

Question 1

Consider the Euler initial-value problem

$$x^2y'' - 2xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 3.$$

A power x^m is a useful trial function, but the reason it works should also be understood.

Tasks

1. For $x > 0$, set $t = \ln x$ and $Y(t) = y(e^t)$. Derive the formulas for xy' and x^2y'' and transform the equation into one with constant coefficients.
2. Find a fundamental pair, compute its Wronskian, and solve the IVP exactly. Verify both initial values.
3. State the maximal real interval of this IVP and determine the behavior of its solution at each end of that interval.
4. Locate and classify every stationary point on that interval. Give the exact minimum value and explain how two monotone power modes can combine into a nonmonotone solution.

Question 1 – Solution

Strategy. Logarithmic coordinates turn powers into exponentials and make the characteristic equation systematic.

Step 1: Transform derivatives correctly. The chain rule gives $Y' = xy'$ and $Y'' = xy' + x^2y''$, hence

$$xy' = Y', \quad x^2y'' = Y'' - Y'.$$

The equation becomes $Y'' - 3Y' - 4Y = 0$, with characteristic polynomial $m^2 - 3m - 4 = (m - 4)(m + 1)$.

Step 2: Solve and verify the data. The powers x^4, x^{-1} form a pair with $W = x^4(-x^{-2}) - 4x^3x^{-1} = -5x^2 \neq 0$ for $x > 0$. Writing $y = C_1x^4 + C_2x^{-1}$ gives $C_1 + C_2 = 2$, $4C_1 - C_2 = 3$, so

$$y = x^4 + x^{-1}.$$

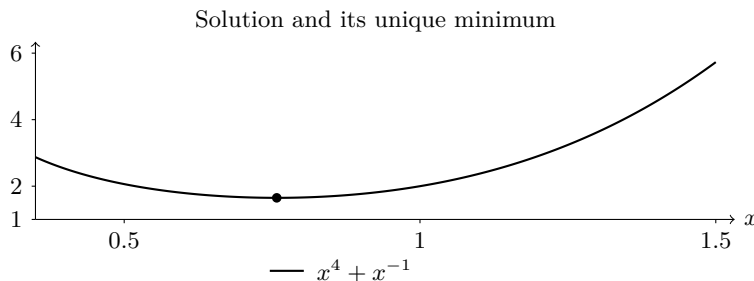
At 1, its value is 2 and derivative $4x^3 - x^{-2}$ is 3.

Step 3: Identify the maximal interval. The normalized coefficients are continuous on $(0, \infty)$, and the explicit solution exists throughout it. Since $y \rightarrow +\infty$ as $x \rightarrow 0^+$, no finite continuation through 0 is possible. Thus the maximal real interval through 1 is $(0, \infty)$. At infinity, $y \sim x^4 \rightarrow +\infty$; at zero, $y \sim x^{-1}$.

Step 4: Find the unique minimum. The equation $y' = 0$ is $4x^5 = 1$, giving $x_* = 4^{-1/5}$. Since $y'' = 12x^2 + 2x^{-3} > 0$, it is the unique global minimum. Using $x_*^{-1} = 4x_*^4$,

$$y(x_*) = 5 \cdot 4^{-4/5}.$$

One mode increases and the other decreases. Their derivatives cancel at exactly one point, although neither mode has a stationary point of its own.



Question 2

For $x > 0$, consider

$$x^2 y'' - 3xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 3.$$

A repeated characteristic exponent needs a second independent solution and careful treatment at the singular point.

Tasks

1. Derive the repeated exponent and obtain a second independent solution using logarithmic coordinates. Verify independence rather than merely listing two functions.
2. Solve the IVP and find every positive zero of its solution.
3. Determine whether this IVP solution admits a continuous, C^1 , or C^2 extension to 0. Compute the relevant one-sided limits.
4. For the general positive-half-line solution, classify which choices permit a C^2 extension through 0 that solves the original equation. Explain whether the data $y(0) = y'(0) = 0$ determine that extension uniquely.

Question 2 – Solution

Strategy. A repeated exponential in logarithmic time produces a logarithmic factor in the original variable.

Step 1: Construct a genuine fundamental pair. With $t = \ln x$, the equation becomes $Y'' - 4Y' + 4Y = 0$, whose characteristic polynomial is $(m - 2)^2$. Thus

$$y = x^2(A + B \ln x).$$

For $u = x^2$, $v = x^2 \ln x$, the Wronskian is $W = x^2(2x \ln x + x) - 2x(x^2 \ln x) = x^3 \neq 0$ on $x > 0$.

Step 2: Use both initial values. At 1, $y(1) = A$ and $y'(1) = 2A + B$. Hence $A = B = 1$, giving

$$y = x^2(1 + \ln x).$$

Because $x^2 > 0$, the only positive zero is $x = e^{-1}$.

Step 3: Test successive derivatives at zero. For this solution,

$$y' = x(3 + 2 \ln x), \quad y'' = 5 + 2 \ln x.$$

As $x \rightarrow 0^+$, $y \rightarrow 0$ and $y' \rightarrow 0$, but $y'' \rightarrow -\infty$. Defining $y(0) = 0$ gives a right-hand C^1 extension with derivative 0: indeed $y(x)/x = x(1 + \ln x) \rightarrow 0$. It can be joined to zero on the negative side as a C^1 function, but no C^2 extension is possible.

Step 4: Classify classical extensions. In general, $y'' = 2A + B(2 \ln x + 3)$, so a finite second-derivative limit requires $B = 0$. On $x < 0$ the general solution is $x^2(C + D \ln |x|)$. A C^2 join similarly forces $D = 0$, and matching second derivatives forces $C = A$. Thus precisely $y = Ax^2$ extend through 0 as C^2 solutions of the original equation. Each satisfies $y(0) = y'(0) = 0$, so these data do not give uniqueness at the singular point.

Question 3

Study the logarithmically oscillating IVP

$$x^2 y'' + xy' + 4y = 0, \quad y(1) = 0, \quad y'(1) = 2, \quad x > 0.$$

Oscillations in $\ln x$ behave differently from oscillations in x .

Tasks

1. Find the real general solution and the specified IVP solution.
2. Determine all its positive zeros and the ratio of consecutive zeros in increasing order. Explain where the zeros accumulate.
3. Decide whether the IVP solution is bounded near 0 and whether it has a continuous extension there. Extend the conclusion about limits to every nonzero real solution of this equation.
4. Set $z = xy'/2$. Determine the curve traced by (y, z) and its direction as x increases. Locate the initial point and distinguish this curve from the graph of y against x .

Question 3 – Solution

Strategy. Use $t = \ln x$ to reveal a harmonic oscillator and then translate its time scale back to x .

Step 1: Solve the transformed oscillator. The transformed equation is $Y'' + 4Y = 0$, so $y = A \cos(2 \ln x) + B \sin(2 \ln x)$. At $x = 1$, $A = 0$ and $2B = y'(1) = 2$, hence

$$y(x) = \sin(2 \ln x), \quad x > 0.$$

The characteristic exponents are $m = \pm 2i$.

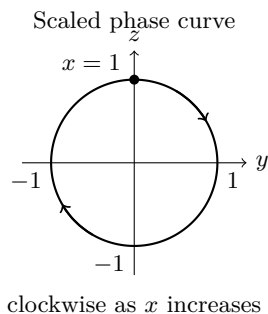
Step 2: Locate the geometric sequence of zeros. Zeros occur when $2 \ln x = k\pi$:

$$x_k = e^{k\pi/2}, \quad k \in \mathbb{Z}, \quad x_{k+1}/x_k = e^{\pi/2}.$$

They accumulate at 0 as $k \rightarrow -\infty$ and grow without bound as $k \rightarrow \infty$. There is no smallest positive zero.

Step 3: Separate boundedness from a limit. The IVP solution has $|y| \leq 1$, but sequences with $\ln x = \pi/4 - k\pi$ and $\ln x = 3\pi/4 - k\pi$ tend to 0 while giving values 1 and -1 . Thus no limit at zero exists. Any nonzero real solution is a nonzero-amplitude sinusoid in $\ln x$ and likewise attains two distinct limiting values along sequences approaching zero. Only the zero solution extends continuously there.

Step 4: Read the scaled phase curve. For the IVP, $z = \cos(2 \ln x)$, so $y^2 + z^2 = 1$. In logarithmic time, $\dot{y} = 2z$, $\dot{z} = -2y$; at $(y, z) = (0, 1)$ the motion is to the right. The circle is traversed clockwise as x increases, because $\ln x$ increases with x . The horizontal coordinate here is y , not x ; equal scales preserve the unit circle.



Question 4

Solve on the negative half-line:

$$x^2 y'' + xy' - \frac{1}{4}y = 0, \quad y(-4) = 3, \quad y'(-4) = -\frac{1}{8}.$$

Using fractional powers without specifying a real branch can invalidate a solution.

Tasks

1. Use $t = \ln(-x)$ to derive a real fundamental pair on $x < 0$. Explain why the positive-half-line expression $x^{1/2}$ is unsuitable as a real-valued basis there.
2. Translate both initial values into logarithmic coordinates and solve the IVP. Check the derivative sign directly in the original variable.
3. Give its maximal real interval and both endpoint behaviors.
4. Find its unique stationary point and minimum. Explain how increasing logarithmic time relates to increasing x on this negative interval.

Question 4 – Solution

Strategy. Work with positive $-x$, retaining the sign in the chain rule.

Step 1: Choose a real branch. For $x = -e^t$, $dx/dt = x$, so $Y' = xy'$ and $Y'' = xy' + x^2y''$ still hold. The equation becomes $Y'' - Y/4 = 0$. A real pair is $\boxed{(-x)^{1/2}, (-x)^{-1/2}}$ for $x < 0$. The expression $x^{1/2}$ has no real value on this interval; powers of $|x| = -x$ avoid that problem.

Step 2: Translate the slope correctly. At $t = \ln 4$, $Y = 3$ and $Y' = xy' = (-4)(-1/8) = 1/2$. Writing $y = A\sqrt{-x} + B/\sqrt{-x}$ yields $2A + B/2 = 3$ and $A - B/4 = 1/2$, hence $A = 1$, $B = 2$. Thus

$$\boxed{y = \sqrt{-x} + \frac{2}{\sqrt{-x}}, \quad y' = -\frac{1}{2\sqrt{-x}} + \frac{1}{(-x)^{3/2}}.}$$

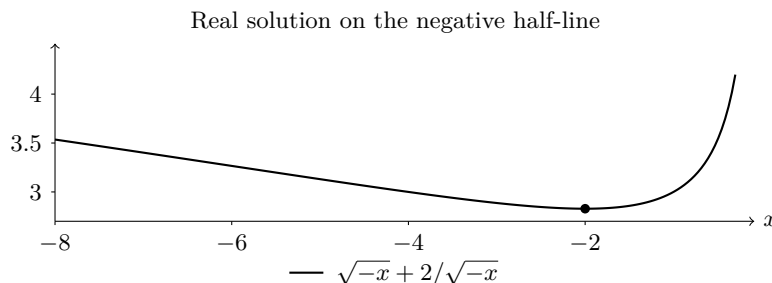
At -4 these give 3 and $-1/8$, as required.

Step 3: State the maximal interval. The solution is smooth throughout $(-\infty, 0)$, and its divergence at 0 prevents extension through that point. This is the maximal real interval containing -4 . As $x \rightarrow -\infty$, $y \sim \sqrt{-x} \rightarrow \infty$; as $x \rightarrow 0^-$, $y \sim 2/\sqrt{-x} \rightarrow \infty$.

Step 4: Locate the minimum and orient time. Put $r = -x > 0$. The derivative is $(2 - r)/(2r^{3/2})$, so it is negative for $x < -2$ and positive for $-2 < x < 0$. Therefore

$$\boxed{x_* = -2, \quad y(x_*) = 2\sqrt{2}}$$

is the unique global minimum. Since $dt/dx = 1/x < 0$, increasing x corresponds to decreasing t on this half-line. This reverses the relation between the two horizontal directions, not the derivative formulas.



Question 5

For $x > 0$, solve the resonantly forced Euler problem

$$x^2 y'' - 3xy' + 4y = x^2 \ln x, \quad y(1) = y'(1) = 0.$$

A trial expression $x^2(A + B \ln x)$ resembles the forcing but may not produce it.

Tasks

1. Transform the equation using $t = \ln x$ and then $Y = e^{2t}Z$. Derive the resulting equation for Z .
2. Find the general solution. Explain exactly why the proposed trial expression fails and why a cubic logarithmic term appears.
3. Apply the initial conditions and verify the final solution by computing the transformed residual.
4. Determine whether the IVP solution extends continuously, as a C^1 function, or as a C^2 function to 0. Give the relevant limiting expressions.

Question 5 – Solution

Strategy. Remove the repeated characteristic exponential before integrating the forcing.

Step 1: Factor the logarithmic-time operator. The equation is $(D - 2)^2 Y = e^{2t}t$, where $D = d/dt$. Since $(D - 2)(e^{2t}Z) = e^{2t}Z'$, a second application gives

$$Z'' = t.$$

Step 2: Integrate through the resonance. Two integrations yield $Z = t^3/6 + C_1 + C_2t$, hence

$$y = x^2 \left(C_1 + C_2 \ln x + \frac{(\ln x)^3}{6} \right).$$

The proposed trial has $Z = A + Bt$ and therefore $Z'' = 0$: it belongs entirely to the homogeneous solution space. Integrating the degree-one forcing twice produces degree three, accounting for the cubic logarithm and its factor $1/6$.

Step 3: Impose and verify the initial data. At $x = 1$, $t = 0$, so $y(1) = C_1$ and $y'(1) = 2C_1 + C_2$. Thus both constants vanish and

$$y = \frac{x^2(\ln x)^3}{6}.$$

For this solution, $Z = t^3/6$ satisfies $Z'' = t$ exactly; therefore the original left-hand side is $e^{2t}t = x^2 \ln x$. The two initial values are zero.

Step 4: Inspect regularity at the singular point. Writing $t = \ln x$,

$$y' = x \left(\frac{t^3}{3} + \frac{t^2}{2} \right), \quad y'' = \frac{t^3}{3} + \frac{3t^2}{2} + t.$$

As $x \rightarrow 0^+$, the power of x dominates every logarithmic power, so $y \rightarrow 0$, $y' \rightarrow 0$, and $y(x)/x \rightarrow 0$. Assigning $y(0) = 0$ therefore gives a C^1 right extension, which may be joined to zero on the left as a C^1 function. But $y'' \rightarrow -\infty$, so no C^2 extension is possible.

Question 6

For $x > 0$, consider

$$x^2 y'' + xy' = \frac{x^2}{1+x^2}.$$

Instead of prescribing data at a positive point, seek the solution that is bounded near 0 and extends continuously with $y(0) = 0$.

Tasks

1. Integrate the equation to obtain its general solution in definite-integral form on $(0, \infty)$, displaying both arbitrary constants.
2. Impose boundedness and the specified value at zero. Prove that they select exactly one solution on the positive half-line.
3. Derive its power series about zero, give its first three nonzero terms, and determine the radius and both endpoint outcomes. Justify termwise integration.
4. Prove that the selected solution has a C^2 extension to zero, compute $y'(0)$ and $y''(0)$, and certify a two-term approximation on $[0, 1]$ using a rigorous remainder bound.

Question 6 – Solution

Strategy. Integrate the weighted derivative first, then use regularity at zero to remove the logarithmic homogeneous mode.

Step 1: Integrate the Euler operator. Dividing by x gives $(xy')' = x/(1+x^2)$. Hence

$$xy' = \frac{1}{2} \ln(1+x^2) + C, \quad y = \frac{1}{2} \int_1^x \frac{\ln(1+s^2)}{s} ds + C \ln x + D.$$

These formulas display both constants and solve the equation for every $x > 0$.

Step 2: Apply the singular-endpoint conditions. Since $\ln(1+s^2)/s \sim s$ at zero, its integral there converges. Boundedness therefore forces $C = 0$; the value $y(0) = 0$ fixes the remaining constant. The unique selected solution is

$$y(x) = \frac{1}{2} \int_0^x \frac{\ln(1+s^2)}{s} ds.$$

Any two bounded solutions differ by a constant, and their common value at zero makes that constant zero.

Step 3: Expand the integral with justification. The divided logarithmic series $\sum_{n \geq 1} (-1)^{n-1} s^{2n-1}/n$ is uniformly convergent on $|s| \leq \rho < 1$, by comparison with $\sum_{n \geq 1} \rho^{2n-1}/n$. Termwise integration therefore yields, for $|x| < 1$,

$$y(x) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n^2} = \frac{x^2}{4} - \frac{x^4}{16} + \frac{x^6}{36} - \dots$$

The ratio of successive nonzero terms tends to x^2 , so the radius is 1. At $x = \pm 1$ the series converges absolutely by comparison with $\sum 1/n^2$. The series and the even integral extension agree there by continuity: the series is uniformly convergent on $[-1, 1]$ by the same comparison.

Step 4: Recover derivatives and a certificate. The local power series supplies a smooth even extension with $y'(0) = 0$ and $y''(0) = 1/2$. With the forcing defined continuously as zero at 0, the original equation holds there as well. On $[0, 1]$, the alternating term magnitudes decrease, so for $P_4 = x^2/4 - x^4/16$,

$$0 \leq y(x) - P_4(x) \leq \frac{x^6}{36}.$$

The bound includes the endpoint $x = 1$; it is a certificate, not a claim that the two-term approximation is highly accurate there.

Question 7

Let $L > 0$ be fixed. Seek nonzero real solutions of the two-endpoint Euler problem

$$x^2 y'' + x y' + \lambda y = 0, \quad y(1) = y(e^L) = 0,$$

where λ is real. No general theory of boundary-value eigenproblems is required; solve the Euler equation directly.

Tasks

1. Transform the problem to $0 \leq t \leq L$ and prove that $\lambda \leq 0$ permits only the zero solution.
2. Find every $\lambda > 0$ permitting a nonzero solution, and give the corresponding solution families.
3. For each admissible value, impose $y'(1) = 1$. Find the normalized solution and all its zeros, including the number strictly inside $(1, e^L)$.
4. Explain why uniqueness of ordinary initial-value problems does not imply uniqueness for these two endpoint conditions. For $L = \pi$, describe the first three normalized profiles in the coordinate $t = \ln x$.

Question 7 – Solution

Strategy. In logarithmic coordinates, the two endpoint conditions reduce to elementary trigonometric or hyperbolic equations.

Step 1: Exclude nonpositive parameters. Set $Y(t) = y(e^t)$. Then $Y'' + \lambda Y = 0$, with $Y(0) = Y(L) = 0$. If $\lambda = 0$, $Y = A + Bt$ and both constants must vanish. If $\lambda = -\mu^2 < 0$, $Y = A \cosh(\mu t) + B \sinh(\mu t)$. The first condition gives $A = 0$; since $\sinh(\mu L) > 0$, the second gives $B = 0$.

Step 2: Find all nonzero families. For $\lambda = \omega^2 > 0$, the first condition leaves $Y = B \sin(\omega t)$. A nonzero B requires $\sin(\omega L) = 0$, so

$$\lambda_n = (n\pi/L)^2, \quad y = C \sin\left(\frac{n\pi \ln x}{L}\right), \quad n = 1, 2, \dots$$

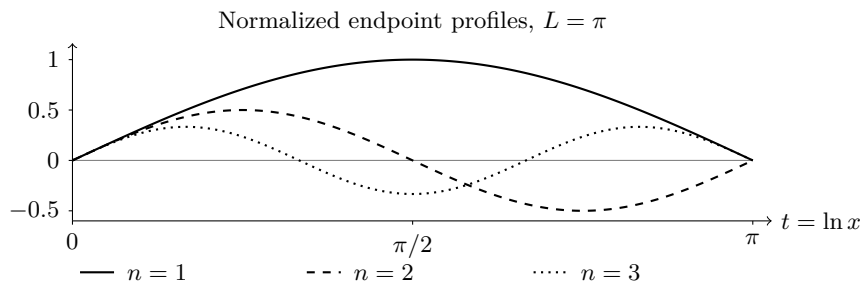
There are no other real parameter values admitting nonzero solutions.

Step 3: Normalize and locate the zeros. At $x = 1$, $y'(1) = Y'(0) = Cn\pi/L$. Thus

$$y_n(x) = \frac{L}{n\pi} \sin\left(\frac{n\pi \ln x}{L}\right).$$

Its zeros on the specified interval are $x_k = e^{kL/n}$ for $k = 0, \dots, n$. Exactly $n - 1$ lie strictly inside; they are equally spaced in $\ln x$, not in x .

Step 4: Distinguish endpoint and initial data. The original endpoint conditions do not specify the initial slope. At an admissible parameter, every amplitude satisfies both endpoints, including zero; hence those data are not unique. Once $y'(1) = 1$ is imposed, the usual IVP uniqueness applies. For $L = \pi$, the first three normalized profiles are $Y_n(t) = \sin(nt)/n$, $n = 1, 2, 3$, with 0, 1, 2 interior zeros, respectively. The figure uses logarithmic horizontal coordinates.



Question 8

For fixed $a \in \mathbb{R}$ and parameter $\varepsilon \geq 0$, consider on $x > 0$

$$x^2 y'' + (1 - 2a)xy' + (a^2 - \varepsilon^2)y = 0.$$

Two distinct power solutions become nearly identical when their characteristic exponents approach one another.

Tasks

1. Find the characteristic exponents for $\varepsilon > 0$ and explain why simply taking their two power solutions to the limit does not produce a fundamental pair at $\varepsilon = 0$.
2. Construct solutions $u_\varepsilon, v_\varepsilon$ satisfying $u_\varepsilon(1) = 1$, $u'_\varepsilon(1) = a$, $v_\varepsilon(1) = 0$, $v'_\varepsilon(1) = 1$.
3. Compute their Wronskian and prove independence for every $\varepsilon \geq 0$, using the limiting pair at zero.
4. Determine that limiting pair and justify uniform convergence of each basis function on every compact interval $[r, R] \subset (0, \infty)$ as $\varepsilon \rightarrow 0$. Explain how the logarithmic repeated-root solution emerges.

Question 8 – Solution

Strategy. Normalize a symmetric sum and a divided difference before allowing the characteristic roots to merge.

Step 1: Identify the degenerating pair. The power equation is $(m - a)^2 - \varepsilon^2 = 0$, giving $m = a \pm \varepsilon$. Both $x^{a+\varepsilon}$ and $x^{a-\varepsilon}$ tend to x^a as $\varepsilon \rightarrow 0$. Their separate limits are the same function, so they cannot remain a fundamental pair.

Step 2: Normalize before taking the limit. For $\varepsilon > 0$, set $t = \ln x$ and define

$$u_\varepsilon = x^a \cosh(\varepsilon t), \quad v_\varepsilon = x^a \frac{\sinh(\varepsilon t)}{\varepsilon}.$$

These are the half-sum and the difference divided by 2ε of the two powers. At $x = 1$ they have the four required value/slope data by direct differentiation.

Step 3: Check the determinant exactly. Differentiation and $\cosh^2 s - \sinh^2 s = 1$ give

$$W(u_\varepsilon, v_\varepsilon) = x^{2a-1} > 0.$$

At $\varepsilon = 0$, define $u_0 = x^a$ and $v_0 = x^a \ln x$. They solve the repeated-root equation, have the same normalized data, and their Wronskian is again x^{2a-1} . Independence holds for every parameter value.

Step 4: Justify the limit uniformly. On $[r, R]$, let $T = \max(|\ln r|, |\ln R|)$ and $M = \max_{[r, R]} x^a$. For $0 < \varepsilon \leq 1$, Taylor's theorem for $\cosh$ and $\sinh$ gives

$$|u_\varepsilon - u_0| \leq \frac{M\varepsilon^2 T^2 \cosh T}{2}, \quad |v_\varepsilon - v_0| \leq \frac{M\varepsilon^2 T^3 \cosh T}{6}.$$

Both bounds tend to zero uniformly. The divided difference tends to $\partial_m x^m|_{m=a} = x^a \ln x$, recovering the independent logarithmic solution that the two unnormalized limits lose.

Question 9

For real α and $\beta > 0$, consider a nonzero solution on $x > 0$ of

$$x^2 y'' + (1 - 2\alpha)xy' + (\alpha^2 + \beta^2)y = 0.$$

Its power envelope may tend to zero even while its derivatives keep oscillating.

Tasks

1. Find the real general solution and show that its j th derivative has the form $x^{\alpha-j}$ times a sinusoid in $\ln x$ with nonzero amplitude.
2. For each integer $k \geq 0$, prove a necessary and sufficient condition on α for a nonzero solution to have a C^k extension to $x = 0$. Treat the equality case explicitly.
3. Apply your criterion to $y = x^{5/2} \sin(\ln x)$. Compute y'' and determine exactly which integer orders of smoothness are possible at zero.
4. Join this particular solution to zero for $x \leq 0$. Verify that the joined function is a classical C^2 solution of the corresponding Euler equation on the whole real line. Explain the implication for initial data at the singular point.

Question 9 – Solution

Strategy. Track both the power envelope and the nonvanishing oscillation amplitude after every differentiation.

Step 1: Differentiate the complex powers. The exponents are $\alpha \pm i\beta$, so the real solutions are

$$y = x^\alpha (A \cos(\beta \ln x) + B \sin(\beta \ln x)).$$

The j th derivative of $x^{\alpha+i\beta}$ is $\prod_{\ell=0}^{j-1} (\alpha - \ell + i\beta) x^{\alpha-j+i\beta}$. Since $\beta > 0$, none of these factors vanishes. A nonzero real solution therefore retains a nonzero sinusoidal amplitude at every derivative order.

Step 2: Establish the exact regularity threshold. If $\alpha > k$, all derivatives through order k tend to zero. Their extensions are consistent derivatives: for $j < k$, $y^{(j)}(x)/x = O(x^{\alpha-j-1}) \rightarrow 0$. If $\alpha = k$, the k th derivative has constant nonzero amplitude and no limit. If $\alpha < k$, that derivative has an unbounded envelope and attains unbounded values along a sequence approaching zero. Hence

$$\boxed{\text{A nonzero solution extends as } C^k \text{ to } 0 \iff \alpha > k.}$$

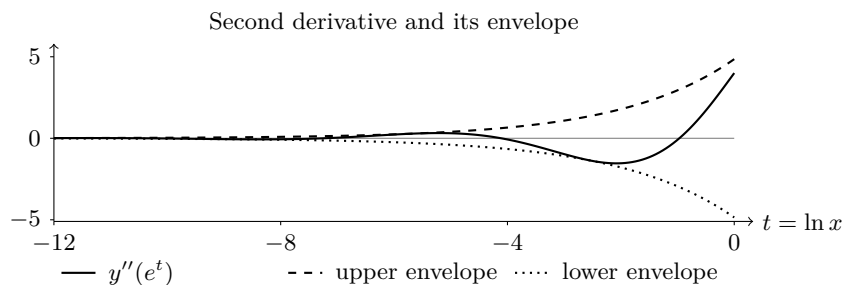
This includes $k = 0$; bounded oscillation at equality is not continuity.

Step 3: Inspect the concrete example. For $\alpha = 5/2$, $\beta = 1$,

$$\boxed{y'' = x^{1/2} \left(\frac{11}{4} \sin(\ln x) + 4 \cos(\ln x) \right).}$$

The solution extends as C^0, C^1, C^2 , with all three center values zero, but not as C^3 or any higher integer order. The plot shows $y''(e^t)$ and its decaying envelope as $t \rightarrow -\infty$.

Step 4: Verify the joined classical solution. The equation is $x^2 y'' - 4xy' + (29/4)y = 0$. On $x > 0$ the formula solves it; on $x < 0$ the zero function does. The zero limits through the second derivative make the join C^2 , and at 0 every term in the original equation is zero. Both this nonzero joined solution and the identically zero solution have $y(0) = y'(0) = 0$. Classical IVP uniqueness fails at this singular point.



Question 10

For $x \geq 1$, consider a finite interval of forcing in an Euler equation:

$$x^2 y'' - xy' + y = g(x), \quad y(1) = y'(1) = 0, \quad g(x) = \begin{cases} x, & 1 \leq x < e, \\ 0, & x \geq e. \end{cases}$$

Seek a solution that is C^1 across $x = e$ and solves the equation on each side. No classical second derivative at the switching point is assumed.

Tasks

1. Set $t = \ln x$ and $Y = e^t Z$. Derive a general definite-integral solution for zero initial data, valid for piecewise continuous forcing g .
2. Apply it to the specified forcing and give explicit formulas on $1 \leq x \leq e$ and $x \geq e$.
3. Verify continuity of y and y' at e . Compute the one-sided second derivatives and explain why a C^2 solution at that point is impossible.
4. Determine the large- x behavior after the forcing stops. Explain why the solution does not return to zero just because the right-hand side becomes zero.

Question 10 – Solution

Strategy. Remove the repeated exponent, integrate in logarithmic time, and match the finite-forcing transition.

Step 1: Derive the integral response. With $D = d/dt$, the transformed equation is $(D-1)^2 Y = g(e^t)$, so $Z'' = e^{-t}g(e^t)$. The data at $t = 0$ give $Z(0) = Z'(0) = 0$. Thus

$$Z(t) = \int_0^t (t-s)e^{-s}g(e^s)ds, \quad \boxed{y(x) = x \int_1^x \ln(x/r) \frac{g(r)}{r^2} dr.}$$

The second formula follows from $r = e^s$, including $ds = dr/r$.

Step 2: Evaluate the finite forcing interval. Here $Z'' = 1$ for $0 < t < 1$ and $Z'' = 0$ for $t > 1$. Integrating with the zero initial data gives $Z = t^2/2$ up to 1, then $Z = t - 1/2$. Therefore

$$\boxed{y(x) = \begin{cases} \frac{1}{2}x(\ln x)^2, & 1 \leq x \leq e, \\ x(\ln x - \frac{1}{2}), & x \geq e. \end{cases}}$$

Both formulas have the same value at e , and the first satisfies both initial conditions at 1.

Step 3: Check the join and the derivative jump. For $Y = e^t Z$, $y' = Z + Z'$ and $y'' = (Z' + Z'')/x$. At $t = 1$, both sides have $Z = 1/2$ and $Z' = 1$, so

$$y(e) = e/2, \quad y'(e) = 3/2, \quad y''(e^-) = 2/e, \quad y''(e^+) = 1/e.$$

The finite jump in forcing produces a jump $-1/e$ in y'' , while y, y' remain continuous. The unequal one-sided limits rule out C^2 regularity, irrespective of any value assigned to $g(e)$.

Step 4: Interpret the surviving homogeneous motion. After e , the solution is the nonzero homogeneous combination $x \ln x - x/2$. Thus $\boxed{y(x)/(x \ln x) \rightarrow 1}$ as $x \rightarrow \infty$. The completed forcing interval leaves nonzero value and slope at the switch. Those become initial data for the homogeneous equation; zero subsequent forcing does not reset them to zero.