

## Question 1

Consider the real power series

$$F(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}.$$

A student reports a radius of convergence and assumes that this alone specifies where the series represents a finite function.

### Tasks

1. Identify the center and radius of convergence. Explain what the ratio test says inside and outside the radius, and what it leaves undecided.
2. Test both boundary points separately. Give the exact interval of convergence and distinguish absolute from conditional convergence.
3. Derive a closed formula for the sum inside the open interval by integrating a geometric series. Include the constant fixed by the value at the center.
4. Determine the sum at the included boundary point without substituting into an unjustified endpoint identity. Sketch the convergence interval and explain why the excluded endpoint behaves differently.

## Question 1 – Solution

**Strategy.** Normalize the displacement from the center, test the endpoints as numerical series, then justify any boundary value independently.

**Step 1: Apply the ratio test away from the center.** Write  $z = (x - 2)/3$ . For  $z \neq 0$ , the ratio of successive term magnitudes is  $|z|n/(n+1) \rightarrow |z|$ . Thus the center is 2 and  $R = 3$ . The series converges absolutely for  $|z| < 1$  and diverges for  $|z| > 1$ . At the center all terms vanish. The ratio test is inconclusive at  $z = \pm 1$ .

**Step 2: Resolve the two endpoints.** At  $x = -1$ , the series is  $\sum_{n \geq 1} (-1)^n/n$ , which converges by alternation but not absolutely. At  $x = 5$ , it is the divergent harmonic series. Therefore

$$I = [-1, 5), \quad \text{absolute on } (-1, 5), \quad \text{conditional only at } -1.$$

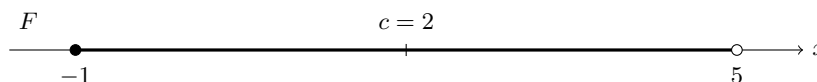
A radius specifies distances, not whether either boundary point is included.

**Step 3: Integrate the geometric series.** On any compact subinterval of  $(-1, 1)$ , the geometric series converges uniformly, so integration from 0 to  $z$  gives

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \int_0^z \frac{du}{1-u} = -\log(1-z).$$

This also holds for negative  $z$  with the usual oriented integral. Both sides vanish at  $z = 0$ , fixing the constant. Hence  $F(x) = -\log(1 - (x - 2)/3)$  for  $-1 < x < 5$ .

**Step 4: Establish the boundary value.** The finite geometric identity gives  $\sum_{k=0}^{N-1} (-u)^k = (1 - (-u)^N)/(1 + u)$ . Integrating over  $[0, 1]$ , its difference from  $\log 2$  has magnitude at most  $\int_0^1 u^N du = 1/(N+1) \rightarrow 0$ . Negating the resulting alternating harmonic sum proves  $F(-1) = -\log 2$ . At  $x = 5$  there is no alternation and harmonic divergence persists; the interior logarithm also tends to  $+\infty$  there.



## Question 2

A power series with large gaps between nonzero coefficients is written as

$$L(x) = \sum_{k=0}^{\infty} 2^k x^{2^k}.$$

The first terms are  $x + 2x^2 + 4x^4 + 8x^8 + \cdots$ . Use the ordinary definition of a power series even though many coefficients vanish.

### Tasks

1. Write its coefficient rule in the form  $\sum_{n=0}^{\infty} a_n x^n$ . Explain why applying  $|a_{n+1}/a_n|$  to all consecutive coefficients is inappropriate.
2. Prove the exact radius using comparison when  $|x| < 1$  and the necessary term test when  $|x| > 1$ .
3. Test both endpoints, paying attention to the parity of  $2^k$ . Does the endpoint  $x = -1$  produce an alternating tail?
4. Let  $x_m = e^{-1/2^m}$  for integers  $m \geq 0$ . Use a single term to bound  $L(x_m)$  from below, and decide whether the sparse exponents make  $L$  bounded as  $x \rightarrow 1^-$ .

## Question 2 – Solution

**Strategy.** Treat the missing coefficients explicitly. A convergent positive comparison series controls the interior, while selected large terms control boundary growth.

**Step 1: Identify the coefficients.** We have  $a_n = n$  when  $n = 2^k$  for an integer  $k \geq 0$ , and  $a_n = 0$  otherwise, including  $n = 0$ . Consecutive coefficient ratios repeatedly have zero denominators, so the usual coefficient-ratio shortcut is unavailable. This does not prevent other convergence tests from determining the radius.

**Step 2: Establish the radius directly.** For  $r = |x| < 1$ ,

$$\sum_{k=0}^{\infty} 2^k r^{2^k} \leq \sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2} < \infty.$$

Thus the series converges absolutely throughout  $|x| < 1$ . If  $|x| > 1$ , the magnitudes  $2^k |x|^{2^k}$  do not tend to zero; indeed they grow without bound. Therefore  $\boxed{R = 1}$ . The comparison sum follows by differentiating the geometric series inside its radius.

**Step 3: Test parity at the endpoints.** At  $x = 1$ , the terms are  $2^k$  and fail the term test. At  $x = -1$ , the first term is  $-1$ , but every term with  $k \geq 1$  is  $+2^k$ , since  $2^k$  is even. There is no alternating tail. Both endpoint series diverge, so  $\boxed{I = (-1, 1)}$ , with absolute convergence at every point of this interval.

**Step 4: Prove unbounded growth despite the gaps.** For  $0 < x_m < 1$ , every term is positive. Keeping just the term indexed by  $k = m$  gives

$$\boxed{L(x_m) \geq 2^m x_m^{2^m} = 2^m / e \rightarrow \infty.}$$

Also  $x_m \uparrow 1$ . Thus  $L$  is not bounded near 1. In fact the sum is increasing on  $[0, 1)$  because each term is increasing, so this sequence of lower bounds proves  $L(x) \rightarrow +\infty$  as  $x \rightarrow 1^-$ . Sparse powers do not offset the growth of the coefficients and their near-boundary terms.

**Question 3**

Let

$$F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \quad G(x) = \int_0^x F(t) dt.$$

Use termwise differentiation and integration only where their hypotheses hold. No closed formula for  $F$  itself is required.

**Tasks**

1. Determine the radius and exact convergence interval of the series for  $F$ , including the type of convergence at each endpoint.
2. Obtain power series for  $F'$  and  $G$ . Determine each radius and interval, testing the endpoints afresh rather than copying them from  $F$ .
3. Sum the derivative series for  $|x| < 1$  using the logarithmic series, treating  $x = 0$  separately. Find its limit as  $x \rightarrow -1^+$  and as  $x \rightarrow 1^-$ .
4. Prove  $F$  is continuous on  $[-1, 1]$ . Determine its one-sided endpoint slopes and explain why a finite endpoint value need not have a finite slope. Compare the three convergence intervals in a diagram.

## Question 3 – Solution

**Strategy.** Differentiation and integration preserve the radius but can alter endpoints; continuity and the mean value theorem settle endpoint slopes.

**Step 1: Check the original series.** The ratio test gives  $R = 1$ . At  $x = 1$  and  $x = -1$ , the absolute values sum to  $\sum 1/n^2 < \infty$ . Thus  $I_F = [-1, 1]$ , absolutely throughout. For  $|x| > 1$  the terms do not tend to zero.

**Step 2: Differentiate and integrate inside the radius.** For  $|x| < 1$ ,

$$F'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \quad G(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2(n+1)}.$$

The constant in  $G$  is zero because  $G(0) = 0$ . Both new radii are 1. For the derivative series,  $x = 1$  gives harmonic divergence;  $x = -1$  gives conditional alternating convergence. Hence  $I_{F'} = [-1, 1)$ . The integrated series converges absolutely at both endpoints, so  $I_G = [-1, 1]$ . Here these interval labels refer to the displayed series, not an assumed endpoint differentiation rule.

**Step 3: Sum the derivative in the open interval.** Using  $\sum_{n \geq 1} x^n/n = -\log(1-x)$  gives

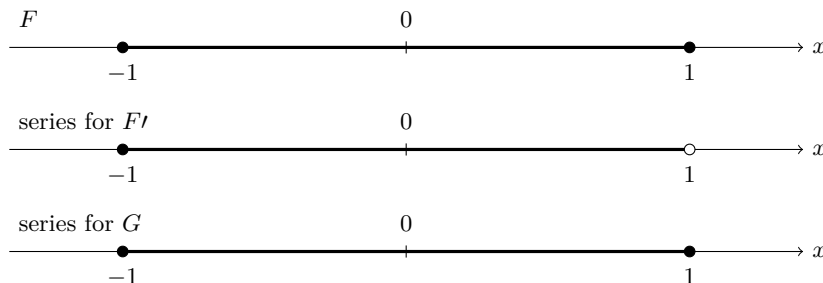
$$F'(x) = \frac{-\log(1-x)}{x} \quad (0 < |x| < 1), \quad F'(0) = 1.$$

The value at zero follows from the constant term and also from the limit. Consequently  $\lim_{x \rightarrow -1+} F'(x) = \log 2$ , whereas  $\lim_{x \rightarrow 1-} F'(x) = +\infty$ .

**Step 4: Justify what happens at the boundary.** Since  $|x^n/n^2| \leq 1/n^2$  on  $[-1, 1]$ , the uniform majorant test gives uniform convergence there and continuity of  $F$ . Apply the mean value theorem between an interior point and either endpoint. The intervening derivative then tends to the corresponding limit found above. Therefore

$$F'_+(-1) = \log 2, \quad \lim_{x \rightarrow 1-} \frac{F(1) - F(x)}{1 - x} = +\infty.$$

So the finite value  $F(1)$  has no finite left derivative. Endpoint behavior must be proved separately even when the power-series radius is unchanged.



**Question 4**

For an integer  $N \geq 0$ , define

$$S_N(x) = \sum_{n=0}^N x^n, \quad f(x) = \frac{1}{1-x}, \quad |x| < 1.$$

A student says that choosing a sufficiently large  $N$  makes this approximation uniformly accurate throughout the entire convergence interval.

**Tasks**

1. Derive the exact remainder  $f - S_N$  from a finite geometric identity. State its sign for positive and negative  $x$ .
2. Find the least  $N$  guaranteeing absolute error at most  $10^{-3}$  for every  $x \in [-1/2, 1/2]$ . State both the largest exponent and the number of retained terms.
3. Express the relative error  $|f - S_N|/|f|$ . For any fixed  $N$ , find a point in  $(0, 1)$  where this error is exactly  $1/2$ .
4. Decide whether convergence is uniform on  $(-1, 1)$ , for either absolute or relative error, and justify the distinction from convergence at each fixed point. Plot  $S_2, S_5, S_{10}$  and  $f$  on  $[0, 0.85]$ .

## Question 4 – Solution

**Strategy.** An exact remainder supplies a sharp interval bound and exposes what fails when points approach the boundary as the degree changes.

**Step 1: Retain the exact geometric remainder.** Multiplication gives  $(1-x)S_N = 1 - x^{N+1}$ . Therefore

$$f(x) - S_N(x) = \frac{x^{N+1}}{1-x}.$$

For  $0 < x < 1$  it is positive. For  $-1 < x < 0$ , its sign is  $(-1)^{N+1}$ ; at  $x = 0$  it is zero. The denominator is positive throughout the interval.

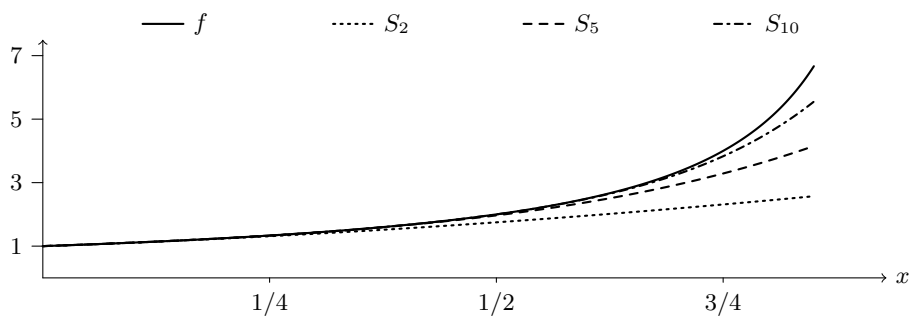
**Step 2: Find the least degree on the compact interval.** For  $|x| \leq 1/2$ ,

$$|f - S_N| \leq \frac{(1/2)^{N+1}}{1 - 1/2} = 2^{-N}.$$

Equality holds at  $x = 1/2$ , so this is the exact maximum, not merely a convenient estimate. Since  $2^{-9} > 10^{-3}$  and  $2^{-10} < 10^{-3}$ ,  $N = 10$  is least. This retains 11 terms, including the constant.

**Step 3: Select a moving point with fixed relative error.** The relative error simplifies to  $|x|^{N+1}$ . Taking  $x_N = 2^{-1/(N+1)}$  gives a point in  $(0, 1)$  with relative error exactly  $1/2$  for every  $N \geq 0$ . These points approach 1 as  $N$  increases.

**Step 4: Distinguish fixed-point from uniform accuracy.** At each fixed  $|x| < 1$ , both errors tend to zero. Yet for every fixed  $N$ ,  $x^{N+1}/(1-x) \rightarrow +\infty$  as  $x \rightarrow 1^-$ , so the supremum of the absolute error on  $(-1, 1)$  is infinite. The supremum of relative error is 1. Neither error converges uniformly to zero on that whole interval. The plot shows slower approximation near the right boundary; on any fixed  $[-r, r]$ ,  $r < 1$ , the bound  $r^{N+1}/(1-r) \rightarrow 0$  does give uniform convergence.



**Question 5**

Consider two series centered at zero:

$$A(x) = \sum_{n=0}^{\infty} x^n, \quad B(x) = 1 - \sum_{n=1}^{\infty} \frac{x^n}{2^n}.$$

Let  $C(x) = \sum_{n=0}^{\infty} c_n x^n$  be the series obtained by their Cauchy product, using finite coefficient convolutions.

**Tasks**

1. Find the sums and exact convergence intervals of  $A$  and  $B$  separately.
2. Compute  $c_n$  for every  $n \geq 0$ , including the constant coefficient. Determine the radius and interval of the resulting series  $C$ .
3. Identify the interval where the Cauchy-product theorem directly proves  $C = AB$ . Explain the rational cancellation that permits the new series to converge on a larger interval.
4. Evaluate the series  $C$  at  $x = 1$  and  $x = 3/2$ . At each point, decide whether this value can legitimately be called the product of the two original series sums.

## Question 5 – Solution

**Strategy.** Compute the product coefficients before interpreting the enlarged domain; a new convergent series does not make a divergent factor converge.

**Step 1: Sum and test the two factors.** Geometric summation gives

$$A(x) = \frac{1}{1-x} \quad (|x| < 1), \quad B(x) = 1 - \frac{x/2}{1-x/2} = \frac{1-x}{1-x/2} \quad (|x| < 2).$$

At each respective endpoint the terms fail to tend to zero. Thus the exact intervals are  $(-1, 1)$  and  $(-2, 2)$ , both absolutely convergent inside.

**Step 2: Compute the finite coefficient sums.** The coefficients of  $A$  are all 1; those of  $B$  are  $b_0 = 1$  and  $b_n = -2^{-n}$  for  $n \geq 1$ . Hence

$$c_n = \sum_{k=0}^n b_k = 1 - \sum_{k=1}^n 2^{-k} = 2^{-n} \quad (n \geq 0).$$

In particular  $c_0 = 1$ . Therefore  $C(x) = \sum_{n \geq 0} (x/2)^n = 1/(1-x/2)$  for  $|x| < 2$ . Its radius is 2 and both endpoints diverge.

**Step 3: State where multiplication is justified.** Both factors converge absolutely on  $(-1, 1)$ , so the Cauchy-product theorem proves equality there. Their rational formulas give

$$\frac{1}{1-x} \frac{1-x}{1-x/2} = \frac{1}{1-x/2}.$$

Cancellation removes the factor responsible for  $A$ 's boundary at 1. The coefficient series  $C$  consequently has radius 2, exceeding the smaller input radius. The theorem guarantees convergence on the common interior; it does not require the resulting radius to equal the smaller one.

**Step 4: Interpret the additional points carefully.** The new series gives  $C(1) = 2$ ,  $C(3/2) = 4$ . At  $x = 1$ ,  $A$  diverges and  $B$  sums to zero; an undefined series sum cannot be multiplied by zero to produce 2. At  $x = 3/2$ ,  $A$  again diverges, although  $B$  converges to  $-2$ . Thus neither value is a product of the two original series sums. They are valid values of the independently convergent product-coefficient series and its rational formula.

**Question 6**

Let  $f(x) = 1/(4 - x)$ , defined for real  $x \neq 4$ . Represent this same function by geometric power series with two different centers,  $c_1 = 1$  and  $c_2 = -1$ . Do not use a Taylor-coefficient formula.

**Tasks**

1. Rewrite the denominator about each center and obtain the two power series explicitly.
2. Find both radii and exact real convergence intervals, testing their endpoints. Mark centers and intervals in a diagram.
3. At  $x = -4$ , decide which series converges and find its sum. At  $x = 5$ , decide whether either series converges even though  $f(5)$  exists.
4. Explain why the two series agree wherever both converge, why changing the center changes the radius here, and why the domain of the rational function is not the convergence interval of either series.

## Question 6 – Solution

**Strategy.** Factor out the distance from each center to the denominator's zero, then apply the geometric-series condition to the normalized displacement.

**Step 1: Expand about each center.** Using  $4 - x = 3 - (x - 1)$  and  $4 - x = 5 - (x + 1)$  gives

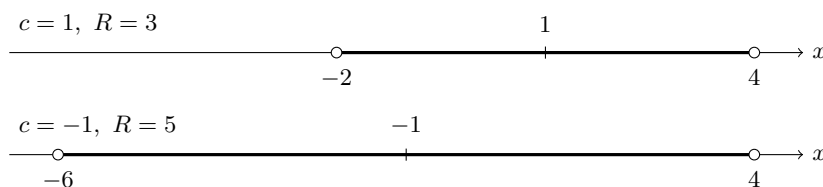
$$f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{3^{n+1}}, \quad f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{5^{n+1}}.$$

Each equality is initially asserted only when its geometric ratio has magnitude less than one.

**Step 2: Determine the two intervals.** The first ratio is  $(x-1)/3$ , so  $R_1 = 3$  and  $I_1 = (-2, 4)$ . The second is  $(x+1)/5$ , so  $R_2 = 5$  and  $I_2 = (-6, 4)$ . At each endpoint the terms have constant nonzero magnitude ( $1/3$  or  $1/5$ ), so all four endpoint tests give divergence. Both series converge absolutely inside their respective intervals.

**Step 3: Test the specified points.** At  $x = -4$ , the first ratio is  $-5/3$ , so its terms do not tend to zero. The second ratio is  $-3/5$ , so it converges and sums to  $\boxed{(1/5)/(1+3/5) = 1/8 = f(-4)}$ . At  $x = 5$ , the ratios are  $4/3$  and  $6/5$ , so both series diverge even though  $\boxed{f(5) = -1}$  is a valid rational-function value.

**Step 4: Compare representations and domains.** On the common interval  $(-2, 4)$ , each convergent geometric identity gives the same function  $1/(4-x)$ , so their sums agree. Their right endpoints coincide with the denominator's zero, but this point is distance 3 from one center and distance 5 from the other. The left boundaries follow from the same absolute-ratio conditions, even though  $f$  is finite there. The rational function has domain  $\mathbb{R} \setminus \{4\}$ , while each series represents it on only its own convergence interval. Evaluation of the rational formula outside that interval is not evaluation of the series.



**Question 7**

Suppose  $y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n$  has positive radius  $R$ . Consider  $H(x) = xy'(x) + y(x)$  only for  $|x-2| < R$ . A student claims that  $H(x) = \sum_{n=0}^{\infty} (n+1)a_n(x-2)^n$ .

**Tasks**

1. Derive the correct coefficient of  $(x-2)^n$  in  $H$  for every  $n \geq 0$ , reindexing the derivative carefully.
2. Write the constant, linear, quadratic and cubic coefficients explicitly in terms of the  $a_n$ .
3. Identify the student's missing contribution and characterize exactly which series  $y$  make the claimed identity valid throughout the open interval.
4. Apply your formula to  $a_n = 3^{-n-1}$ . Sum  $y$  geometrically, compute  $H$  directly from this closed form, and verify the coefficients agree. State the resulting series' radius.

## Question 7 – Solution

**Strategy.** Work in  $h = x - 2$  and remember that multiplication by  $x$  means multiplication by  $h + 2$ , not merely by  $h$ .

**Step 1: Reindex before multiplying.** Inside the radius,  $y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}h^n$ . Now  $hy' = \sum_{n=1}^{\infty} na_nh^n$ ; its constant coefficient is zero. Adding  $2y'$  and  $y$  yields

$$H = \sum_{n=0}^{\infty} ((n+1)a_n + 2(n+1)a_{n+1})h^n.$$

The formula also works for  $n = 0$ , where the constant term of  $xy'$  comes entirely from  $2y'$ .

**Step 2: Display the first four coefficients.** The requested coefficients are

$$a_0 + 2a_1, \quad 2a_1 + 4a_2, \quad 3a_2 + 6a_3, \quad 4a_3 + 8a_4.$$

As a direct constant-term check,  $H(2) = 2y'(2) + y(2) = 2a_1 + a_0$ . Keeping the same summation index through a differentiation would miss the shift from  $a_n$  to  $a_{n+1}$ .

**Step 3: Determine exactly when the shortcut works.** The claimed series actually represents  $hy' + y$ . Its difference from  $H$  is  $2y'$ . The identity therefore holds throughout the interval exactly when  $y' \equiv 0$ , or equivalently  $a_n = 0$  for every  $n \geq 1$ . One justification of the equivalence is uniqueness of power-series coefficients, obtained by successive differentiation at the center. Any constant  $a_0$  is allowed. Equality at a single isolated point would not imply this identity.

**Step 4: Check against a geometric closed form.** For  $a_n = 3^{-n-1}$ ,  $y = (1/3) \sum (h/3)^n = 1/(3-h) = 1/(5-x)$  when  $|h| < 3$ . Hence

$$H = \frac{x}{(5-x)^2} + \frac{1}{5-x} = \frac{5}{(5-x)^2}.$$

The coefficient formula gives  $(n+1)/3^{n+1} + 2(n+1)/3^{n+2} = 5(n+1)/3^{n+2}$ . This agrees with the differentiated geometric identity  $5/(3-h)^2 = \sum_{n \geq 0} 5(n+1)h^n/3^{n+2}$ . The radius is  $\boxed{3}$ ; the factors growing linearly in  $n$  do not change it.

**Question 8**

Consider a series of polynomials,

$$P(x) = \sum_{n=0}^{\infty} \left( \frac{x^2 - 4}{2} \right)^n.$$

It is geometric in  $(x^2 - 4)/2$ , but its displayed terms are not single monomials in  $x$ . Investigate how its convergence set compares with that of an ordinary power series representing the same sum.

**Tasks**

1. Determine the exact real convergence set of the displayed series, including all boundary tests. Explain why it need not be one interval centered at zero.
2. Sum the series wherever it converges. Compare the domain of that rational formula with the convergence set just found.
3. Starting afresh from the rational formula, derive an ordinary power series centered at zero. Find its radius and endpoint behavior.
4. Compare the two series at  $x = 0$  and  $x = 1$ , and state exactly where their sums are known to agree. Diagram both convergence sets and explain why one should not silently substitute the new domain for the original one.

## Question 8 – Solution

**Strategy.** Apply the geometric test to the substituted variable first; then derive and test a separate series in powers of  $x$ .

**Step 1: Pull back the geometric convergence condition.** The displayed series converges exactly when  $|(x^2 - 4)/2| < 1$ , or  $2 < x^2 < 6$ . Thus its convergence set is

$$(-\sqrt{6}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{6}).$$

At  $x = \pm\sqrt{2}$  the ratio is  $-1$ ; at  $x = \pm\sqrt{6}$  it is  $1$ . In each case the terms fail to approach zero. Elsewhere outside the two intervals the ratio has magnitude greater than one. These polynomial terms do not have the ordinary form  $a_n(x - c)^n$  for a fixed center.

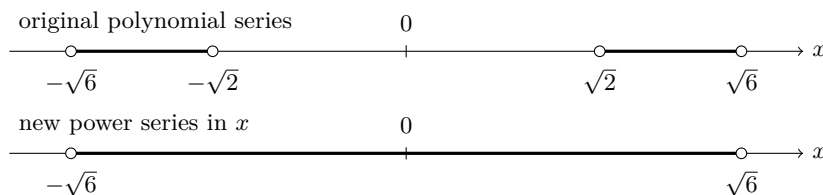
**Step 2: Sum on the justified set.** Geometric summation gives  $P(x) = 2/(6 - x^2)$  on the two intervals above. This rational formula is defined for every real  $x \neq \pm\sqrt{6}$ , which is a much larger set. Its being finite at other points says nothing by itself about convergence of the displayed polynomial series there.

**Step 3: Construct a distinct centered power series.** Factoring the denominator differently gives

$$\frac{2}{6 - x^2} = \frac{1/3}{1 - x^2/6} = \sum_{n=0}^{\infty} \frac{x^{2n}}{3 \cdot 6^n}, \quad |x| < \sqrt{6}.$$

Its odd coefficients vanish. Its radius is  $\sqrt{6}$  and both endpoints diverge because the terms there equal  $1/3$ . The new convergence interval is  $(-\sqrt{6}, \sqrt{6})$ .

**Step 4: Compare actual sums, not just formulas.** At  $x = 0$ , the original terms are  $(-2)^n$ , so the original series diverges; the new series sums to  $1/3$ . At  $x = 1$ , the original ratio is  $-3/2$  and its series diverges, while the new series sums to  $2/5$ . On  $\sqrt{2} < |x| < \sqrt{6}$ , both converge and both equal  $2/(6 - x^2)$ . The new representation adds valid points; it does not retroactively change the convergence set of the original sequence of partial sums.



**Question 9**

For a real parameter  $p$ , consider

$$F_p(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n n^p}.$$

Use this family to investigate which endpoint patterns can be designed by changing coefficient decay, and which require changing coefficient signs.

**Tasks**

1. Find the center and radius for every real  $p$ , including  $p \leq 0$ .
2. Give a complete classification of convergence and absolute convergence at both endpoints as  $p$  varies. Include the threshold cases  $p = 0$  and  $p = 1$ .
3. Determine all  $p$  yielding each of the intervals  $(-2, 4)$ ,  $[-2, 4)$  and  $[-2, 4]$ . Supply one concrete value for each pattern.
4. Can any member of this family have convergence interval  $(-2, 4]$ ? Prove your answer, then construct another power series centered at 1 with radius 3 and exactly that interval.

## Question 9 – Solution

**Strategy.** The exponential coefficient scale fixes the radius, while power-law decay and signs decide the endpoints.

**Step 1: Determine the radius independently of  $p$ .** For  $x \neq 1$ , the successive absolute-term ratio is

$$\frac{|x-1|}{3} \left( \frac{n}{n+1} \right)^p \rightarrow \frac{|x-1|}{3}.$$

Thus the center is 1 and  $\boxed{R=3}$  for every finite real  $p$ . At the center the sum is zero. The open interval  $(-2, 4)$  is always a region of absolute convergence, and outside the closed interval the series diverges.

**Step 2: Classify the endpoints.** At  $x = 4$ , the series is  $\sum 1/n^p$ , convergent exactly when  $p > 1$ . At  $x = -2$ , it is  $\sum (-1)^n/n^p$ : it converges for  $p > 0$ , since the magnitudes then decrease to zero, and fails the term test for  $p \leq 0$ . Absolute convergence at either endpoint occurs exactly for  $p > 1$ . Thus the left endpoint is conditional for  $0 < p \leq 1$ , including  $p = 1$ ; at  $p = 0$  neither endpoint's terms tend to zero.

**Step 3: List all parameter regimes.** Combining the tests gives

| parameter range | convergence interval | example |
|-----------------|----------------------|---------|
| $p \leq 0$      | $(-2, 4)$            | $p = 0$ |
| $0 < p \leq 1$  | $[-2, 4)$            | $p = 1$ |
| $p > 1$         | $[-2, 4]$            | $p = 2$ |

No other parameter cases remain. Changing  $p$  affects the boundary tests, not the radius or the open interval of absolute convergence.

**Step 4: Reverse the endpoint pattern by signs.** Within the given family, convergence at the right endpoint implies  $p > 1$ , which forces absolute convergence at the left endpoint too. Hence  $(-2, 4]$  is impossible in this family. A different series that works is

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n 3^n}}.$$

Its radius is 3. At  $x = 4$  it is alternating harmonic and converges conditionally; at  $x = -2$  it is the divergent positive harmonic series. This explicitly separates a restriction of the chosen family from a restriction on power series in general.

**Question 10**

An unknown real power series  $\sum_{n=0}^{\infty} a_n(x-2)^n$  converges at  $x = 5$  and diverges at  $x = -2$ . No coefficient formula is supplied. Use only what these two observations imply; boundary convergence may be conditional.

**Tasks**

1. Find the strongest possible bounds on its radius of convergence  $R$ . Explain why either inequality may be an equality.
2. Classify the behavior forced at  $x = 0, -1, 11/2, 6, 7$ : absolute convergence, divergence, or not determined by the information. Give reasons using distances from the center.
3. Construct two explicit series satisfying both observations that establish all the undetermined cases in your table. Verify the observations and the disputed points for each example.
4. Show every radius in your proposed closed range can occur. Use explicit examples for the endpoints and a general construction for the intermediate radii.

## Question 10 – Solution

**Strategy.** Convergence at a point controls smaller distances absolutely; divergence controls larger distances. Boundary tests cannot be transferred automatically to the opposite side.

**Step 1: Bound the radius sharply.** The convergent point is distance 3 from the center, so  $R \geq 3$ . The divergent point is distance 4, so  $R \leq 4$ : if  $R > 4$  that point would be inside the region of absolute convergence. Thus  $\boxed{3 \leq R \leq 4}$ . Convergence at distance 3 could be boundary convergence, and divergence at distance 4 could be boundary divergence.

**Step 2: Record only forced conclusions.**

| $x$  | $ x - 2 $ | conclusion           |
|------|-----------|----------------------|
| 0    | 2         | absolute convergence |
| -1   | 3         | not determined       |
| 11/2 | 7/2       | not determined       |
| 6    | 4         | not determined       |
| 7    | 5         | divergence           |

The absolute convergence at distance 2 follows from the convergent point at distance 3. The opposite point at distance 3 need not share an endpoint test, and distances between 3 and 4 are not settled by the bounds.

**Step 3: Construct examples proving the uncertainty.** Take, starting at  $n = 1$  and setting  $a_0 = 0$ ,

$$U(x) = \sum_{n \geq 1} \frac{(-1)^n (x - 2)^n}{n 3^n}, \quad V(x) = \sum_{n \geq 1} \frac{(-1)^n (x - 2)^n}{n 4^n}.$$

For  $U$ ,  $x = 5$  gives alternating harmonic convergence and  $x = -2$  fails the term test;  $R_U = 3$ . At  $x = -1$  it is positive harmonic and diverges, and at 11/2 and 6 its terms do not tend to zero. For  $V$ ,  $x = 5$  converges absolutely, while  $x = -2$  is positive harmonic and diverges;  $R_V = 4$ . At  $-1$  and 11/2 it converges absolutely, and at 6 it converges conditionally. These two examples resolve every “not determined” entry without changing the given observations.

**Step 4: Realize the entire radius range.** The examples above realize  $R = 3$  and  $R = 4$ . For any

$3 < \rho < 4$ , use  $\boxed{\sum_{n=0}^{\infty} ((x - 2)/\rho)^n}$ . Its radius is  $\rho$ ; at 5 its ratio has magnitude  $3/\rho < 1$ , and at  $-2$  it has magnitude  $4/\rho > 1$ . Thus every radius in  $[3, 4]$  occurs, and no narrower range follows from the two observations alone.