

Question 1

Consider the real differential equation

$$(x^2 - 1)y^{(4)} + 2xy''' - 3y' = \ln x.$$

A leading coefficient that vanishes at one point requires more care than simply counting derivatives.

Tasks

1. Classify the equation by order, linearity and homogeneity. Write its normalized form wherever that form is defined over the reals.
2. At $x_0 = 1/2$, prescribe $y = 1$, $y' = 0$, $y'' = -2$ and $y''' = 3$. State the largest interval containing x_0 on which the standard linear IVP theorem guarantees a unique solution, and explain why four data are needed.
3. A proposed C^4 solution near $x = 1$ has $y(1) = 0$, $y'(1) = 2$, $y''(1) = 0$, $y'''(1) = 1$. Test these data directly in the original equation.
4. Replace only the last datum by $y'''(1) = 3$. Explain what this compatibility check establishes and what it does not establish. Does a vanishing leading coefficient make the original equation third order on a neighborhood of 1?

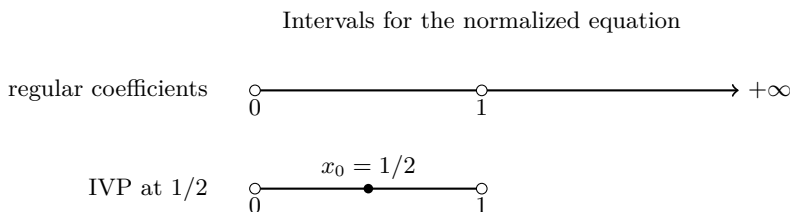
Strategy. Separate the equation's order from the intervals on which its highest derivative can be isolated.

$$y^{(4)} + \frac{2x}{x^2 - 1} y''' - \frac{3}{x^2 - 1} y' = \frac{\ln x}{x^2 - 1}.$$

Step 2: Apply the theorem on its proper interval. All normalized coefficients and the forcing are continuous on $\boxed{(0,1)}$, the largest such interval through $1/2$. The linear IVP theorem gives one solution for arbitrary values of y, y', y'', y''' at that ordinary point. These four data specify the fourth-order initial state. The theorem does not assert that this particular solution cannot extend beyond the interval; it simply gives no guarantee across 1.

$$\boxed{2y'''(1) - 3y'(1) = 0.}$$

Step 4: Interpret compatibility without overclaiming. With $y'''(1) = 3$, the same test gives $6 - 6 = 0$. This removes that contradiction, but alone proves neither existence nor uniqueness; the normalized IVP theorem still does not apply at 1. The coefficient of $y^{(4)}$ is nonzero at nearby points other than 1, so the equation remains fourth order on a neighborhood. At the isolated point, its value imposes a lower-derivative constraint instead of determining $y^{(4)}(1)$.



Question 2

Let $L[y] = y^{(4)}$ and consider $L[y] = 24x$ on $\mathbb{R}$. Define

$$p(x) = \frac{x^5}{5}, \quad u(x) = p(x) + 1 + x^2, \quad v(x) = p(x) + 2x - x^3.$$

For a nonhomogeneous equation, the phrase “superposition of solutions” needs a precise qualification.

Tasks

1. Verify that p, u, v solve the equation. Determine whether $u + v$, $u - v$ and $(u + v)/2$ solve the same equation or its homogeneous counterpart.
2. Find every triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ for which $\alpha u + \beta v + \gamma p$ solves $L[y] = 24x$.
3. Prove that the complete solution set is p plus the space of cubic polynomials. Explain why it is an affine set but not a vector space.
4. Find the unique solution with $y(0) = 1$, $y'(0) = -2$, $y''(0) = 6$, $y'''(0) = 0$, and verify the forcing and all four data.

Question 2 – Solution

Strategy. Apply the linear operator before deciding which combinations preserve the nonzero forcing.

Step 1: Track the right-hand side. Since $p^{(4)} = 24x$ and fourth derivatives of cubics vanish, $L[p] = L[u] = L[v] = 24x$. Linearity gives

$$L[u + v] = 48x, \quad L[u - v] = 0, \quad L[(u + v)/2] = 24x.$$

Thus the sum solves neither of the two stated equations, the difference solves the homogeneous equation, and the average solves the original one.

Step 2: Characterize every admissible combination. The proposed combination has image $24x(\alpha + \beta + \gamma)$. Because $24x$ is not identically zero on $\mathbb{R}$,

$$\boxed{\alpha u + \beta v + \gamma p \text{ solves the equation} \iff \alpha + \beta + \gamma = 1.}$$

The forcing's isolated zero at 0 does not remove this identity requirement.

Step 3: Describe the complete solution set. If y is any solution, $(y - p)^{(4)} = 0$. Four integrations show that $y - p$ is a polynomial of degree at most three. Conversely, adding any such polynomial to p preserves the equation. Hence

$$\boxed{y = p + c_0 + c_1x + c_2x^2 + c_3x^3.}$$

This is a translate of a four-dimensional vector space, hence affine. It is not itself a vector space: the zero function does not satisfy the nonzero forcing, and sums generally double the forcing.

Step 4: Fit and check the initial state. The data give $c_0 = 1$, $c_1 = -2$, $2c_2 = 6$, $6c_3 = 0$, so

$$\boxed{y = \frac{x^5}{5} + 1 - 2x + 3x^2.}$$

The added cubic has fourth derivative zero. Its value and first three derivatives at 0 are $1, -2, 6, 0$; p contributes zero to all four. The coefficients are uniquely fixed.

Question 3

For $y^{(4)} = 0$ on $\mathbb{R}$, consider the shifted functions

$$f_0 = 1, \quad f_1 = x - 1, \quad f_2 = \frac{(x-1)^2}{2}, \quad f_3 = \frac{(x-1)^3}{6}.$$

Initial derivatives and polynomial coefficients coincide only for an appropriately normalized basis.

Tasks

1. Verify that the four functions solve the equation and compute their Wronskian for arbitrary x .
2. Construct the solution with $y(1) = 2$, $y'(1) = -1$, $y''(1) = 3$ and $y'''(1) = 4$. Explain why the coefficients in this basis can be read directly from the data.
3. Express the same solution in the unshifted basis $1, x, x^2, x^3$ and verify the four data in that form.
4. Prove that the map sending a solution to $(y(1), y'(1), y''(1), y'''(1))$ is a linear isomorphism onto $\mathbb{R}^4$. What does this say about the number of free constants and the effect of changing the basis?

Question 3 – Solution

Strategy. Build the derivative matrix and use its normalization at the initial point.

Step 1: Verify solutions and a nonzero determinant. Every function is cubic or lower, so its fourth derivative is zero. With $h = x - 1$, their derivative matrix is

$$F(x) = \begin{pmatrix} 1 & h & h^2/2 & h^3/6 \\ 0 & 1 & h & h^2/2 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is triangular with unit diagonal, giving $\boxed{W = \det F = 1}$ at every x .

Step 2: Use the identity matrix at the center. Since $F(1) = I$, the coefficient vector is exactly the prescribed derivative vector. Therefore

$$\boxed{y = 2 - h + \frac{3}{2}h^2 + \frac{2}{3}h^3, \quad h = x - 1.}$$

The factorials in the basis make its initial derivative matrix the identity; without them, derivative values would not equal coefficients.

Step 3: Change coordinates without changing the solution. Expansion gives

$$\boxed{y = \frac{23}{6} - 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3.}$$

Here $y' = -2 - x + 2x^2$, $y'' = -1 + 4x$, $y''' = 4$. At 1, these are $-1, 3, 4$, while $y(1) = 2$, verifying all data independently of the shifted expression.

Step 4: Establish the initial-data isomorphism. Differentiation and evaluation are linear. For any vector in $\mathbb{R}^4$, the corresponding combination of f_0, f_1, f_2, f_3 realizes it, proving surjectivity. If all four derivatives at 1 vanish, the coefficient vector in this basis is zero, proving injectivity. Thus the solution space has dimension four. A change of basis changes coordinates, not the number of free constants, the initial data themselves or the resulting function.

Question 4

Four functions $y_1, \dots, y_4$ solve the same homogeneous equation

$$y^{(4)} - \frac{2x}{1+x^2}y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

on $\mathbb{R}$, where p_0, p_1, p_2 are continuous. Their Wronskian at zero is $W(0) = -6$.

Tasks

1. Derive the Wronskian identity $W' = -p_3W$ for a normalized fourth-order homogeneous equation, by differentiating the determinant row by row.
2. Determine $W(x)$ for the stated equation and decide whether the four functions form a fundamental set on the whole real line.
3. Explain why every four-component initial derivative vector at any finite x_0 is realized by exactly one linear combination of these functions.
4. A computation reports $\widetilde{W}(x) = -6(1+x^2)(1-x)$. It has the correct value at 0. Give two independent reasons why it cannot be the Wronskian of this common solution family.

Question 4 – Solution

Strategy. The coefficient of the next-to-highest derivative controls the entire Wronskian, not just its value at one point.

Step 1: Differentiate the determinant. The Wronskian rows contain derivative orders 0, 1, 2, 3. Differentiating any of the first three rows duplicates the following row, giving determinant zero. In the last-row term, substitute $y_j^{(4)} = -p_3 y_j''' - p_2 y_j'' - p_1 y_j' - p_0 y_j$ for every column. All terms except the multiple of the original last row again duplicate a row. Therefore

$$W' = -p_3 W.$$

Step 2: Determine the Wronskian globally. Here $p_3 = -2x/(1+x^2)$, so

$$W(x) = -6 \exp\left(\int_0^x \frac{2s}{1+s^2} ds\right) = -6(1+x^2).$$

It never vanishes. The four solutions are linearly independent, and the regular fourth-order homogeneous solution space has dimension four; hence they form a fundamental set on $\mathbb{R}$.

Step 3: Interpret invertibility of the derivative matrix. At any finite x_0 , the matrix with columns $(y_j, y_j', y_j'', y_j''')(x_0)$ has determinant $W(x_0) \neq 0$. It therefore maps one and only one coefficient vector to any prescribed initial derivative vector. The resulting combination solves the equation; the linear IVP theorem shows it is the unique solution for those data on $\mathbb{R}$.

Step 4: Reject a plausible-looking candidate. First, $\widetilde{W}(1) = 0$ contradicts the nonvanishing formula forced by $W(0) \neq 0$ on this regular connected interval. Second, it fails the differential identity even at the initial point: $\widetilde{W}'(0) = 6$, whereas $-p_3(0)\widetilde{W}(0) = 0$. Matching one determinant value is insufficient; a valid Wronskian must obey the identity throughout the interval.

Question 5

You may use the fact that the function

$$\phi(s) = \begin{cases} \exp\left(-\frac{1}{1-4s^2}\right), & |s| < 1/2, \\ 0, & |s| \geq 1/2 \end{cases}$$

is C^∞ on $\mathbb{R}$, with every derivative zero at $s = \pm 1/2$. Define $f_1(x) = \phi(x+2)$, $f_2(x) = \phi(x)$ and $f_3(x) = \phi(x-2)$.

Tasks

1. Locate the supports of the three functions and prove that they are linearly independent on $\mathbb{R}$.
2. Prove that their three-by-three Wronskian is identically zero. Use their derivative supports, not a long symbolic expansion.
3. Could these functions all solve one equation $y''' + p_2y'' + p_1y' + p_0y = 0$ with continuous coefficients on $\mathbb{R}$? Prove your answer using uniqueness of an initial-value problem.
4. State the extra hypothesis that makes a zero Wronskian imply dependence for solutions of a linear equation. Explain why the example does not contradict that theorem.

Question 5 – Solution

Strategy. Compare independence of arbitrary smooth functions with independence inside one regular homogeneous solution space.

Step 1: Prove independence by separated values. The supports are $[-5/2, -3/2]$, $[-1/2, 1/2]$ and $[3/2, 5/2]$, respectively. They are disjoint, and each function has value e^{-1} at its center $-2, 0, 2$. If $c_1f_1 + c_2f_2 + c_3f_3 = 0$ identically, evaluation at those three centers gives $c_1 = c_2 = c_3 = 0$. The functions are therefore linearly independent.

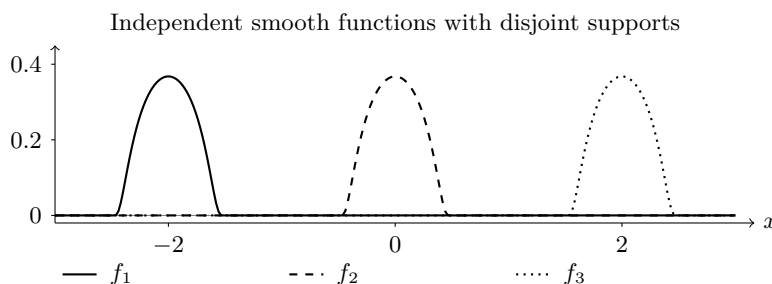
Step 2: Compute the determinant structurally. Outside a function's support, it and its derivatives vanish; at the support boundary, the supplied smoothness facts give the same conclusion. At every x , at most one column of

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix}$$

can be nonzero. Thus $\boxed{W(f_1, f_2, f_3)(x) = 0 \text{ for all } x}$.

Step 3: Test the common-equation hypothesis. Suppose all three solved the stated regular homogeneous equation. The function f_1 has $f_1(0) = f_1'(0) = f_1''(0) = 0$. The zero solution has the same data, so uniqueness on $\mathbb{R}$ would force $f_1 \equiv 0$. But $f_1(-2) = e^{-1} > 0$, a contradiction. Such continuous coefficients cannot exist.

Step 4: State the theorem with its hypotheses. For three solutions of the same normalized third-order homogeneous linear equation with continuous coefficients on a connected interval, a zero Wronskian at one point gives a nonzero coefficient combination with zero initial derivative vector. Uniqueness then makes that combination identically zero, proving dependence. Our functions are arbitrary smooth functions, not solutions of one such equation. Their identically zero Wronskian alone therefore does not imply dependence.



Question 6

For $y^{(4)} = 0$ on $[0, 1]$, prescribe four scalar endpoint conditions:

$$y(0) = 0, \quad y(1) = 0, \quad y'(0) + y'(1) = 0, \quad y''(1) - y''(0) = \eta,$$

where η is a real parameter. Counting four conditions is not the same as verifying that they determine four constants independently.

Tasks

1. Write the general solution and translate all four conditions into a linear system for its coefficients. Determine the rank of the coefficient matrix.
2. Classify all η according to existence and uniqueness, and give every solution whenever solutions exist.
3. Replace the fourth condition by $y'(0) = 1$, keeping the first three. Find the resulting solution and prove it is unique.
4. Explain why none of these conclusions contradicts the standard fourth-order IVP theorem. Describe how the family in the compatible original problem varies while retaining all four endpoint conditions.

Question 6 – Solution

Strategy. Test the endpoint functionals on the cubic solution space instead of relying on their number.

Step 1: Form the coefficient system. Write $y = a + bx + cx^2 + dx^3$. The conditions become

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \end{pmatrix}.$$

The fourth row is six times the third minus twelve times the second plus twelve times the first. The first three rows are independent, so the rank is $\boxed{3}$.

Step 2: Classify the parameter completely. The first three equations imply $a = 0$, $b + c + d = 0$ and $2b + 2c + 3d = 0$; subtracting twice the middle equation gives $d = 0$. Thus the last equation can hold only if $\eta = 0$. For that value, $c = -b$ and

$$y = Kx(1 - x), \quad K \in \mathbb{R}.$$

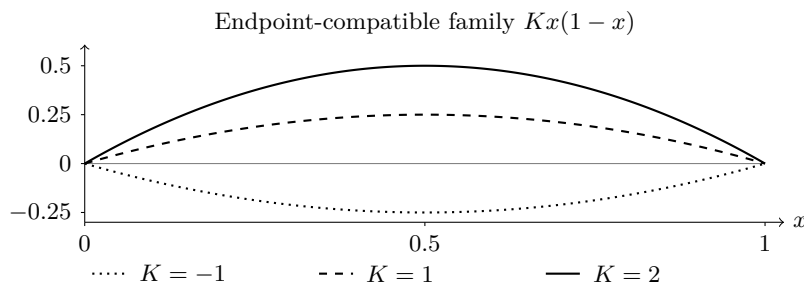
There are infinitely many solutions if $\eta = 0$ and none if $\eta \neq 0$. No value of η gives a unique solution under the original conditions.

Step 3: Replace the redundant information. The condition $y'(0) = 1$ gives $b = 1$. The first three conditions already imply $a = d = 0$ and $c = -b$, so

$$y = x(1 - x)$$

is the unique solution after replacement. The new condition fixes the one remaining free parameter.

Step 4: Distinguish endpoint data from a full initial state. The IVP theorem concerns y, y', y'', y''' specified at one ordinary point. These endpoint conditions mix values at two points and can be dependent or inconsistent on the solution space. In the compatible family, changing K changes height and curvature while preserving zero endpoint values, opposite endpoint slopes, and equal endpoint curvatures. The figure shows three such members; their different initial derivative vectors explain why IVP uniqueness is not violated.



Question 7

Consider the singular third-order equation

$$xy''' - 2y'' = 0$$

on the whole real line. A classical solution here means a C^3 function satisfying the original equation at every point, including 0.

Tasks

1. Solve on each half-line by first setting $z = y''$. Display all constants before matching across zero.
2. Classify every global C^3 solution and determine the dimension of this solution space.
3. For prescribed $y(0) = \alpha$, $y'(0) = \beta$, $y''(0) = \gamma$, determine exactly when solutions exist and how many free parameters remain.
4. Determine which global solutions are C^4 . Does demanding this extra derivative restore uniqueness for compatible initial data at zero? Explain why the regular third-order dimension theorem cannot be used across this singular point.

Question 7 – Solution

Strategy. Solve separately where the leading coefficient is nonzero, then impose the actual regularity required at the join.

Step 1: Solve on both regular components. For $x \neq 0$, $xz' - 2z = 0$ gives $z = Cx^2$. Two integrations, with the constant absorbed into the quartic coefficient, yield

$$y = \begin{cases} A_+x^4 + B_+x + C_+, & x > 0, \\ A_-x^4 + B_-x + C_-, & x < 0. \end{cases}$$

The constants on the two components are initially independent.

Step 2: Impose a classical join. Continuity of y and y' forces $C_+ = C_- = C$ and $B_+ = B_- = B$. The second and third derivatives are $12A_{\pm}x^2$ and $24A_{\pm}x$, both tending to zero, so C^3 regularity imposes no relation between A_+ and A_- . Every global solution is therefore

$$y(x) = C + Bx + \begin{cases} A_+x^4, & x \geq 0, \\ A_-x^4, & x < 0, \end{cases}$$

with four independent constants. At zero the equation reduces to $-2y''(0) = 0$, which these functions satisfy. This is a four-dimensional solution space.

Step 3: Classify the initial data. Every solution has $y(0) = C$, $y'(0) = B$ and $y''(0) = 0$. Thus $\boxed{\gamma = 0}$ is necessary and sufficient. When it holds, $C = \alpha$, $B = \beta$ and both A_+, A_- remain arbitrary. When $\gamma \neq 0$, no classical solution exists.

Step 4: Check the extra derivative. The one-sided fourth derivatives are $24A_+$ and $24A_-$. A C^4 join requires and is guaranteed by $A_+ = A_- = A$, giving the polynomial $C + Bx + Ax^4$. One free parameter A still survives for the compatible initial data, so uniqueness is not restored. The leading coefficient vanishes at 0, and normalization introduces $-2/x$. The hypotheses guaranteeing dimension three and arbitrary unique initial data hold only on the separate half-lines.

Question 8

For the homogeneous equation $y''' + y' = 0$ on $\mathbb{R}$, consider the three functions $1, \sin x, \cos x$. A student believes that a nonzero solution cannot have both its value and its first derivative zero at the same point.

Tasks

1. Verify the three functions as solutions and compute their Wronskian. Explain why they form a fundamental set on $\mathbb{R}$.
2. Find the solution with $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 1$, and verify the full initial state.
3. Determine all zeros of this solution and their exact multiplicities. Decide whether finitely or infinitely many such zeros are possible here.
4. Correct the student's claim for a regular homogeneous equation of general order n . State which consecutive derivatives must vanish at a point to force the entire solution to be zero, and explain why the present example does not satisfy that condition.

Question 8 – Solution

Strategy. Use the full initial derivative vector, rather than only the value and slope, to apply uniqueness.

Step 1: Verify a fundamental set. Each candidate satisfies $y''' + y' = 0$. Their Wronskian is

$$W = \det \begin{pmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{pmatrix} = \boxed{-1}.$$

Thus the three solutions are independent. The regular third-order equation has a three-dimensional homogeneous solution space, so they form a fundamental set.

Step 2: Determine the unique IVP solution. Write $y = A + B \sin x + C \cos x$. The data give $A + C = 0$, $B = 0$ and $-C = 1$. Hence

$$\boxed{y = 1 - \cos x.}$$

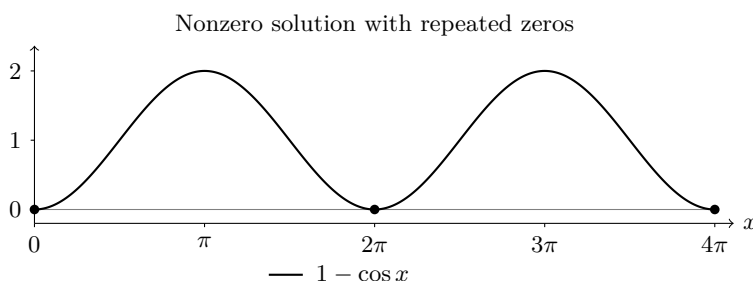
Its first two derivatives are $\sin x$ and $\cos x$, giving initial vector $(0, 0, 1)$; its third derivative $-\sin x$ verifies the equation.

Step 3: Classify all zero contacts. The equation $1 - \cos x = 0$ holds exactly at $x = 2k\pi$, $k \in \mathbb{Z}$. At each such point, $y' = 0$ but $y'' = 1 \neq 0$, so every zero has multiplicity exactly two. There are infinitely many, with no finite accumulation point. Their repeated tangency does not make the function identically zero.

Step 4: State the correct uniqueness consequence. For a normalized homogeneous linear equation of order n with continuous coefficients on a connected interval, if

$$\boxed{y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0,}$$

then uniqueness of the zero-data IVP forces $y \equiv 0$ throughout that interval. Here $n = 3$, and the second derivative at every zero is 1, so the required three-component vector is not zero. Value and slope alone are insufficient; an arbitrary number of separated double zeros is not excluded by the IVP theorem.



Question 9

Consider the nonlinear third-order IVP

$$y''' = \sqrt{|y|}, \quad y(0) = y'(0) = y''(0) = 0.$$

For each delay $\tau \geq 0$, investigate a candidate that is zero for $x \leq \tau$ and has the form $C(x - \tau)^p$ for $x > \tau$, with $C > 0$ and $p > 3$.

Tasks

1. Explain why the standard linear IVP theorem does not apply, despite a nonzero leading coefficient and a continuous right-hand side.
2. Find the exponent p and coefficient C that make the positive branch satisfy the equation.
3. Verify the joined function as a classical solution on $\mathbb{R}$ and check all three initial data for every $\tau \geq 0$. Determine its exact smoothness at the joining point.
4. Explain how this family proves nonuniqueness. Contrast it with the linear equation $y''' = y$ under the same initial data, and examine the ratio $\sqrt{h}/h$ as $h \rightarrow 0^+$ to identify a regularity issue in the nonlinear dependence on y .

Question 9 – Solution

Strategy. Construct actual delayed solutions; continuity of a nonlinear right-hand side alone is not a uniqueness proof.

Step 1: Check linearity, not just the leading coefficient. The equation is third order but nonlinear because $\sqrt{|y|}$ is not a linear function of the unknown. A leading coefficient of 1 and continuity do not make the linear existence-and-uniqueness theorem applicable.

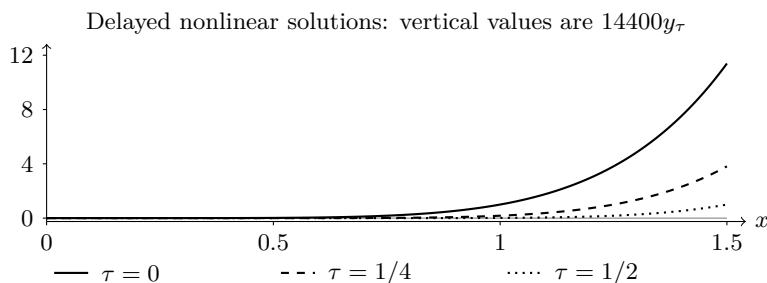
Step 2: Balance exponents and amplitudes. For $s = x - \tau > 0$, the proposed branch has $y''' = Cp(p-1)(p-2)s^{p-3}$ and $\sqrt{|y|} = \sqrt{C}s^{p/2}$. Since $p > 3$ and $C > 0$, equality requires $p-3 = p/2$, giving $p = 6$. Then $120C = \sqrt{C}$, so $C = 1/14400$.

Step 3: Verify the join and the initial state. Define

$$y_\tau(x) = \begin{cases} 0, & x \leq \tau, \\ (x - \tau)^6/14400, & x > \tau. \end{cases}$$

Derivatives through order five approach zero from the right and equal zero on the left; difference quotients at the join give the same derivatives. Thus $y_\tau \in C^5$, and the equation holds at the join too, since $y''' = \sqrt{|y|} = 0$ there. The sixth derivative jumps from 0 to $6!/14400 = 1/20$, so it is not C^6 at τ . Because $0 \leq \tau$, all three initial data vanish, including when $\tau = 0$.

Step 4: Identify the precise contrast. Distinct delays give distinct solutions with identical initial data; the identically zero function is another solution. This proves nonuniqueness constructively. For the linear equation $y''' - y = 0$, continuous normalized coefficients and the same zero initial vector force the unique zero solution. Here $\sqrt{h}/h = 1/\sqrt{h} \rightarrow \infty$, so the nonlinear right-hand side is not locally Lipschitz in y at zero. Nonlinearity alone does not always cause nonuniqueness; the displayed family establishes the failure in this specific case.



Question 10

Two solutions of $y^{(4)} = 0$ on $\mathbb{R}$ have initial derivative errors

$$\delta_j = \tilde{y}^{(j)}(0) - y^{(j)}(0), \quad |\delta_j| \leq \varepsilon \quad (j = 0, 1, 2, 3),$$

where $\varepsilon \geq 0$. Let $e = \tilde{y} - y$. We want a guaranteed error bound over a whole interval, not just at the initial point.

Tasks

1. Find $e(x)$ exactly in terms of the four initial errors.
2. For $T > 0$, prove the sharp uniform bound

$$\max_{|x| \leq T} |e(x)| \leq \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6} \right).$$

Show that the constant cannot be improved when only the stated error information is available.

3. At $T = 2$, find the largest ε guaranteeing solution error at most 10^{-2} for every admissible initial error vector. Evaluate the guarantee for $\varepsilon = 10^{-3}$.
4. If only $\delta_0, \delta_1, \delta_2$ are controlled, can any finite uniform guarantee depending on those three errors alone hold on $[-2, 2]$? Construct a decisive example and interpret what the result says about a complete initial state.

Question 10 – Solution

Strategy. The linear initial-data map is explicit here, so its worst-case amplification can be determined exactly.

Step 1: Propagate all four initial errors. The difference satisfies $e^{(4)} = 0$ and $e^{(j)}(0) = \delta_j$. Therefore

$$e(x) = \delta_0 + \delta_1 x + \frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}.$$

The factorials are required because the data are derivatives, not monomial coefficients.

Step 2: Prove and attain the uniform bound. For $|x| \leq T$, the triangle inequality gives

$$|e(x)| \leq \varepsilon \left(1 + |x| + \frac{|x|^2}{2} + \frac{|x|^3}{6} \right) \leq \varepsilon \left(1 + T + \frac{T^2}{2} + \frac{T^3}{6} \right).$$

Choose all four $\delta_j = \varepsilon$. At $x = T$, every term has the same nonnegative sign, attaining equality. Thus the stated worst-case constant is exact, including the trivial case $\varepsilon = 0$.

Step 3: Translate the certificate into a tolerance. At $T = 2$, the amplification factor is $1+2+2+8/6 = 19/3$. Consequently the largest admissible tolerance is

$$\varepsilon_{\max} = \frac{10^{-2}}{19/3} = \frac{3}{1900}.$$

Any larger value fails for the equal-sign error vector from Step 2. For $\varepsilon = 10^{-3}$, the exact worst-case bound is $19/3000 \approx 0.0063333 < 0.01$.

Step 4: Show why one uncontrolled component matters. Take $y = 0$ and $\tilde{y} = Mx^3/6$ for arbitrary real M . Both solve the equation and have the same value, first derivative and second derivative at zero, but their third-derivative difference is M . At $x = 2$, the solution error is $4|M|/3$, which is unbounded as $|M| \rightarrow \infty$. Thus controlling only three components cannot give the proposed finite guarantee. For the fourth-order IVP, the complete initial derivative vector is needed for this quantitative stability bound.