

Question 1

Consider the differential equation $y'' - xy = 0$ near $x = 0$. Write $y = \sum_{n=0}^{\infty} a_n x^n$; the coefficients are not the derivatives themselves.

Tasks

1. Derive the coefficient recurrence, stating the exceptional equation at degree zero. Explain how the indices split into three chains.
2. Construct the normalized solutions $u(0) = 1$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 1$. Give the first four nonzero terms of each.
3. Prove that both series converge for every real x and form a fundamental pair. State the solution with arbitrary initial values $y(0) = A$, $y'(0) = B$.
4. For the degree-nine truncation U_9 of u , prove a uniform error bound below 5.88×10^{-7} on $[-1, 1]$. A numerical sample of the error is not a proof of this bound.

Question 1 – Solution

Strategy. Keep the lowest index separate, then use the recurrence both to construct solutions and to bound their tails.

Step 1: Align the powers. The constant coefficient gives $a_2 = 0$. For $n \geq 1$, $(n+2)(n+1)a_{n+2} = a_{n-1}$, equivalently

$$a_{m+3} = \frac{a_m}{(m+3)(m+2)} \quad (m \geq 0).$$

The chains start at a_0, a_1, a_2 ; the last is identically zero.

Step 2: Build the normalized series. With empty products interpreted as 1,

$$u(x) = \sum_{k \geq 0} \frac{x^{3k}}{\prod_{j=1}^k (3j)(3j-1)} = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \frac{x^9}{12960} + \cdots,$$

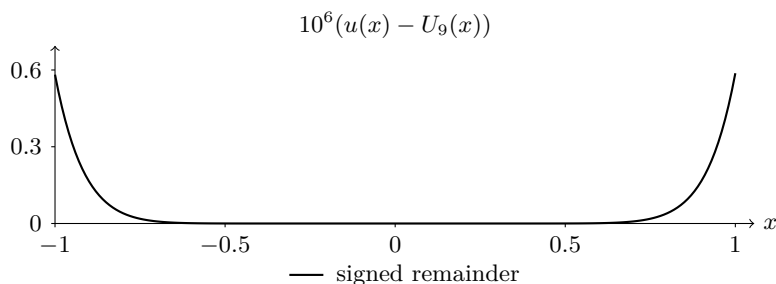
$$v(x) = \sum_{k \geq 0} \frac{x^{3k+1}}{\prod_{j=1}^k (3j+1)(3j)} = x + \frac{x^4}{12} + \frac{x^7}{504} + \frac{x^{10}}{45360} + \cdots.$$

Step 3: Establish actual solutions and independence. For either series, the ratio of consecutive nonzero term magnitudes tends to zero at every fixed x . Thus both have infinite radius, may be differentiated termwise, and satisfy the equation and stated initial data. Their Wronskian satisfies $W' = uv'' - u''v = 0$ and $W(0) = 1$. Hence $y = Au + Bv$ is the unique solution for those initial values, on $\mathbb{R}$.

Step 4: Bound all omitted terms. The first omitted coefficient of u is $1/1710720$. On $|x| \leq 1$, ratios within the remaining tail are at most $1/(15 \cdot 14) = 1/210$. Therefore

$$|u(x) - U_9(x)| \leq \frac{|x|^{12}}{1710720} \frac{1}{1 - 1/210} < 5.88 \times 10^{-7}.$$

The plot magnifies the actual signed remainder; the geometric estimate covers both signs of x .



Question 2

The equation

$$y'' + 2xy' + 2y = 0$$

admits a useful relation between its power-series construction and a first-order identity. Let $u(0) = 1$, $u'(0) = 0$ and $v(0) = 0$, $v'(0) = 1$.

Tasks

1. Derive the recurrence for a Maclaurin solution. Compute the first four nonzero terms of u and v , and identify their parity.
2. Find a quantity involving y, y' and x whose derivative is the left-hand side. Use it to express both normalized solutions through exponentials and a definite integral.
3. Prove convergence of both series on the whole real line directly from their coefficients. Explain why the series and integral constructions agree.
4. Compute the Wronskian of the normalized pair and determine whether any nontrivial linear combination can have both its value and derivative zero at a finite point.

Question 2 – Solution

Strategy. Compare the even and odd coefficient chains with an integrated form of the equation.

Step 1: Separate the two chains. Coefficient matching gives $(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$, so

$$a_{n+2} = -\frac{2}{n+2}a_n \quad (n \geq 0).$$

Thus $u = 1 - x^2 + x^4/2 - x^6/6 + \cdots$ is even, while $v = x - 2x^3/3 + 4x^5/15 - 8x^7/105 + \cdots$ is odd.

Step 2: Integrate a conserved quantity. Since $(y' + 2xy)' = 0$, $y' + 2xy = C$. At 0, $C = y'(0)$. Multiplying by e^{x^2} and integrating gives

$$y(x) = e^{-x^2} \left[y(0) + y'(0) \int_0^x e^{t^2} dt \right].$$

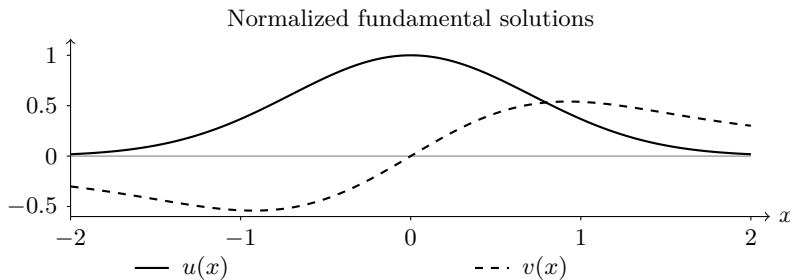
Consequently
$$u = e^{-x^2}, \quad v = e^{-x^2} \int_0^x e^{t^2} dt.$$

Step 3: Justify convergence and equality. The even coefficients are $(-1)^k/k!$, and the odd ones are $(-2)^k/(2k+1)!!$, where $(2k+1)!! = 1 \cdot 3 \cdots (2k+1)$. Successive nonzero term ratios are $x^2/(k+1)$ and $2x^2/(2k+3)$, respectively; both tend to zero. Thus the series define solutions for all real x . The integral formulas solve the same equation with the same initial data, so uniqueness identifies the constructions.

Step 4: Test independence at every point. Differentiating v gives $v' = -2xv + 1$, while $u' = -2xu$. Hence

$$W(u, v) = uv' - u'v = u = e^{-x^2} > 0.$$

At any finite point, the two-by-two system for the coefficients of a combination with zero value and derivative has nonzero determinant. Both coefficients must vanish. The plotted pair remains independent even where one curve crosses zero.



Question 3

Solve the initial-value problem

$$(1-x)y'' - y' = 0, \quad y(0) = 0, \quad y'(0) = 1$$

by a series about 0, then investigate what happens when that series is used far from its center.

Tasks

1. Derive the recurrence and determine every Maclaurin coefficient.
2. Identify the represented function and its maximal real interval containing 0. Independently determine the power series' radius and both endpoint outcomes.
3. Explain why the initial series fails at $x = -2$ even though the solution exists there. Is this a singularity of the solution at -2 ?
4. Reexpand about $x = -1$, deriving the new coefficient recurrence from the differential equation. Give the first four nonconstant terms, the new convergence interval including endpoints, and the value at $x = -2$.

Question 3 – Solution

Strategy. Distinguish a solution's real domain from the disk reached by a particular Taylor center.

Step 1: Determine the Maclaurin coefficients. For $n \geq 0$, coefficient matching gives $(n+2)(n+1)a_{n+2} - (n+1)^2a_{n+1} = 0$. With $a_0 = 0, a_1 = 1$, this yields

$$a_{n+2} = \frac{n+1}{n+2}a_{n+1}, \quad y(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Step 2: Identify two different domains. The differential equation says $[(1-x)y']' = 0$. Initial data give $y' = 1/(1-x)$ and $y = -\ln(1-x)$ on $(-\infty, 1)$, its maximal real interval through 0. The series has radius 1. It diverges at 1 (harmonic series) and converges at -1 (alternating series), so its interval is $[-1, 1)$. Integrating the geometric series identifies its sum on $(-1, 1)$; at -1 , the alternating harmonic value is $-\ln 2$.

Step 3: Diagnose the failed evaluation. At -2 , the terms $(-2)^n/n$ do not tend to zero. The series therefore diverges although $y(-2) = -\ln 3$ is finite. The obstruction is the center's distance to the singularity at 1, not a singularity at -2 .

Step 4: Move the expansion center. Let $h = x+1$ and $y = \sum b_n h^n$. The equation is $(2-h)y'' - y' = 0$, giving $b_{n+2} = (n+1)b_{n+1}/[2(n+2)]$. Since $b_0 = -\ln 2, b_1 = 1/2$,

$$y = -\ln 2 + \sum_{n \geq 1} \frac{h^n}{n2^n} = -\ln 2 + \frac{h}{2} + \frac{h^2}{8} + \frac{h^3}{24} + \frac{h^4}{64} + \cdots.$$

The radius in h is 2, with convergence at $h = -2$ and divergence at $h = 2$; in x the interval is $[-3, 1)$. Now $x = -2$ is interior, and the sum is $-\ln 2 - \ln(3/2) = -\ln 3$.

Question 4

Consider a nonhomogeneous equation centered away from the origin:

$$y'' + (x - 2)y' = e^{x-2}, \quad y(2) = 0, \quad y'(2) = 1.$$

You may use the analytic ordinary-point theorem: a linear equation with nonzero leading coefficient and analytic normalized coefficients and forcing in a complex disk has a convergent Taylor solution throughout that disk.

Tasks

1. Set $h = x - 2$ and derive the recurrence for $y = \sum_{n \geq 0} a_n h^n$, explicitly accounting for the constant coefficient of hy' .
2. Compute a_0 through a_6 and the resulting polynomial $P_6(h)$. Identify any coefficient that vanishes despite nonzero neighboring coefficients.
3. Define the residual $\mathcal{R} = P_6'' + hP_6' - e^h$. Find its leading nonzero term. Explain why a degree-six solution truncation need not satisfy the differential equation through degree six.
4. Produce a definite-integral representation of the exact solution and justify infinite radius of convergence for its series. Check both initial values in your representation.

Question 4 – Solution

Strategy. Expand every term in the shifted variable and keep the forcing coefficients in the same index convention.

Step 1: Match the forcing coefficient. Derivatives with respect to x and h agree. The coefficient of h^n in hy' is na_n for $n \geq 0$; it is zero when $n = 0$. Thus

$$a_{n+2} = \frac{1/n! - na_n}{(n+2)(n+1)} \quad (n \geq 0), \quad a_0 = 0, \quad a_1 = 1.$$

Step 2: Compute the initial segment. Successive applications give $a_2 = 1/2$, $a_3 = 0$, $a_4 = -1/24$, $a_5 = 1/120$, $a_6 = 1/144$. Therefore

$$P_6(h) = h + \frac{h^2}{2} - \frac{h^4}{24} + \frac{h^5}{120} + \frac{h^6}{144}.$$

The cubic coefficient vanishes because $1/1! - a_1 = 0$, not because the solution has a fixed parity.

Step 3: Interpret the residual correctly. Direct differentiation cancels the coefficients through h^4 . At degree five, hP'_6 contributes $1/24$ while e^h contributes $1/120$; hence

$$\mathcal{R}(h) = \frac{h^5}{30} + O(h^6).$$

The omitted term a_7h^7 would contribute $42a_7h^5$ to y'' . Indeed, the recurrence gives $a_7 = -1/1260$, cancelling that residual term. Two differentiations lower the degree of an omitted term by two.

Step 4: Recover and validate the exact solution. Writing $v = y'$ gives $v' + hv = e^h$. An integrating factor yields

$$y(2+h) = \int_0^h e^{-s^2/2} \left(1 + \int_0^s e^{t+t^2/2} dt \right) ds.$$

At $h = 0$ the outer integral is zero and its integrand is 1, verifying the initial values. Differentiating gives the first-order equation for v , hence the original equation. Its leading coefficient is 1 and all coefficients and forcing are entire, so the stated theorem applies to disks of every radius. The Taylor series is therefore entire.

Question 5

For real λ , consider

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

and seek analytic solutions near $x = 0$. A polynomial solution would remain meaningful even where the leading coefficient vanishes.

Tasks

1. Derive the recurrence and explain how it separates even and odd coefficients.
2. Classify all real λ for which a nonzero polynomial solution exists. Prove necessity from the highest-degree term and sufficiency from the recurrence; include degree zero.
3. For $\lambda = 20$, $y(0) = 3/8$ and $y'(0) = 0$, find the exact solution and verify the differential equation by substitution.
4. Find all its real zeros exactly, determine its values at $x = \pm 1$, and explain why this particular solution extends through those points although the normalized equation is singular there.

Question 5 – Solution

Strategy. Use coefficient termination to distinguish polynomial solutions from general local series.

Step 1: Derive the parity recurrence. Matching x^n gives

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} a_n \quad (n \geq 0).$$

Even and odd indices evolve independently from a_0 and a_1 .

Step 2: Characterize termination completely. If a nonzero polynomial has degree m , its highest coefficient obeys $[\lambda - m(m+1)]a_m = 0$, so $\lambda = m(m+1)$ for an integer $m \geq 0$. Conversely, choose the starting coefficient of parity m nonzero and the other zero. For indices of that parity below m , no numerator vanishes; at $n = m$ it does, terminating the chain. Thus these and only these parameter values work, including $m = 0$, $\lambda = 0$ (constants).

Step 3: Construct and verify the requested polynomial. For $\lambda = 20$, the recurrence yields $a_2 = -15/4$, $a_4 = 35/8$, and $a_6 = 0$. Hence

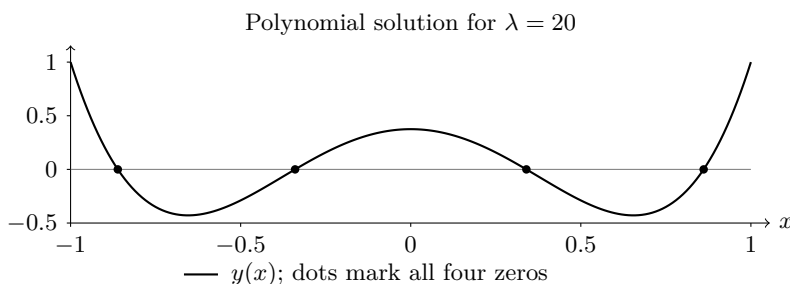
$$y(x) = \frac{35x^4 - 30x^2 + 3}{8}.$$

Here $y' = (35x^3 - 15x)/2$ and $y'' = (105x^2 - 15)/2$. Substitution in $(1 - x^2)y'' - 2xy' + 20y$ cancels every coefficient. Initial data also hold, so uniqueness near 0 identifies the solution.

Step 4: Locate roots and extend the solution. Solving the quadratic in x^2 gives four distinct zeros:

$$x = \pm \sqrt{\frac{15 - 2\sqrt{30}}{35}}, \quad x = \pm \sqrt{\frac{15 + 2\sqrt{30}}{35}}.$$

They lie in $(-1, 1)$, and $y(-1) = y(1) = 1$. The polynomial satisfies the original equation identically for every real x , including ± 1 . Singular normalized coefficients do not forbid this special solution from extending through them.



Question 6

The leading coefficient vanishes at the proposed expansion center in

$$xy'' + 2y' + xy = 0.$$

A student therefore needs to check which initial data a Maclaurin solution can actually accommodate.

Tasks

1. Substitute $y = \sum_{n \geq 0} a_n x^n$. Find the exceptional constant-coefficient equation and the recurrence for all remaining coefficients.
2. Determine every analytic solution near 0 and identify its series in terms of a familiar function, including the value assigned at 0.
3. Decide whether $y(0) = 1$, $y'(0) = 1$ can be satisfied by an analytic solution or even by a twice continuously differentiable solution of the original equation near 0.
4. On $x > 0$, substitute $v = xy$ to find the full two-parameter solution. Determine which members extend boundedly to 0 and explain why only one free parameter survives in the analytic family.

Question 6 – Solution

Strategy. Apply the original equation at its singular center before invoking any ordinary-point intuition.

Step 1: Retain the exceptional equation. The coefficient of x^0 is $2a_1 = 0$. For $n \geq 1$, the coefficient of x^n is $(n+1)(n+2)a_{n+1} + a_{n-1} = 0$. Equivalently,

$$a_1 = 0, \quad a_{m+2} = -\frac{a_m}{(m+2)(m+3)} \quad (m \geq 0).$$

Thus a_0 is free but a_1 is not.

Step 2: Identify all analytic solutions. Every odd coefficient vanishes. The even chain gives

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!} = a_0 \frac{\sin x}{x} \quad (x \neq 0), \quad y(0) = a_0.$$

The ratio test gives infinite radius. Termwise substitution verifies the original equation, including $x = 0$. Conversely, every analytic solution must obey the recurrence, so the displayed family is exhaustive.

Step 3: Test the proposed initial values. The data $y(0) = 1$, $y'(0) = 1$ are incompatible with $a_1 = 0$. More generally, a C^2 solution satisfies the equation at 0, where it reduces to $2y'(0) = 0$. Even if the equation were initially imposed only at nearby nonzero points, continuity of y, y', y'' would give the same constraint by taking the limit. Such a C^2 solution cannot have the proposed data.

Step 4: Explain the lost second parameter. With $v = xy$, $v'' = xy'' + 2y'$, so $v'' + v = 0$ for $x > 0$. Consequently

$$y(x) = A \frac{\sin x}{x} + B \frac{\cos x}{x} \quad (x > 0).$$

The second term behaves like B/x as $x \rightarrow 0^+$; boundedness forces $B = 0$. The remaining term extends analytically with value A and slope 0. The ordinary-point theorem does not guarantee two arbitrary analytic initial coefficients at a center where the leading coefficient is zero.

Question 7

Consider the nonlinear initial-value problem

$$y' = 1 + y^2, \quad y(0) = 0.$$

Seek a Maclaurin solution $y = \sum_{n \geq 0} a_n x^n$. For the radius question, you may use these facts: sine and cosine are entire; the only complex zeros of cosine are $z = \pi/2 + k\pi$, $k \in \mathbb{Z}$; and a Taylor series reaches its nearest nonremovable complex singularity.

Tasks

1. Derive the nonlinear coefficient recurrence using a Cauchy product. Compute all coefficients through degree seven.
2. Prove that the resulting solution is odd, and explain why the square y^2 cannot be handled by squaring individual coefficients separately.
3. Identify the exact solution by a method independent of the coefficient calculation. Determine its maximal real interval containing 0 and the exact Taylor radius.
4. For the degree-seven polynomial P_7 , prove $0 < P_7(x) < y(x)$ when $0 < x < \pi/2$. Explain why a fixed polynomial cannot approximate this solution uniformly on that whole open interval.

Question 7 – Solution

Strategy. Nonlinearity couples coefficients by convolution; an exact solution then validates convergence and reveals blow-up.

Step 1: Convolve rather than square termwise. The constant forcing contributes only at $n = 0$:

$$(n+1)a_{n+1} = \delta_{n0} + \sum_{k=0}^n a_k a_{n-k}, \quad a_0 = 0,$$

where $\delta_{n0} = 1$ for $n = 0$ and 0 otherwise. This gives

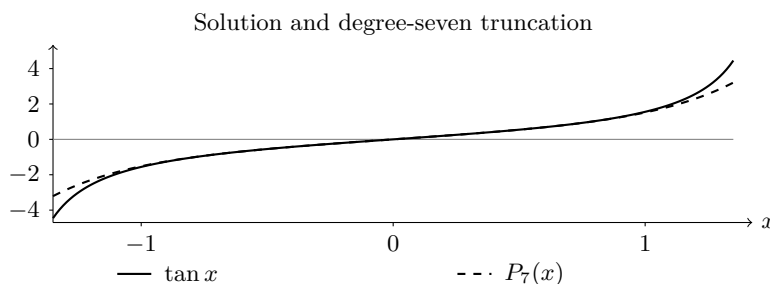
$$P_7(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315},$$

with $a_2 = a_4 = a_6 = 0$.

Step 2: Establish parity and coefficient signs. If y solves the IVP, $-y(-x)$ does too; local uniqueness gives oddness. The recurrence also proves by induction that all even coefficients vanish and all odd coefficients are positive: each required convolution contains positive odd-index products. For example, the coefficient of x^4 in y^2 is $2a_1a_3$, not a_2^2 . Cross terms are essential.

Step 3: Identify the function and its radius. Separation yields $\arctan y = x$ near 0, so $y = \tan x$. It satisfies the equation and initial condition on $(-\pi/2, \pi/2)$ and blows up at both endpoints, making this the maximal real interval through 0. Since $\sin(\pm\pi/2) \neq 0$, the nearest zeros of cosine are nonremovable poles of the quotient. The supplied facts give $R = \pi/2$. Within this radius, the analytic solution's coefficients obey the recurrence, which uniquely determines them; hence it is the constructed series. At $x = \pi/2$ the positive-term series diverges: a finite endpoint sum would bound $\tan x$ below that endpoint, contradicting blow-up. Oddness gives divergence at $-\pi/2$ as well.

Step 4: Interpret truncation and blow-up. For $0 < x < \pi/2$, every omitted odd term is positive, so $0 < P_7(x) < \tan x$. But P_7 stays bounded as $x \rightarrow (\pi/2)^-$ while $\tan x \rightarrow +\infty$. Their difference is unbounded on that interval. Every fixed Taylor truncation has this defect; convergence on smaller compact intervals does not imply uniform approximation up to a blow-up boundary.



Question 8

An unknown equation has the form

$$y'' + \alpha xy' + (\beta + \gamma x^2)y = 0,$$

where α, β, γ are real constants. Two analytic solutions have measured Taylor expansions

$$u(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + O(x^8), \quad v(x) = x - x^3 + \frac{1}{2}x^5 - \frac{1}{6}x^7 + O(x^9).$$

The displayed coefficients are exact.

Tasks

1. Determine α, β, γ by substituting only the coefficients needed from these data. Show the recovery is unique.
2. Derive the full recurrence for a generic series solution of the recovered equation, including its conventions at the lowest indices.
3. Use the substitution $y = e^{-x^2}w$ to identify every solution exactly. Verify that the two measured series are a fundamental pair and determine their radii of convergence.
4. Predict the coefficients of x^8 in u and x^9 in v . Can an additional exact measurement assigning the value $1/20$ to the former coefficient be consistent with the equation and the original data?

Question 8 – Solution

Strategy. Infer the differential operator from low-order data, then validate all orders by a transformation.

Step 1: Recover the three constants. The constant coefficient in the equation for u gives $-2 + \beta = 0$. The x coefficient for v gives $-6 + \alpha + \beta = 0$. The x^2 coefficient for u gives $6 - 2\alpha - \beta + \gamma = 0$. These successive equations uniquely force

$$\boxed{\beta = 2, \quad \alpha = 4, \quad \gamma = 4.}$$

Step 2: State a recurrence valid from the start. With $a_{-2} = a_{-1} = 0$ for the shifted contribution,

$$\boxed{a_{n+2} = -\frac{(4n+2)a_n + 4a_{n-2}}{(n+2)(n+1)} \quad (n \geq 0).}$$

The starting coefficients $a_0 = y(0)$ and $a_1 = y'(0)$ are free. In particular, $a_2 = -a_0$ and $a_3 = -a_1$; no negative-index coefficient is an additional parameter.

Step 3: Verify the recovered equation globally. Writing $E = e^{-x^2}$, differentiation gives

$$y' = E(w' - 2xw), \quad y'' = E(w'' - 4xw' + (4x^2 - 2)w).$$

Thus $y'' + 4xy' + (2 + 4x^2)y = Ew''$. The equation becomes $w'' = 0$, yielding

$$\boxed{y = e^{-x^2}(C_1 + C_2x), \quad u = e^{-x^2}, \quad v = xe^{-x^2}.}$$

Their initial data match the measured series, and their Wronskian is $E^2 = e^{-2x^2} > 0$. Both functions are entire, so both Taylor radii are infinite. The transformation also proves the entire measured series, not just the first displayed terms, are consistent.

Step 4: Test a new exact measurement. The exponential expansion gives $[x^8]u = 1/4! = 1/24$ and $[x^9]v = 1/24$. The recurrence produces the same values. Since the recovered parameters and each solution's initial data are fixed uniquely, $1/20$ is incompatible; it cannot be accommodated by choosing later coefficients independently.

Question 9

For

$$y'' + y = x, \quad y(0) = y'(0) = 0,$$

a power-series approximation is to be certified on $[0, 1]$. For a trial function p , define its differential residual by $\mathcal{R}_p = p'' + p - x$.

Tasks

1. Derive the coefficient recurrence and construct the degree-seven Taylor polynomial P_7 of the exact solution.
2. Compute $\mathcal{R}_{P_7}$ exactly. Derive a definite-integral formula for the error $e = y - P_7$ from its differential equation and initial values.
3. Prove the signed error enclosure $-x^9/362880 \leq e(x) \leq 0$ on $[0, 1]$. Identify the exact solution and compare this certificate with the alternating Taylor remainder.
4. Show that the residual alone cannot control approximation error if initial data are not also checked. Construct trial functions with the same residual as P_7 but arbitrarily large error at $x = 1$.

Question 9 – Solution

Strategy. Convert the residual into a forced error equation, keeping track of both initial errors.

Step 1: Construct the solution truncation. Matching powers gives $(n+2)(n+1)a_{n+2} + a_n = \delta_{n1}$, with $a_0 = a_1 = 0$. Thus all even coefficients vanish and

$$P_7(x) = \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040}.$$

Step 2: Derive the error equation. Direct differentiation yields $\mathcal{R}_{P_7} = x^7/5040$. Since P_7 satisfies both initial values, $e'' + e = -x^7/5040$ with $e(0) = e'(0) = 0$. The zero-data solution is

$$e(x) = -\frac{1}{5040} \int_0^x \sin(x-t)t^7 dt.$$

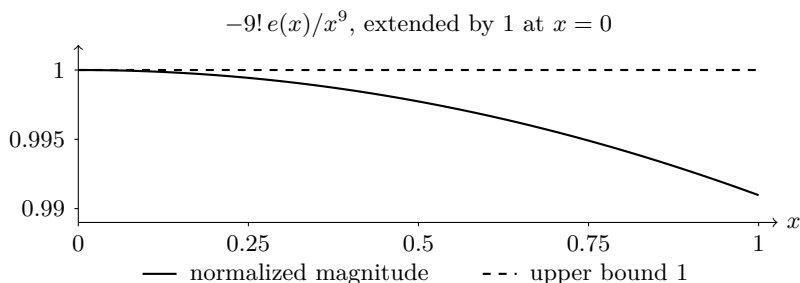
Differentiating this integral twice verifies the equation and the two initial conditions, so uniqueness proves the formula.

Step 3: Bound the signed integral. For $0 \leq t \leq x \leq 1$, $0 \leq \sin(x-t) \leq x-t$. Therefore

$$-\frac{1}{5040} \int_0^x (x-t)t^7 dt = -\frac{x^9}{362880} \leq e(x) \leq 0.$$

The exact solution is $y = x - \sin x$, as substitution and initial values verify. Its convergent sine expansion has first omitted term $-x^9/9!$, so the alternating estimate agrees with the integral certificate. For $x > 0$ the integral is strictly positive before its minus sign, giving $e(x) < 0$.

Step 4: Expose the role of initial data. Let $p_M = P_7 + M \sin x$ for any real M . Since $(\sin x)'' + \sin x = 0$, every p_M has exactly the same residual. But $p'_M(0) = M$, and $y(1) - p_M(1) = e(1) - M \sin 1$ is unbounded as $|M| \rightarrow \infty$. Residual size certifies error only together with suitable initial or boundary errors and a stability estimate.



Question 10

Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases} \quad y'' + y = f(x), \quad y(0) = y'(0) = 0.$$

You may use the established facts that f is infinitely differentiable, even, positive away from 0, and $f^{(n)}(0) = 0$ for every $n \geq 0$.

Tasks

1. Determine every derivative of the solution at 0. What Taylor series does formal coefficient matching produce?
2. Derive a definite-integral representation of the actual solution and verify the equation and both initial values. Explain its existence and uniqueness on the whole real line.
3. Prove that the solution is even and strictly positive for $0 < |x| < \pi$. What does this imply about whether its Taylor series represents it near 0?
4. A student argues that constant coefficients make 0 an ordinary point, so the Taylor method must work. Identify the missing hypothesis and distinguish a solution of the formally expanded equation from a solution of the original equation.

Question 10 – Solution

Strategy. Test the forcing's analyticity before treating its Taylor coefficients as the forcing itself.

Step 1: Compute all center derivatives. The continuous forcing gives a C^2 solution; since f is smooth, repeated use of $y'' = f - y$ makes y smooth as well. Differentiation gives

$$y^{(n+2)}(0) = f^{(n)}(0) - y^{(n)}(0) = -y^{(n)}(0).$$

Starting with $y(0) = y'(0) = 0$, induction gives every derivative zero. Its Taylor series is therefore

$$\sum_{n \geq 0} 0x^n = 0, \text{ with infinite radius as a series.}$$

Step 2: Recover the actual forced solution. Variation of constants gives, for every real x ,

$$y(x) = \int_0^x \sin(x-t)f(t) dt.$$

Here $y'(x) = \int_0^x \cos(x-t)f(t) dt$ and $y''(x) = f(x) - y(x)$. At 0 both integrals vanish. The formula is defined on every finite real segment; any difference between two solutions satisfies $z'' + z = 0$ with zero initial values, hence is zero. This proves global existence and uniqueness.

Step 3: Prove nonzero values despite zero Taylor data. Because f is even, $y(-x)$ solves the same IVP; uniqueness gives $y(-x) = y(x)$. For $0 < x < \pi$ and $0 < t < x$, both $\sin(x-t)$ and $f(t)$ are strictly positive, so the integral is positive. Evenness extends this conclusion to $0 < |x| < \pi$. The zero Taylor series cannot equal y on any neighborhood of 0, even though it converges everywhere.

Step 4: Locate the missing analyticity assumption. The leading coefficient is nonzero and the homogeneous coefficients are analytic, but the forcing is not analytic at 0: its Taylor sum is zero while it is positive nearby. The analytic ordinary-point theorem for a nonhomogeneous equation requires analytic forcing too. Replacing f by its Taylor series changes the equation to $y'' + y = 0$. The formal zero series solves that changed equation, not the original equation away from 0.