

Question 1

Let $f(x) = \ln x$, and expand about $x = 2$. A student proposes

$$P_4(x) = \ln 2 + \sum_{n=1}^4 \frac{(-1)^{n-1}(n-1)!}{2^n} (x-2)^n.$$

A Taylor coefficient and a derivative value are different kinds of data.

Tasks

1. Derive $f^{(n)}(2)$ for $n \geq 1$, correct the proposed polynomial, and identify its first incorrect coefficient.
2. Find the full Taylor series and its interval of convergence, including both endpoints. Justify that its sum equals $\ln x$ on the open interval.
3. Use the Lagrange remainder to give a uniform absolute error bound for the corrected fourth-degree polynomial on $[1.8, 2.2]$.
4. Decide whether the corrected polynomial lies above or below $\ln x$ on each side of 2 in that interval. Support the conclusion with a signed remainder.

Question 1 – Solution

Strategy. Divide derivatives by factorials, then distinguish series convergence from finite-polynomial accuracy.

Step 1: Normalize the coefficients. For $n \geq 1$, $f^{(n)}(x) = (-1)^{n-1}(n-1)!/x^n$. Thus

$$T_4(2+h) = \ln 2 + \frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{24} - \frac{h^4}{64}.$$

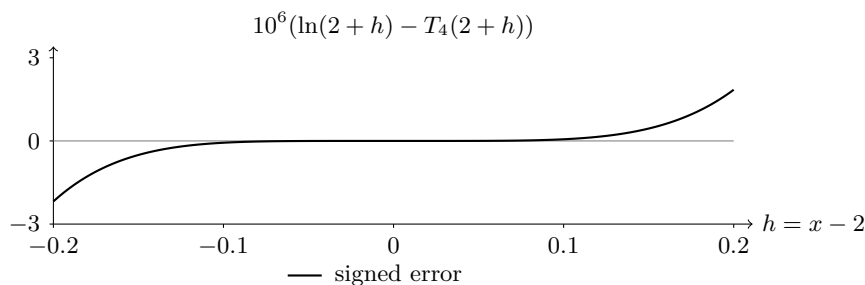
The student omitted division by $n!$; the first error is the h^2 coefficient, $-1/4$ instead of $-1/8$.

Step 2: Establish the represented function. Integrating the geometric series for $1/(2+h)$ from 0 to h gives $\ln(2+h) = \ln 2 + \sum_{n \geq 1} (-1)^{n-1} h^n / (n 2^n)$ for $|h| < 2$. Uniform convergence on the integration segment justifies this step. The term ratio tends to $|h|/2$, so the radius is 2; $h = -2$ gives the divergent negative harmonic series and $h = 2$ gives the alternating harmonic series. The convergence interval is $(0, 4]$ in x .

Step 3: Certify a uniform error. Since $f^{(5)}(x) = 24/x^5$, on $[1.8, 2.2]$,

$$|f(x) - T_4(x)| \leq \frac{24}{1.8^5} \frac{0.2^5}{120} = \frac{1}{5 \cdot 9^5} < 3.39 \times 10^{-6}.$$

Step 4: Determine the error sign. For $x \neq 2$, $f(x) - T_4(x) = h^5/(5\xi^5)$ for some positive ξ between 2 and x . Thus T_4 is above f on $[1.8, 2)$ and below it on $(2, 2.2]$; equality holds at 2. The plot magnifies this signed error.



Question 2

For $0 \leq x \leq 1$, let $T_n(x) = \sum_{k=0}^n (-x)^k / k!$ approximate e^{-x} . The number of terms in T_n is $n + 1$.

Tasks

1. Prove that the Maclaurin series represents e^{-x} on $[0, 1]$, using a remainder estimate that is uniform in x .
2. Find the smallest degree certified by the bound $|e^{-x} - T_n(x)| \leq 1/(n+1)!$ to achieve error at most 10^{-6} throughout $[0, 1]$.
3. A sufficient bound need not identify the true smallest degree. Prove in this case that the degree you found really is minimal, using the alternating tail at $x = 1$ to exclude every smaller degree.
4. Produce a rational enclosure for e^{-1} from consecutive partial sums at that degree, and explain which endpoint is the upper bound.

Question 2 – Solution

Strategy. Use Taylor's theorem for convergence and alternating tails for both upper and lower error bounds.

Step 1: Control all points simultaneously. Every derivative of e^{-x} has absolute value $e^{-x} \leq 1$ on $[0, 1]$. Taylor's theorem gives

$$|e^{-x} - T_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \leq \frac{1}{(n+1)!} \rightarrow 0.$$

This proves equality to the series and uniform convergence on the stated interval.

Step 2: Find a certified degree. Because $9! = 362880 < 10^6 < 3628800 = 10!$, the smallest degree certified by this bound is $\boxed{n=9}$, using ten terms. Degree 8 is not certified by the upper bound alone.

Step 3: Prove actual minimality. At $x = 1$, the alternating tail has the sign of its first term, and its magnitude is strictly greater than the difference of its first two magnitudes:

$$|e^{-1} - T_n(1)| > \frac{1}{(n+1)!} - \frac{1}{(n+2)!} = \frac{n+1}{(n+2)!} =: b_n.$$

Indeed, the remaining tail starts positive after factoring out its first sign and grouping consecutive pairs. Also $b_{n+1}/b_n = (n+2)/[(n+1)(n+3)] < 1$ for $n \geq 0$. Hence for every $0 \leq n \leq 8$, the error exceeds $b_8 = 9/10! > 10^{-6}$. No smaller degree works on all of $[0, 1]$.

Step 4: Enclose the value rationally. Odd partial sums lie below the limit and even ones above. Therefore

$$\boxed{\frac{16687}{45360} = T_9(1) < e^{-1} < T_{10}(1) = \frac{16481}{44800}}.$$

The interval width is exactly $1/10!$; the upper endpoint comes from the positive first omitted term after T_9 .

Question 3

For $x \neq 0$, define

$$H(x) = \frac{\sin x - x \cos x}{x^3}.$$

Near the origin the numerator suffers substantial cancellation.

Tasks

1. Derive the full Maclaurin series of H by combining the sine and cosine series before dividing. State its convergence domain.
2. Determine the unique continuous extension at 0, and find $H''(0)$ and $H^{(4)}(0)$ for that extension.
3. Construct its fourth-degree Taylor polynomial and certify its absolute error on $[-1, 1]$ by an alternating-series estimate.
4. Prove that the extended function is even and strictly positive on $[-1, 1]$. Explain why the series is useful for evaluating the original quotient near 0.

Question 3 – Solution

Strategy. Expose the first nonzero numerator term before removing the apparent singularity.

Step 1: Cancel and reindex. The coefficient of x^{2k+1} in the numerator is $(-1)^k[1/(2k+1)! - 1/(2k)!]$. It is zero when $k = 0$. Setting $j = k - 1$ for the surviving terms gives

$$H(x) = \sum_{j=0}^{\infty} \frac{(-1)^j 2(j+1)}{(2j+3)!} x^{2j}.$$

Sine and cosine converge absolutely for every real x ; the new series also has infinite radius by the ratio test. It agrees with the quotient when $x \neq 0$.

Step 2: Fill the removable singularity. The new series defines an analytic extension with

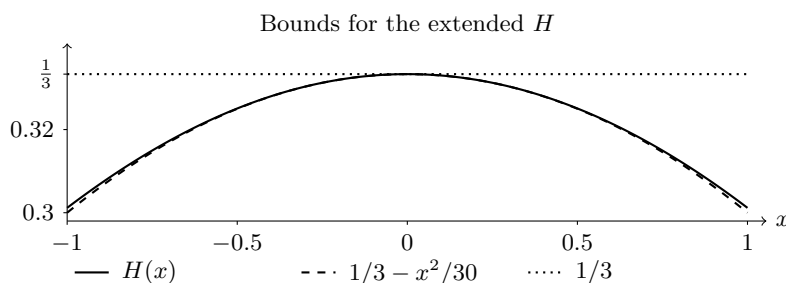
$$H(0) = \frac{1}{3}, \quad H''(0) = 2! \left(-\frac{1}{30} \right) = -\frac{1}{15}, \quad H^{(4)}(0) = 4! \frac{1}{840} = \frac{1}{35}.$$

Continuity forces the value $1/3$, so no other continuous extension is possible.

Step 3: Bound the truncated tail. The polynomial is $T_4(x) = 1/3 - x^2/30 + x^4/840$. For $|x| \leq 1$, successive term magnitudes have ratio $[(j+2)/(j+1)]x^2/[(2j+5)(2j+4)] \leq 1/10$. They decrease to zero, so

$$|H(x) - T_4(x)| \leq \frac{|x|^6}{45360} \leq \frac{1}{45360}.$$

Step 4: Interpret symmetry and stability. Only even powers occur. Alternating bounds give $H(x) \geq 1/3 - x^2/30 \geq 3/10 > 0$ on $[-1, 1]$. The numerator subtracts two quantities of order x to obtain one of order x^3 ; the series evaluates the finite limiting value without that subtraction. This is a cancellation concern, not a failure of the exact quotient.



Question 4

Let $F(x) = e^{x^2} \cos x$. A product of truncated series contains both useful terms and terms beyond the intended degree.

Tasks

1. Obtain the sixth-degree Maclaurin polynomial P_6 of F by explicit coefficient convolution.
2. Find $F^{(6)}(0)$ and explain why the remainder has order x^8 , rather than merely order x^7 .
3. Write $A = 1 + x^2 + x^4/2 + x^6/6$ and $B = 1 - x^2/2 + x^4/24 - x^6/720$. Compute $AB - P_6$ exactly.
4. On $|x| \leq 1/2$, prove the concrete bound $|F(x) - P_6(x)| < 0.0005$. Account separately for the exponential tail, cosine tail, and high-degree terms in AB .

Question 4 – Solution

Strategy. Convolve by degree, then bound the two truncation errors and the discarded product terms separately.

Step 1: Collect only the requested powers. Using the entire exponential and cosine series,

$$[x^2]F = 1 - \frac{1}{2} = \frac{1}{2}, \quad [x^4]F = \frac{1}{2} - \frac{1}{2} + \frac{1}{24} = \frac{1}{24},$$

$$[x^6]F = \frac{1}{6} - \frac{1}{4} + \frac{1}{24} - \frac{1}{720} = -\frac{31}{720}.$$

Thus $\boxed{P_6(x) = 1 + x^2/2 + x^4/24 - 31x^6/720}$.

Step 2: Recover a derivative and its parity. We have $F^{(6)}(0) = 6!(-31/720) = -31$. Both factors are even and analytic everywhere, so all odd coefficients vanish. The next possible power after degree six is eight, proving $F - P_6 = O(x^8)$ near 0.

Step 3: Retain the discarded product explicitly. Direct multiplication gives

$$AB - P_6 = -\frac{23}{360}x^8 + \frac{1}{160}x^{10} - \frac{1}{4320}x^{12}.$$

These terms belong to the product of polynomials, not to P_6 .

Step 4: Certify the error on the whole interval. Write $E = e^{x^2}$ and $C = \cos x$. Then $EC - P_6 = (E - A)C + A(C - B) + (AB - P_6)$. For $|x| \leq 1/2$, $|C| \leq 1$, $0 < A \leq e^{1/4} < 4/3$ (compare the exponential series with the geometric series). Taylor's theorem and the alternating cosine tail give

$$|E - A| \leq \frac{4}{3} \frac{|x|^8}{24}, \quad |C - B| \leq \frac{|x|^8}{8!}.$$

Consequently,

$$|F - P_6| \leq \frac{4/3}{24 \cdot 256} + \frac{4/3}{40320 \cdot 256} + \frac{23}{360 \cdot 256} + \frac{1}{160 \cdot 1024} + \frac{1}{4320 \cdot 4096} < 0.000473 < 0.0005.$$

The bound is conservative but rigorous; no unbounded O -term is used as a numerical certificate.

Question 5

Consider the nonelementary integral

$$I = \int_0^{1/2} e^{-t^2} dt, \quad S_m = \sum_{k=0}^m \frac{(-1)^k}{2^{2k+1} k! (2k+1)}.$$

An accuracy certificate is required; a decimal from a calculator is insufficient.

Tasks

1. Derive the series for I and justify integrating the Maclaurin series term by term on the whole integration interval.
2. Find the smallest m for which the first-omitted-term bound certifies $|I - S_m| \leq 10^{-6}$, and state the number of terms used.
3. Give an exact rational enclosure for I using that partial sum and the next one. Report an approximation with its certified absolute error.
4. Explain the sign of the error geometrically by comparing e^{-t^2} with the degree- $2m$ polynomial under the integral on $[0, 1/2]$.

Question 5 – Solution

Strategy. Integrate a uniformly convergent series, then use the integrated alternating tail as an error certificate.

Step 1: Justify the interchange. On $[0, 1/2]$, $|(-t^2)^k/k!| \leq (1/4)^k/k!$, whose sum is finite. The Weierstrass test gives uniform absolute convergence. Integrating yields

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+1}k!(2k+1)}.$$

The positive term magnitudes decrease strictly to zero.

Step 2: Choose a certified truncation. The first omitted magnitude is $b_m = 1/[2^{2m+3}(m+1)!(2m+3)]$. It decreases with m . For $m = 3$, $b_3 = 1/110592 > 10^{-6}$, whereas

$$b_4 = \frac{1}{2703360} < 3.70 \times 10^{-7} < 10^{-6}.$$

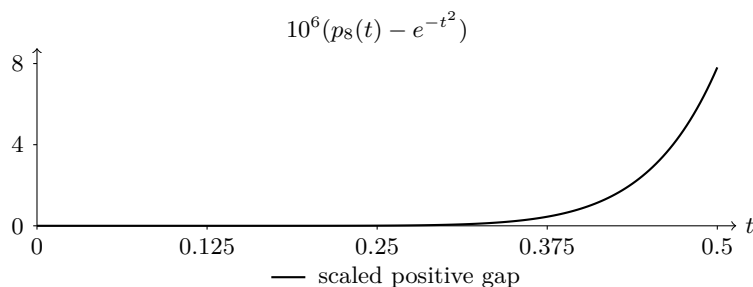
Thus $\boxed{m = 4}$ is the smallest index certified by this bound, using five terms.

Step 3: Supply an exact enclosure. The even partial sum is an upper bound and the next odd sum a lower bound:

$$S_4 - \frac{1}{2703360} = S_5 < I < S_4 = \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376} + \frac{1}{110592}.$$

Numerically, $S_4 \approx 0.4612813637$. Reporting $I \approx 0.46128136$ introduces less than 4×10^{-9} additional rounding error, so its total absolute error is less than 3.74×10^{-7} .

Step 4: Interpret the signed area. Set $p_8(t) = 1 - t^2 + t^4/2 - t^6/6 + t^8/24$. The alternating tail is strictly negative for $t > 0$ in this interval, so $p_8(t) > e^{-t^2}$. Hence $S_4 - I$ is the positive area between the curves. The scaled gap below makes the very small difference visible; it vanishes at $t = 0$.



Question 6

Define a function on the real line by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

A student claims: “Every infinitely differentiable function agrees with its Taylor series near the center.”

Tasks

1. Prove $|x|^{-m}e^{-1/x^2} \rightarrow 0$ as $x \rightarrow 0$ for every nonnegative integer m .
2. Prove that f is infinitely differentiable at 0 and that every derivative there vanishes. Use a polynomial-in- $1/x$ description away from 0 and difference quotients at 0.
3. Find the Maclaurin series, its radius of convergence, and the set of points where it equals f .
4. Explain precisely why Taylor’s theorem is still valid but fails to prove equality to the infinite series here. Identify the remainder $R_n(x)$ at any fixed nonzero x .

Question 6 – Solution

Strategy. Separate the existence of every derivative from the limiting behavior of the Taylor remainders.

Step 1: Exponential decay beats every power. Put $u = 1/x^2 \rightarrow \infty$. Choose an integer $k > m/2$. Since $e^u \geq u^k/k!$,

$$|x|^{-m} e^{-1/x^2} = u^{m/2} e^{-u} \leq k! u^{m/2-k} \rightarrow 0.$$

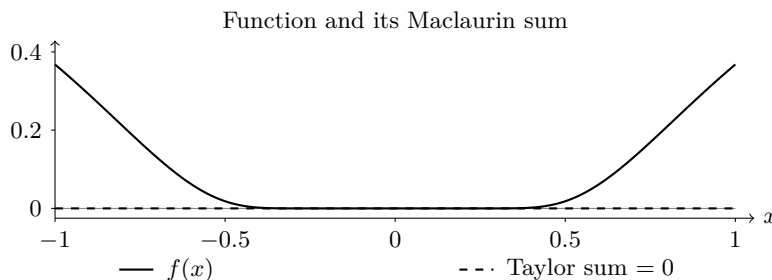
Step 2: Prove smoothness, including the center. For $x \neq 0$, induction gives $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$, where $P_0(z) = 1$ and

$$P_{n+1}(z) = -z^2 P'_n(z) + 2z^3 P_n(z).$$

Define each candidate derivative to be zero at 0. Step 1 proves its continuity there. Its difference quotient at 0 is $P_n(1/h)e^{-1/h^2}/h \rightarrow 0$, again by Step 1. Inductively each candidate is the derivative of the preceding one, so $f \in C^\infty(\mathbb{R})$ and $f^{(n)}(0) = 0$ for every $n \geq 0$.

Step 3: Compare the series with the function. Every Taylor polynomial is zero. Thus the Maclaurin series is $\sum_{n=0}^{\infty} 0x^n = 0$, with infinite radius. It equals f only at $x = 0$, since $f(x) > 0$ for every nonzero x . Infinite radius does not by itself identify the represented function.

Step 4: Locate the missing hypothesis. Taylor's finite theorem remains valid on the segment from 0 to any fixed x . But $R_n(x) = f(x) - T_n(x) = e^{-1/x^2} > 0$ for $x \neq 0$, independently of n . Thus $R_n(x) \not\rightarrow 0$. Smoothness provides each finite remainder formula, not a derivative-growth bound forcing remainders to vanish. The graph and the zero Taylor sum touch to every order at 0 without agreeing nearby.



Question 7

The direct Maclaurin expansion of $\ln(1+x)$ converges slowly at $x=1$. Seek a faster Taylor-series evaluation of $\ln 2$ using

$$L(z) = \ln\left(\frac{1+z}{1-z}\right), \quad |z| < 1.$$

Count terms by the number of nonzero summands, not by polynomial degree.

Tasks

1. Derive the Maclaurin series for L by differentiating and integrating a geometric series, with a justification on $|z| < 1$.
2. Select a positive z for which $L(z) = \ln 2$. For the first N terms, prove the tail bound

$$\frac{2z^{2N+1}}{(2N+1)(1-z^2)}$$

and determine the error sign.

3. Find the smallest N for which this bound is strictly below 10^{-6} at the selected z . Give the resulting approximation and a rigorous enclosure.
4. For the direct alternating harmonic sum with N terms, how many terms does the next-term bound require to certify error strictly below 10^{-6} ? Compare the two certificates and explain the limits of this comparison.

Question 7 – Solution

Strategy. Change the evaluation point to shrink the geometric factor controlling the Taylor tail.

Step 1: Construct the odd-power series. Since $L'(z) = 2/(1 - z^2) = 2 \sum_{k \geq 0} z^{2k}$ and $L(0) = 0$,

$$L(z) = 2 \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}, \quad |z| < 1.$$

The geometric series converges uniformly on any closed segment inside $(-1, 1)$, so termwise integration is justified.

Step 2: Bound a positive tail. The equation $(1+z)/(1-z) = 2$ gives $z = 1/3$. With $A_N = 2 \sum_{k=0}^{N-1} z^{2k+1}/(2k+1)$ and $0 < z < 1$,

$$\begin{aligned} 0 < L(z) - A_N &= 2 \sum_{k=N}^{\infty} \frac{z^{2k+1}}{2k+1} \\ &< \frac{2z^{2N+1}}{(2N+1)(1-z^2)} =: B_N. \end{aligned}$$

Strictness follows because later denominators exceed $2N+1$.

Step 3: Count the accelerated terms. At $z = 1/3$, $B_N = 9/[4(2N+1)3^{2N+1}]$, strictly decreasing in N . Here $B_5 = 9/7794468 > 10^{-6}$ and $B_6 = 9/82904796 < 1.086 \times 10^{-7}$. Thus six terms are the smallest number certified by this bound, and

$$A_6 < \ln 2 < A_6 + \frac{9}{82904796}, \quad A_6 \approx 0.6931470738.$$

Step 4: Make a fair certificate comparison. For the direct alternating sum with N terms, the stated bound is $1/(N+1)$. To make that bound strictly less than 10^{-6} requires $N \geq 1,000,000$. The comparison is six versus one million terms under these particular certificates. It does not assert that a million terms are necessary for the true direct-series error, nor that a loose bound is an optimal algorithm. The transformed series profits from the factor $z^2 = 1/9$.

Question 8

An unknown function belongs to the family $f(x) = e^{ax} \cos(bx)$, where a, b are real. Exact local measurements give

$$f'(0) = 1, \quad f''(0) = -3.$$

A separate report claims $f'''(0) = -10$. No measurement uncertainty is assumed.

Tasks

1. Recover every admissible parameter pair (a, b) from the first two measurements. Explain the remaining ambiguity and whether it changes f .
2. Find the fourth-degree Maclaurin polynomial of every compatible function.
3. Test the separate third-derivative report and decide whether all three measurements can arise from one member of the stated family.
4. Prove a uniform remainder bound below 5.2×10^{-6} for your polynomial on $[-0.1, 0.1]$. You may use $f(x) = \operatorname{Re} e^{(a+ib)x}$ and $|\operatorname{Re} z| \leq |z|$.

Question 8 – Solution

Strategy. Turn measured derivatives into algebraic constraints before predicting higher coefficients.

Step 1: Recover the parameters. Differentiation at 0 gives $f'(0) = a$ and $f''(0) = a^2 - b^2$. Hence $a = 1$ and $b^2 = 4$, so

$$(a, b) = (1, 2) \text{ or } (1, -2).$$

The sign cannot be recovered because cosine is even; both pairs produce the same function $e^x \cos 2x$.

Step 2: Predict the Taylor polynomial. Using real parts of $(1 + 2i)^n$, the derivative values for $n = 0, \dots, 4$ are $1, 1, -3, -11, -7$. Dividing by $n!$ yields

$$T_4(x) = 1 + x - \frac{3}{2}x^2 - \frac{11}{6}x^3 - \frac{7}{24}x^4.$$

For example, $(1 + 2i)^2 = -3 + 4i$ and $(1 + 2i)^3 = -11 - 2i$.

Step 3: Reject inconsistent exact data. Every pair compatible with the first two measurements forces $f'''(0) = -11$. The reported value -10 is incompatible. Changing the sign of b conjugates the complex powers and leaves their real parts unchanged, so it cannot repair the discrepancy.

Step 4: Bound the fifth derivative uniformly. For real t ,

$$|f^{(5)}(t)| = \left| \operatorname{Re} \left((1 + 2i)^5 e^{(1+2i)t} \right) \right| \leq 5^{5/2} e^t.$$

Taylor's theorem on each segment from 0 to $x \in [-0.1, 0.1]$ therefore gives

$$|f(x) - T_4(x)| \leq \frac{5^{5/2} e^{0.1}}{120} 10^{-5} < 5.2 \times 10^{-6}.$$

For a purely rational check of the last inequality, use $\sqrt{5} < 9/4$ and $e^{0.1} < 1.106$: the latter follows by bounding the tail after $1 + 0.1 + 0.1^2/2$ by $(0.1^3/6)/(1 - 0.025)$. These bounds give less than 5.185×10^{-6} .

Question 9

Let $T_4(x) = 1 - x^2/2 + x^4/24$, the fourth-degree Maclaurin polynomial of $\cos x$. Consider a rational competitor

$$Q(x) = \frac{1 + ax^2}{1 + bx^2}, \quad a, b \in \mathbb{R}.$$

A student believes that replacing a Taylor polynomial by a rational expression with the same local coefficients must improve accuracy.

Tasks

1. Determine a, b so that Q and $\cos x$ agree through degree four at 0, and identify any real poles of Q .
2. Find the leading nonzero terms of $\cos x - T_4(x)$ and $\cos x - Q(x)$. Compute the limiting ratio of their absolute errors as $x \rightarrow 0, x \neq 0$.
3. Use an alternating Taylor enclosure for $\cos 1$ to prove which approximation is closer at $x = 1$, without relying on a calculator value of $\cos 1$.
4. Evaluate the student's claim and explain why matching coefficients alone cannot rank all approximations on a whole interval.

Question 9 – Solution

Strategy. Match coefficients, then examine the first coefficient on which the competing approximations disagree.

Step 1: Solve the matching equations. Near 0, $Q = 1 + (a-b)x^2 + (b^2-ab)x^4 + \dots$. Thus $a-b = -1/2$ and $-b(a-b) = 1/24$, giving

$$b = \frac{1}{12}, \quad a = -\frac{5}{12}, \quad Q(x) = \frac{12-5x^2}{12+x^2}.$$

Since $12+x^2 > 0$, there are no real poles.

Step 2: Compare the leading errors. The rational geometric expansion has x^6 coefficient $-1/288$. Hence

$$\cos x - T_4(x) = -\frac{x^6}{720} + O(x^8), \quad \cos x - Q(x) = \frac{x^6}{480} + O(x^8).$$

Therefore $\lim_{x \rightarrow 0} \frac{|\cos x - Q(x)|}{|\cos x - T_4(x)|} = 3/2$. In particular, the rational approximation is worse for all sufficiently small nonzero x .

Step 3: Make an exact comparison at one. Alternating bounds give

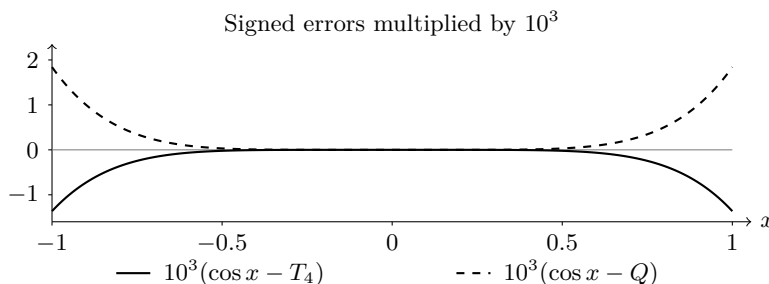
$$T_6(1) = \frac{389}{720} < \cos 1 < T_4(1) = \frac{13}{24}, \quad Q(1) = \frac{7}{13}.$$

Here $Q(1) < 389/720$, and the midpoint of the two approximations is $\frac{1}{2}(7/13 + 13/24) = 337/624 < 389/720$. Thus $\cos 1$ lies above their midpoint and between them: $T_4(1)$ is strictly closer than $Q(1)$.

Step 4: State the proper conclusion. The claim fails both locally and at $x = 1$. Matching four derivatives fixes contact at the center, not the next error coefficient or behavior far away. For $0 < |x| \leq 1$, alternating bounds give $\cos x - T_4(x) < 0$ and

$$\cos x - Q(x) \geq x^6 \left(\frac{1}{312} - \frac{1}{720} \right) > 0,$$

since $T_4 - Q = x^6/[288(1+x^2/12)]$. These signs explain the plotted errors.



Question 10

Let $f \in C^\infty([-r, r])$, with $r > 0$, and suppose constants $C, A > 0$ satisfy

$$|f^{(n)}(t)| \leq CA^n n! \quad (n \geq 0, |t| \leq r).$$

Write $T_N(x) = \sum_{n=0}^N f^{(n)}(0)x^n/n!$. For a boundary example you may use $g(x) = 1/(1+x^2)$ and the identity

$$g^{(n)}(x) = \frac{(-1)^n n!}{2i} \left((x-i)^{-n-1} - (x+i)^{-n-1} \right), \quad i^2 = -1.$$

Tasks

1. Prove that $T_N(x) \rightarrow f(x)$ for $|x| \leq r$ with $A|x| < 1$. Give a uniform error estimate on $[-\rho, \rho]$ when $0 < \rho \leq r$ and $A\rho < 1$.
2. Explain why this argument gives no conclusion at $A|x| = 1$. Show that both convergence and divergence there are possible under the stated derivative hypothesis, using $f = 1$ and g on $[-1, 1]$ with $C = A = 1$.
3. A function has all derivatives zero at 0 but is positive at every nearby nonzero point. Prove it cannot obey such a derivative bound on any neighborhood of 0.
4. Suppose the displayed bound is known only at $t = 0$. What can you conclude about convergence of the Taylor series, and what additional conclusion can you no longer justify? Supply a counterexample.

Question 10 – Solution

Strategy. A bound on derivatives throughout a segment controls Taylor's remainder; center-only data control coefficients.

Step 1: Force the remainders to vanish. Taylor's theorem uses some ξ between 0 and x . The hypothesis gives

$$|f(x) - T_N(x)| \leq \frac{CA^{N+1}(N+1)!}{(N+1)!}|x|^{N+1} = \boxed{C(A|x|)^{N+1}}.$$

This tends to zero whenever $|x| \leq r$ and $A|x| < 1$. On the specified smaller interval the uniform bound is $C(A\rho)^{N+1} \rightarrow 0$.

Step 2: Test the boundary honestly. At $A|x| = 1$ the bound is only C , which does not tend to zero. The constant function 1 satisfies the bound with $C = A = 1$ and its Taylor series converges at ± 1 . For real x , $|x \pm i| \geq 1$, so the given identity yields $|g^{(n)}(x)| \leq n!$. Yet the Maclaurin series of g is $\sum_{k \geq 0} (-1)^k x^{2k}$; at $x = \pm 1$ its terms fail to tend to zero. Thus neither boundary outcome is forced.

Step 3: Rule out the bound for a flat function. If such constants existed on some $[-r, r]$, choose $0 < |x| < \min(r, 1/A)$. All Taylor polynomials would be zero, but Step 1 would force $f(x) = 0$, contradicting positivity. Thus no finite positive C, A can work on any such neighborhood.

Step 4: Separate coefficients from function values. A bound only at 0 gives $|f^{(n)}(0)/n!| \leq CA^n$. Geometric comparison proves absolute convergence of the Taylor series for $|x| < 1/A$, but does not prove that its sum is $f(x)$. The smooth function e^{-1/x^2} extended by zero at 0 has every center derivative zero, so it satisfies every such center-only bound; its Taylor series is zero while the function is positive elsewhere.