambda x}$ in the general solution.
2. Compute the coefficients of e^x and e^{2x} explicitly from a, b, c, d using these polynomials.
3. Give necessary and sufficient conditions on the initial derivatives for boundedness on $[0, \infty)$. Express them also as two relations obtained from $(D+1)(D+2)y = 0$.
4. Start from $(a, b, c, d) = (1, -1, 1, -1)$ and perturb only d by a nonzero amount ε . Find the limit

$$\lim_{x \rightarrow \infty} e^{-2x} y_\varepsilon(x).$$

Explain the long-term consequence of arbitrarily small such errors.

Question 5 – Solution

Strategy. Use polynomial interpolation at the roots to extract growing modes directly from the initial data.

Step 1: Construct mode selectors. The polynomial factors as $(r-1)(r+1)(r-2)(r+2)$. Thus y is a linear combination of $e^{\lambda x}$ for $\lambda = -2, -1, 1, 2$. Since $Q(D)e^{\mu x} = Q(\mu)e^{\mu x}$ and $\Pi_\lambda(\mu)$ is 1 if $\mu = \lambda$ and 0 otherwise,

$$\Pi_\lambda(D)y = A_\lambda e^{\lambda x}.$$

Evaluating at zero reads off A_λ without solving all four coefficients simultaneously.

Step 2: Extract the unstable coefficients. The relevant polynomials are

$$\begin{aligned}\Pi_1(r) &= -\frac{(r+2)(r+1)(r-2)}{6} = \frac{-r^3 - r^2 + 4r + 4}{6}, \\ \Pi_2(r) &= \frac{(r+2)(r+1)(r-1)}{12} = \frac{r^3 + 2r^2 - r - 2}{12}.\end{aligned}$$

Hence

$$\boxed{A_1 = \frac{4a + 4b - c - d}{6}, \quad A_2 = \frac{d + 2c - b - 2a}{12}.$$

Step 3: Describe exactly the decaying initial states. Forward boundedness requires $A_2 = A_1 = 0$; distinct positive exponential rates cannot cancel. The remaining modes decay. Equivalently,

$$\boxed{c + 3b + 2a = 0, \quad d + 3c + 2b = 0.}$$

These follow from $y'' + 3y' + 2y = 0$ and its derivative for the two negative-root modes. Conversely, substituting these relations into A_1, A_2 makes both zero, so the relations are sufficient as well.

Step 4: Quantify the loss of cancellation. The unperturbed solution is e^{-x} . Replacing $d = -1$ by $-1 + \varepsilon$ produces $A_1 = -\varepsilon/6$ and $A_2 = \varepsilon/12$. All modes other than e^{2x} disappear after multiplication by e^{-2x} in the limit, giving

$$\boxed{\lim_{x \rightarrow \infty} e^{-2x} y_\varepsilon(x) = \varepsilon/12.}$$

Every nonzero error of this form eventually yields exponential growth in magnitude. Exact decay requires precise cancellation of both positive-root modes.

Question 6

Let $\alpha \in \mathbb{R}$ and consider the third-order homogeneous equation

$$(D + 1)(D^2 + 2\alpha D + 1)y = 0$$

on $\mathbb{R}$. Call the equation *forward bounded* if every solution is bounded on $[0, \infty)$, and *forward decaying* if every solution tends to zero as $x \rightarrow +\infty$.

Tasks

1. Find the characteristic roots and identify all values of α where roots coincide, including coincidence with the fixed root -1 .
2. Write a complete real general solution in every parameter regime. Treat $\alpha = 1$ and $\alpha = -1$ explicitly.
3. Classify exactly the parameters for forward boundedness and forward decay. Prove the boundary cases rather than deciding from root real parts alone.
4. At $\alpha = 0, 1, -1$, respectively, give a nondecaying bounded solution, a decaying solution with a nonconstant polynomial factor, and an unbounded solution. Explain the different effects of repeated negative and repeated positive roots.

Question 6 – Solution

Strategy. Track the quadratic roots and separately check when either merges with the fixed root.

Step 1: Locate the collisions. Besides $r = -1$, the roots are $r_{\pm} = -\alpha \pm \sqrt{\alpha^2 - 1}$. They coincide when $\alpha = \pm 1$. Substituting $r = -1$ in the quadratic gives $2 - 2\alpha$, so coincidence with the fixed root occurs only at $\alpha = 1$, producing a triple root -1 .

Step 2: State a complete basis in each case. For $|\alpha| > 1$, all three roots are real and distinct, and

$$y = Ce^{-x} + Ae^{r_+x} + Be^{r_-x}.$$

For $|\alpha| < 1$, let $\beta = \sqrt{1 - \alpha^2} > 0$. Then

$$y = Ce^{-x} + e^{-\alpha x}(A \cos \beta x + B \sin \beta x).$$

The exceptional cases are

$$\begin{aligned}\alpha = 1 : \quad y &= e^{-x}(C + Ax + Bx^2), \\ \alpha = -1 : \quad y &= Ce^{-x} + (A + Bx)e^x.\end{aligned}$$

Each family has three free constants.

Step 3: Classify all forward behavior. If $\alpha > 1$, both quadratic roots are negative since $\sqrt{\alpha^2 - 1} < \alpha$. If $0 < \alpha < 1$, their real part is $-\alpha < 0$. At $\alpha = 1$, even the quadratic polynomial factor is dominated by e^{-x} . Thus every solution decays for every $\alpha > 0$.

At $\alpha = 0$, the modes are $e^{-x}, \cos x, \sin x$: all are bounded, but not all decay. If $-1 < \alpha < 0$, the oscillatory amplitude grows exponentially; choosing a cosine and its phase maxima proves unboundedness. At $\alpha = -1$ there are positive-root modes, and for $\alpha < -1$ both quadratic roots are positive. Consequently,

forward bounded $\iff \alpha \geq 0$,	forward decaying $\iff \alpha > 0$.
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Step 4: Test the special parameters concretely. The requested examples are $\cos x$ at $\alpha = 0$, x^2e^{-x} at $\alpha = 1$, and xe^x at $\alpha = -1$. A finite polynomial factor does not defeat exponential decay at a negative root. A positive root already permits growth, and repetition adds further polynomial amplification.

Question 7

A real solution of

$$y^{(4)} + 5y'' + 4y = 0$$

is sampled at all times $x = n\pi$, $n = 0, 1, 2, \dots$. The measured values are 3 for even n and 1 for odd n .

Tasks

1. Find the complete real general solution and determine exactly what the value samples reveal about its four coefficients.
2. Describe the full subspace of solutions invisible to these samples, meaning $y(n\pi) = 0$ for every non-negative integer n . Prove that infinitely many value samples still leave nonuniqueness.
3. Two additional measurements give $y'(0) = 1$ and $y'(\pi) = 3$. Recover the unique solution and verify every type of measurement.
4. Prove that, for any solution of this equation, the four measurements $y(0), y(\pi), y'(0), y'(\pi)$ uniquely determine it. Explain why the equally spaced value samples alone miss actual oscillations between sampling times.

Question 7 – Solution

Strategy. Evaluate the real modes at the sampling grid before interpreting the amount of information in the data.

Step 1: Read the visible coefficients. The polynomial is $(r^2 + 1)(r^2 + 4)$, so

$$y = A \cos x + B \sin x + C \cos 2x + D \sin 2x.$$

At the sample times, $y(n\pi) = A(-1)^n + C$. Thus $A + C = 3$ and $-A + C = 1$, giving $A = 1$ and $C = 2$. These samples impose no restriction on B, D .

Step 2: Find the invisible subspace. Zero values at both even and odd sample indices force $A = C = 0$, while the sine modes vanish at every $n\pi$. Therefore

$$\{y : y(n\pi) = 0 \ \forall n \geq 0\} = \text{span}\{\sin x, \sin 2x\}.$$

It has dimension two. Adding any member to one fitting signal leaves every recorded value unchanged, even though the functions differ between samples.

Step 3: Recover the missing oscillations. The derivative samples give $B + 2D = 1$ and $-B + 2D = 3$, hence $B = -1, D = 1$. The unique signal is

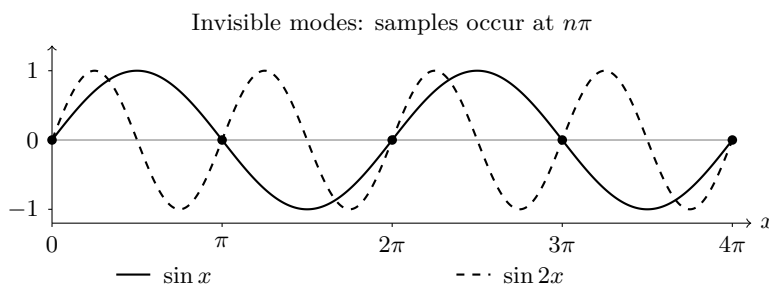
$$y = \cos x - \sin x + 2 \cos 2x + \sin 2x.$$

Its value samples are $(-1)^n + 2$ and its derivative samples at $0, \pi$ are $1, 3$, respectively. Every term solves the equation.

Step 4: Prove general identifiability. For arbitrary measurements $u = y(0), v = y(\pi), p = y'(0)$ and $q = y'(\pi)$,

$$A = \frac{u - v}{2}, \quad C = \frac{u + v}{2}, \quad B = \frac{p - q}{2}, \quad D = \frac{p + q}{4}.$$

These formulas exist and are unique for every real measurement vector. Value samples alone repeat only two linear constraints; their number does not restore the sine information. The figure displays the two independent invisible modes and their common sample zeros.



Question 8

Consider two homogeneous constant-coefficient operators

$$P(D) = (D - 1)^2(D + 2)(D^2 + 1), \quad Q(D) = (D - 1)(D + 2)^2(D^2 + 4).$$

A *common solution* must satisfy both $P(D)y = 0$ and $Q(D)y = 0$ on $\mathbb{R}$.

Tasks

1. Write the complete real general solution of each equation separately.
2. Determine the full common solution space and its dimension. Justify why the smaller multiplicity at each shared root is the relevant one.
3. Find the common solution with $y(0) = 3$ and $y'(0) = 0$. Compute the higher initial derivatives $y''(0)$ and $y'''(0)$ that are then forced.
4. Find the monic real constant-coefficient operator of smallest order that annihilates every solution of either equation. Explain why this last requirement uses larger multiplicities and differs from finding common solutions.

Question 8 – Solution

Strategy. Compare each exponent and its polynomial degree, distinguishing an intersection of solution spaces from a space containing both.

Step 1: List the two five-dimensional families. For P , the roots are 1 twice, -2 , and $\pm i$, giving

$$y_P = (a + bx)e^x + ce^{-2x} + d \cos x + e \sin x.$$

For Q , the roots are 1, -2 twice, and $\pm 2i$, giving

$$y_Q = Ae^x + (B + Cx)e^{-2x} + E \cos 2x + F \sin 2x.$$

All constants in each display are independent.

Step 2: Retain only common modes. Exponential-polynomial modes at different exponents are independent. Equality of the two displays therefore removes both oscillatory pairs, xe^x , and xe^{-2x} . Precisely the simple shared modes survive:

$$y = Ae^x + Be^{-2x}.$$

They are independent, so the common space has dimension two. In polynomial language, the common characteristic factor is the greatest common divisor $(r - 1)(r + 2)$: a mode must satisfy both multiplicity restrictions and hence the smaller one.

Step 3: Fit the common initial data. We need $A + B = 3$ and $A - 2B = 0$, so $A = 2, B = 1$. Thus

$$y = 2e^x + e^{-2x}, \quad y''(0) = 6, \quad y'''(0) = -6.$$

Each exponential is a shared mode, proving both equations directly. The first two data fix the solution within the common space; higher data cannot be chosen independently there.

Step 4: Build the smallest operator containing both kernels. An operator annihilating every solution of either equation must include both repeated real-root chains and both oscillatory pairs. Its characteristic polynomial must be divisible by both P and Q . Their monic least common multiple is

$$R(r) = (r - 1)^2(r + 2)^2(r^2 + 1)(r^2 + 4), \quad \deg R = 8.$$

The multiplicity of each root is now the larger of its two multiplicities. This polynomial is sufficient by the characteristic-root rule and necessary because each listed mode must be killed. Thus its order is minimal and its monic choice unique. Its kernel contains both five-dimensional spaces, whereas the common space is their two-dimensional intersection.

Question 9

Multiplication by x is an operator: $(D - x)u = u' - xu$. On $\mathbb{R}$, consider

$$(D - x)D^2y = 0.$$

Unlike constant-coefficient factors, these factors need not commute.

Tasks

1. Expand the equation. Set $z = y''$ and find its complete real solution space, allowing definite integrals rather than elementary antiderivatives.
2. Define $F(x) = \int_0^x (x - t)e^{t^2/2} dt$. Prove that $1, x, F$ form a fundamental set by computing their Wronskian.
3. Solve the IVP $y(0) = y'(0) = 0, y''(0) = 1$. Prove that its solution is even and satisfies $y(x) \geq x^2/2$ for every real x , with strict inequality when $x \neq 0$.
4. Expand the reversed product $D^2(D - x)y$. Determine all functions that solve both the original and reversed equations, and explain why a characteristic polynomial cannot be used here as if x were constant.

Question 9 – Solution

Strategy. Solve the factors in their given order and use the product rule when reversing them.

Step 1: Reduce the original equation. Expansion gives $y''' - xy'' = 0$. Thus $z' - xz = 0$ and $z = Ce^{x^2/2}$. Integrating twice from zero yields

$$y = A + Bx + CF(x), \quad F(x) = \int_0^x (x-t)e^{t^2/2} dt.$$

This formula holds for negative x as well, with the usual oriented integral.

Step 2: Verify the fundamental set. Differentiating the integral gives $F' = \int_0^x e^{t^2/2} dt$ and $F'' = e^{x^2/2}$, so $F''' = xF''$. The derivative matrix is triangular:

$$W(1, x, F) = \det \begin{pmatrix} 1 & x & F \\ 0 & 1 & F' \\ 0 & 0 & e^{x^2/2} \end{pmatrix} = \boxed{e^{x^2/2} > 0}.$$

These are three independent solutions of a regular third-order linear equation, hence a fundamental set on $\mathbb{R}$.

Step 3: Fit and analyze the initial data. Since $F(0) = F'(0) = 0$ and $F''(0) = 1$, the IVP gives $\boxed{y = F}$. Its second derivative is even, and its value and slope at zero vanish; integrating shows that F is even. For $x > 0$,

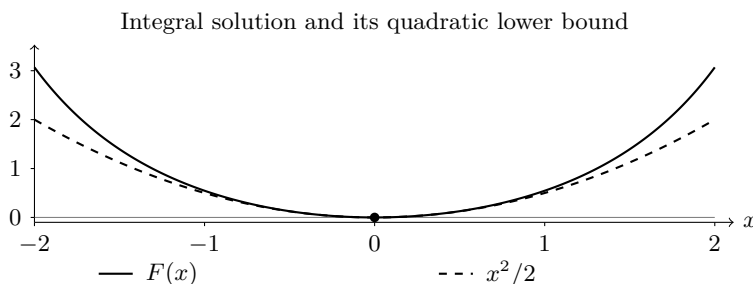
$$F(x) - \frac{x^2}{2} = \int_0^x (x-t)(e^{t^2/2} - 1) dt > 0.$$

The inequality follows because the integrand is positive for $0 < t < x$. Evenness proves the same strict inequality for $x < 0$; equality holds at zero.

Step 4: Respect the order of the factors. The product rule gives

$$D^2(y' - xy) = y''' - xy'' - 2y'.$$

Subtracting the two equations forces $y' = 0$ for any common solution. Conversely, every constant solves both, so their common space consists exactly of constants. The variable coefficient creates the extra $-2y'$ term; treating x as a fixed characteristic root would discard it and give an invalid method.



Question 10

For the fourth-order equation

$$y^{(4)} + 4y = 0,$$

define along a real solution

$$E(x) = y'''(x)y'(x) - \frac{1}{2}(y''(x))^2 + 2y(x)^2.$$

A quantity that is constant along solutions need not control their size.

Tasks

1. Find the characteristic roots and complete real general solution.
2. Differentiate E and prove that it is constant along every solution.
3. Verify that $y = e^x \cos x$ is an unbounded solution with $E = 0$. Give a specific sequence of times proving its unboundedness.
4. Characterize all solutions bounded on $[0, \infty)$ and all solutions bounded on $\mathbb{R}$. Explain why conservation of E does not contradict these classifications and cannot yield a bound of the form $|y(x)|^2 \leq CE(0)$ with a fixed positive C for all solutions.

Question 10 – Solution

Strategy. Compare the root-based behavior with what the conserved expression actually measures.

Step 1: Find growing and decaying oscillatory modes. Since $r^4 + 4 = (r^2 - 2r + 2)(r^2 + 2r + 2)$, the roots are $1 \pm i$ and $-1 \pm i$. Thus

$$y = e^x(A \cos x + B \sin x) + e^{-x}(C \cos x + D \sin x).$$

All roots are simple and the four real modes are independent.

Step 2: Verify the conservation law. The product rule gives exact cancellation:

$$E' = y^{(4)}y' + y'''y'' - y''y''' + 4yy' = y'(y^{(4)} + 4y) = 0.$$

Hence $E(x) = E(0)$ for every x .

Step 3: Test a zero-level growing solution. For $y = e^x \cos x$, the successive derivatives are

$$y' = e^x(\cos x - \sin x), \quad y'' = -2e^x \sin x, \quad y''' = -2e^x(\sin x + \cos x), \quad y^{(4)} = -4y.$$

At zero its initial vector is $(1, 1, 0, -2)$, so $E(0) = -2 + 2 = 0$. Conservation yields $E \equiv 0$. Yet $y(2\pi n) = e^{2\pi n} \rightarrow \infty$, proving unboundedness.

Step 4: Identify the limits of the invariant. On $[0, \infty)$ boundedness requires $A = B = 0$: any nonzero growing sinusoid attains its amplitude along an unbounded sequence, while the other term decays. These conditions also suffice, giving a two-dimensional decaying family. Requiring boundedness backward as well forces $C = D = 0$ by the same phase argument, so the only entire-line bounded solution is zero.

The expression E is not positive definite in the four initial derivatives: it has a mixed product and a negative square. In particular a nonzero state can have $E = 0$. The displayed solution has $|y(0)|^2 = 1$ and $E(0) = 0$, immediately refuting the proposed bound for every positive C . Constancy of an indefinite expression does not guarantee bounded motion.

