

Question 1

Compare the systems $X' = AX$ and $X' = BX$, where

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Both have the characteristic polynomial $(\lambda + 2)^2$.

Tasks

1. Find the eigenspace of each matrix and explain why the repeated characteristic root does not determine the number of independent eigenvectors.
2. Write the complete real solution of each system for initial state $(p, q)^T$. Verify any term containing t .
3. For the B system starting at $(0, 1)$, eliminate time with the correct domain. Find the maximum of $x(t)$ on $t \geq 0$.
4. Compare the two systems' phase trajectories and forward limits. Does every repeated eigenvalue require a polynomial factor in every solution?

Question 1 – Solution

Strategy. The eigenspace dimension decides whether a repeated root already supplies a full basis or needs another solution.

Step 1: Distinguish the eigenspaces. For A , every nonzero vector is a -2 eigenvector, so the eigenspace has dimension two. For B , $(B + 2I)(x, y)^T = (y, 0)^T$, so its eigenspace is $\text{span}(1, 0)^T$, of dimension one. Algebraic multiplicity is two in both cases; the geometric multiplicities differ.

Step 2: Recover both complete families. For initial (p, q) ,

$$X_A = e^{-2t} \begin{pmatrix} p \\ q \end{pmatrix}, \quad X_B = e^{-2t} \begin{pmatrix} p + qt \\ q \end{pmatrix}.$$

For B , first solve $y' = -2y$, then $(e^{2t}x)' = q$. This derives every solution, and direct differentiation verifies the qt term. In particular $te^{-2t}(1, 0)^T$ alone is not the missing solution; the accompanying $e^{-2t}(0, 1)^T$ is essential.

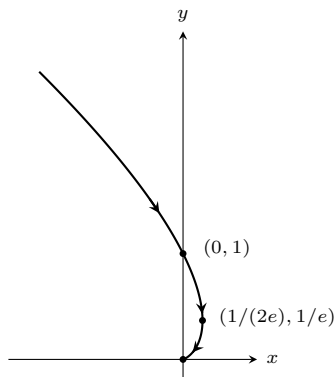
Step 3: Analyze the selected orbit. For B with $(p, q) = (0, 1)$, $x = te^{-2t}$ and $y = e^{-2t} > 0$. Eliminating t gives

$$x = -\frac{1}{2}y \ln y, \quad y > 0.$$

Forward time decreases y . On $t \geq 0$, $x' = e^{-2t}(1 - 2t)$ changes sign at $t = 1/2$, giving $x_{\max} = 1/(2e)$ at $y = e^{-1}$.

Step 4: Interpret multiplicity and geometry. Every nonzero A trajectory is a straight ray toward zero. For B , only the eigenline $y = 0$ gives straight rays; otherwise $x/y = p/q + t$ changes, giving curved trajectories. All solutions of both systems decay. A repeated root needs no polynomial factor when there is a full eigenbasis; even in the defective system, eigenline initial data have $q = 0$ and no t term. The figure shows the selected B orbit, not a generic ray.

$x = -\frac{1}{2}y \ln y$; arrows show increasing time



Question 2

Let

$$X' = NX, \quad N = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Both eigenvalues are zero. A matrix is nilpotent if some positive power of it is zero; investigate the effect on the actual motion.

Tasks

1. Compute N^2 , find the entire equilibrium set, and determine its dimension.
2. Find the normalized evolution matrix and the solution from arbitrary $(p, q)^T$. Verify the evolution law directly.
3. For initial data outside the equilibrium line, find the phase trajectory and its time direction. Can it reach that line at a finite or infinite forward time?
4. Classify all forward-bounded solutions. Decide whether the origin is stable in the sense that sufficiently small initial states remain small for all $t \geq 0$.

Question 2 – Solution

Strategy. A zero eigenvalue can preserve an equilibrium line while a nilpotent part creates unbounded drift parallel to it.

Step 1: Identify the nilpotent part and equilibria. Direct multiplication gives $N^2 = 0$ but $N \neq 0$. The equation $NX = 0$ requires $x = y$, so the equilibria form the one-dimensional line $L = \{(s, s) : s \in \mathbb{R}\}$. The double zero eigenvalue has only one independent eigendirection.

Step 2: Construct the complete flow. The matrix $F(t) = I + tN$ has $F(0) = I$ and $F' = N = NF$, since $N^2 = 0$. Thus

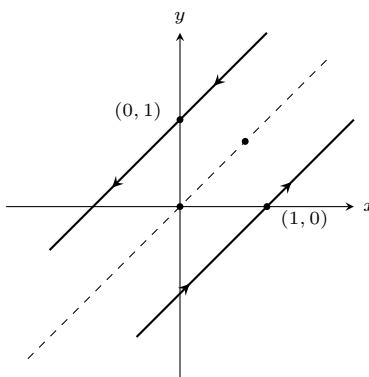
$$X(t) = \begin{pmatrix} p \\ q \end{pmatrix} + t(p - q) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Also $(I + tN)(I + sN) = I + (t + s)N$, verifying the evolution law and $F(t)^{-1} = F(-t)$. All real initial states and all real times are allowed.

Step 3: Find the drifting straight lines. Put $d = p - q$. If $d \neq 0$, then $x - y = d$ for every time and the velocity is the constant vector $d(1, 1)^T$. Thus the full orbit is the entire line $x - y = d$, traversed up/right for $d > 0$ and down/left for $d < 0$. Its distance to L stays $|d|/\sqrt{2} > 0$, so it neither reaches nor approaches the equilibrium line. If $d = 0$, the orbit is one fixed point, not the entire line L .

Step 4: Test boundedness and stability. Exactly $p = q$ gives a forward-bounded solution; all other solutions drift without bound. The origin is unstable: initial $(\varepsilon, 0)$ can be arbitrarily small, yet evolves to $(\varepsilon(1 + t), \varepsilon t)$, eventually leaving any fixed neighborhood. Zero real parts alone do not ensure boundedness or stability when a nonzero nilpotent part is present. The dashed line in the figure consists of separate equilibria.

Solid: drifting orbits; dashed: equilibrium line



Question 3

Consider

$$X' = AX, \quad A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For a repeated eigenvalue λ , a length-two chain means vectors v, w with $(A - \lambda I)v = 0$ and $(A - \lambda I)w = v \neq 0$.

Tasks

1. Find the eigenvalue, its eigenspace, and a chain with $v = (1, 1)^T$.
2. Derive and verify two independent real solutions using this chain. Explain why $te^{\lambda t}v$ alone fails.
3. Solve the IVP and verify the initial state and derivative.
4. Find every generalized vector w compatible with the fixed v . Explain how changing w changes the coefficients without changing the IVP solution.

Question 3 – Solution

Strategy. The generalized vector cancels the extra derivative introduced by the polynomial factor.

Step 1: Build a chain explicitly. The characteristic polynomial is $(\lambda+2)^2$. Let $N = A+2I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; then $N^2 = 0$. Its kernel is $\text{span}(1, 1)^T$. With $v = (1, 1)^T$, choose $w = (1, 0)^T$, since $Nw = v$. The two vectors are independent.

Step 2: Verify the missing solution. Set $U = e^{-2t}v$ and $V = e^{-2t}(w + tv)$. Using $Nv = 0$, $Nw = v$, differentiation gives $U' = AU$ and $V' = AV$. Their determinant is $e^{-4t} \det(v, w) = -e^{-4t} \neq 0$, so they form a complete real basis. For $Z = te^{-2t}v$, however, $Z' - AZ = e^{-2t}v \neq 0$: the polynomial term by itself is not a solution.

Step 3: Determine the IVP coefficients. Since $(2, 1)^T = v + w$, the required solution is $U + V$:

$$X = e^{-2t} \begin{pmatrix} 2+t \\ 1+t \end{pmatrix}.$$

At zero it is $(2, 1)^T$, and its derivative is $(-3, -1)^T$, agreeing with $A(2, 1)^T$. The mode calculation verifies the equation for every time, not just at the initial point.

Step 4: Account for chain freedom. The equation $Nw = v$ means $w_1 - w_2 = 1$, so every choice is $w_c = (1+c, c)^T = w + cv$, $c \in \mathbb{R}$. The new second solution is $V_c = V + cU$. Hence the same IVP becomes $X = (1-c)U + V_c$. In general a combination $aU + bV$ becomes $(a-bc)U + bV_c$. The basis changes, but its span and every physical solution remain the same. Scaling or shifting a chain must be accompanied by the corresponding coefficient change.

Question 4

For $\varepsilon > 0$, let

$$X'_\varepsilon = \begin{pmatrix} -1 & 1 \\ 0 & -1 - \varepsilon \end{pmatrix} X_\varepsilon, \quad X_\varepsilon(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Study the limit as the two distinct eigenvalues merge. Let X_0 denote the solution of the same IVP with $\varepsilon = 0$.

Tasks

1. Solve the IVP for $\varepsilon > 0$ by using the distinct real modes.
2. Find the limit at each fixed time as $\varepsilon \downarrow 0$ and verify the resulting repeated-root solution.
3. For $0 \leq t \leq T$, prove bounds on $|x_\varepsilon - x_0|$ and $|y_\varepsilon - y_0|$ that tend uniformly to zero as $\varepsilon \downarrow 0$. You may use $0 \leq 1 - e^{-u} \leq u$ for $u \geq 0$.
4. At $t = 1/\varepsilon$, compare x_ε/x_0 with 1. Explain how coefficients that diverge and a relative error that persists can coexist with the fixed-time limit.

Question 4 – Solution

Strategy. Keep the difference of nearby exponentials together; an integral representation exposes the limit without subtracting divergent pieces.

Step 1: Solve before the roots merge. Eigenvectors are $(1, 0)^T$ for -1 and $(1, -\varepsilon)^T$ for $-1 - \varepsilon$. Decomposition of $(0, 1)^T$ gives coefficients $1/\varepsilon, -1/\varepsilon$, hence

$$x_\varepsilon = e^{-t} \frac{1 - e^{-\varepsilon t}}{\varepsilon}, \quad y_\varepsilon = e^{-(1+\varepsilon)t}.$$

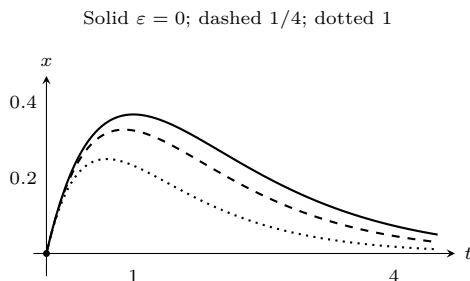
Step 2: Pass to the repeated-root solution. Since $(1 - e^{-\varepsilon t})/\varepsilon = \int_0^t e^{-\varepsilon s} ds$, its limit at each fixed real t is t . Thus $X_0 = e^{-t}(t, 1)^T$. Direct differentiation gives $x'_0 = -x_0 + y_0$, $y'_0 = -y_0$ and the stated initial data. The polynomial factor is the limit of the combined modes.

Step 3: Prove uniform finite-interval control. For $0 \leq t \leq T$, the supplied inequality yields

$$0 \leq x_0 - x_\varepsilon = e^{-t} \int_0^t (1 - e^{-\varepsilon s}) ds \leq \frac{1}{2} \varepsilon t^2 e^{-t} \leq \frac{1}{2} \varepsilon T^2.$$

Similarly $0 \leq y_0 - y_\varepsilon = e^{-t}(1 - e^{-\varepsilon t}) \leq \varepsilon t e^{-t} \leq \varepsilon T$. Both bounds tend uniformly to zero on every fixed finite interval.

Step 4: Separate relative and absolute effects. At $t = 1/\varepsilon$, $x_\varepsilon/x_0 = 1 - e^{-1}$, so the relative discrepancy is e^{-1} , independent of ε . This time moves to infinity, and $x_0 = \varepsilon^{-1} e^{-1/\varepsilon} \rightarrow 0$, so the absolute discrepancy still vanishes there. The separate modal coefficients diverge, but their exponential contributions cancel to a finite limit. Neither divergent coefficients nor this moving-time relative error contradicts the proved finite-interval convergence.



Question 5

Let

$$X' = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Examine the whole trajectory for $-\infty < t < \infty$, including its approach to the origin backward in time.

Tasks

1. Find the repeated eigenvalue and eigenspace, and solve the IVP.
2. Eliminate time and retain the full orbit domain. Find the unique minimum of $x(t)$ and all its finite zeros.
3. Find the limiting normalized directions as $t \rightarrow -\infty$ and $t \rightarrow \infty$. Does approaching an eigenline imply that the trajectory lies on it?
4. Determine whether this orbit ever reaches the origin or repeats a state. Explain why its coordinate reversal is not an oscillation.

Question 5 – Solution

Strategy. The exponential controls overall growth, while the polynomial factor can reverse one coordinate and dominate the limiting direction.

Step 1: Recover the defective solution. The double eigenvalue is 1, with eigenspace $y = 0$. Solving $y' = y$ and then $(e^{-t}x)' = 3$ gives

$$x = 3te^t, \quad y = e^t.$$

The factor 3 is part of the coupling and cannot be omitted from the generalized mode. Both components satisfy the stated initial data.

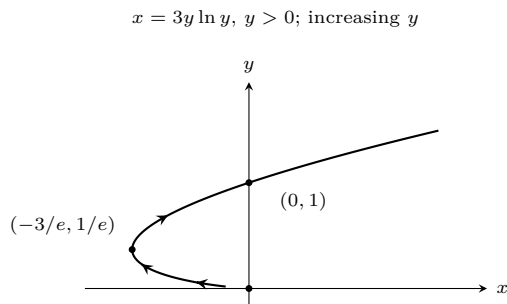
Step 2: Find the curve and minimum. Since $y > 0$, the orbit is $x = 3y \ln y, \quad y > 0$, traversed toward increasing y . Its derivative $x' = 3e^t(t+1)$ vanishes only at $t = -1$, changing from negative to positive. The global minimum is $x = -3/e$ at $y = 1/e$. The only finite zero of x is $t = 0$; its backward limit zero is not another attained zero.

Step 3: Find the two limiting directions. The normalized state is

$$\frac{X}{\|X\|} = \frac{(3t, 1)^T}{\sqrt{9t^2 + 1}}.$$

It tends to $(-1, 0)^T$ backward and $(1, 0)^T$ forward. Although the limiting line is the eigenline $y = 0$, every finite-time state has $y > 0$ and lies off that line. The origin is an unstable defective node; this IVP tends to it only as $t \rightarrow -\infty$.

Step 4: Distinguish reversal from repetition. The origin is never reached at finite time because $e^t > 0$. Also $y = e^t$ is strictly increasing, so two different times cannot give the same state. The single minimum of x comes from the factor te^t , not recurring oscillations. There is no periodic orbit or repeated cycle: a coordinate can reverse even with a positive repeated real eigenvalue. The equal-scale figure marks the minimum and the initial state.



Question 6

Let

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad X' = AX, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Use the Euclidean vector norm. For $N = A + I$, the operator bounds $\|NX\| \leq \|X\|$ and $\|N^2X\| \leq \|X\|$ may be used.

Tasks

1. Compute the powers of N needed to find the normalized evolution and the complete solution.
2. For initial $(0, 0, 1)^T$, find the maximum of each of the first two components on $t \geq 0$ and its time.
3. For arbitrary nonzero initial data, identify when the largest polynomial factor in the solution is quadratic, linear, or constant. Do all solutions decay?
4. Can one constant C satisfy $\|X(t)\| \leq Ce^{-t}\|X(0)\|$ for all $t \geq 0$ and all initial data? Prove instead a bound with decay $e^{-t/2}$ and an explicit constant.

Question 6 – Solution

Strategy. A longer nilpotent chain truncates the exponential after a quadratic term; its degree matters for sharp decay estimates.

Step 1: Truncate the nilpotent exponential. Here N^2 has only entry $(1, 3)$ equal to one; $N^3 = 0$. Direct differentiation verifies $F(t) = e^{-t}(I + tN + t^2N^2/2)$ and $F(0) = I$. Consequently

$$X = e^{-t} \begin{pmatrix} p + qt + rt^2/2 \\ q + rt \\ r \end{pmatrix}.$$

This gives all initial states and shows why a length-three chain needs $t^2/2$, rather than another unrelated eigenvector.

Step 2: Locate the component peaks. For $(p, q, r) = (0, 0, 1)$, the first components are $t^2e^{-t}/2$ and te^{-t} . Their derivatives are $e^{-t}t(2-t)/2$ and $e^{-t}(1-t)$. Thus their maxima are $2e^{-2}$ at $t = 2$ and e^{-1} at $t = 1$, respectively. These peaks are not simultaneous.

Step 3: Classify polynomial degrees. If $r \neq 0$, the leading vector factor is $(r/2)t^2(1, 0, 0)^T$. If $r = 0$ but $q \neq 0$, it is $qt(1, 0, 0)^T$. If $q = r = 0$ and $p \neq 0$, it is the constant $p(1, 0, 0)^T$. Every polynomial factor times e^{-t} tends to zero, so all solutions decay; algebraic multiplicity three does not force degree two for every state.

Step 4: Prove an honest exponential estimate. For initial $(0, 0, 1)$, $e^t\|X(t)\| \geq t^2/2$, ruling out the proposed constant C . The given operator bounds instead imply $\|X(t)\| \leq e^{-t}(1 + t + t^2/2)\|X(0)\|$. Using $e^{-t/2} \leq 1$, $te^{-t/2} \leq 2/e$ and $(t^2/2)e^{-t/2} \leq 8/e^2$ for $t \geq 0$ gives

$$\|X(t)\| \leq (1 + 2/e + 8/e^2)e^{-t/2}\|X(0)\|.$$

A slightly slower exponential absorbs the polynomial transient.

Question 7

An unknown constant real 2×2 matrix A has characteristic polynomial $(\lambda + 1)^2$. Its normalized evolution F is measured exactly at one time:

$$F(1) = e^{-1} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

You may use the Cayley–Hamilton identity $(A + I)^2 = 0$.

Tasks

1. Recover A from the measured matrix and prove uniqueness under the stated spectral assumption.
2. Find $F(t)$ for every real time and verify the evolution and inverse laws.
3. Solve the IVP $X(0) = (1, 0)^T$. Does its first component cross zero for $t \geq 0$?
4. Would measuring only $\det F(1) = e^{-2}$ recover the same matrix? Give a different admissible matrix and explain what information the full observation supplies.

Question 7 – Solution

Strategy. Under the repeated-root assumption the nilpotent exponential is linear in time, making the full one-step observation invertible algebraically.

Step 1: Recover the nilpotent part. Let $N = A + I$. Since $N^2 = 0$, direct verification gives $F(t) = e^{-t}(I + tN)$. Therefore

$$N = eF(1) - I = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}.$$

This N indeed squares to zero and is nonzero. Every matrix with the stated characteristic polynomial has the same formula, so the full measurement determines it uniquely within the specified class.

Step 2: Verify all-time evolution. The normalized evolution is

$$F(t) = e^{-t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}.$$

Its derivative equals AF and $F(0) = I$. Since $N^2 = 0$, multiplying $(I + tN)(I + sN)$ gives $I + (t + s)N$. Thus $F(t + s) = F(t)F(s)$ and $F(t)^{-1} = F(-t)$ for all real times.

Step 3: Inspect the recovered IVP. Selecting the first column gives $X = e^{-t}(1 + t, -t)^T$. At zero the state is $(1, 0)^T$ and the derivative is $(0, -1)^T = A(1, 0)^T$. For $t \geq 0$, the first component stays strictly positive because $1 + t > 0$, though it tends to zero. Its only finite zero on the entire real line is $t = -1$, outside the requested forward interval.

Step 4: Separate area data from the full map. The different matrix $\tilde{A} = -I$ has the same repeated eigenvalue and $\det \tilde{F}(1) = e^{-2}$, but $\tilde{F}(1) = e^{-1}I$. Indeed $\det(I + tN) = 1$, so the nilpotent shear is invisible to this area multiplier. The full observed matrix supplies directional and coupling information, which the determinant alone discards. Uniqueness above uses both the spectral assumption and the complete matrix observation.

Question 8

For $k \geq 0$, consider

$$X' = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix} X.$$

Investigate Euclidean monotonicity and construct an alternative quadratic size when the standard squared norm $x^2 + y^2$ is unsuitable.

Tasks

1. Find the complete solution and classify the eigenspace dimension as k varies. Prove forward decay for all initial data.
2. Classify all k for which $(x^2 + y^2)'$ is negative at every nonzero state, nonpositive with nonzero equality states, or positive at some state.
3. At the boundary parameter between those cases, decide whether the norm strictly decreases between any two distinct times along every nonzero solution. Address the instantaneous equality line.
4. Choose a positive coefficient $c = c(k)$ so that $H = x^2 + cy^2$ has strictly negative derivative at every nonzero state for every $k \geq 0$. Give an explicit choice and proof.

Question 8 – Solution

Strategy. The nilpotent coupling does not change the eigenvalues, but it can change which quadratic size decreases at every state.

Step 1: Solve for every coupling strength. The repeated eigenvalue is -1 . If $k = 0$, the eigenspace is the whole plane; if $k > 0$, it is $y = 0$. For initial (p, q) ,

$$X = e^{-t}(p + kqt, q)^T.$$

The component equations directly verify completeness. Constant and linear factors times e^{-t} both vanish forward, so every initial state decays.

Step 2: Locate the exact norm threshold. The squared-norm derivative is

$$(x^2 + y^2)' = -2x^2 + 2kxy - 2y^2.$$

For $0 \leq k < 2$, $2|xy| \leq x^2 + y^2$ makes this strictly negative off zero. At $k = 2$ it equals $-2(x - y)^2$, with equality on $x = y$. For $k > 2$, its value at $(1, 1)$ is $2k - 4 > 0$. Thus the threshold is $k = 2$, despite identical eigenvalues throughout the family.

Step 3: Distinguish instantaneous equality from an interval. At $k = 2$, a trajectory cannot stay on $x = y$ over an interval unless it is zero. On that line the velocity is $(x, -x)$, which is tangent to the line only when $x = 0$. Integrating $-2(x - y)^2$ over any interval therefore gives strictly negative change for every nonzero solution. The norm strictly decreases between distinct times even if its derivative vanishes at an isolated time.

Step 4: Construct a uniformly valid quadratic choice. Choose $c = 1 + k^2/2 > 0$. Completing the square yields

$$H' = -2(x - ky/2)^2 - (2 + k^2/2)y^2 < 0 \quad (X \neq 0).$$

The quadratic $H = x^2 + (1 + k^2/2)y^2$ is positive definite for every k , including zero. Its decrease is compatible with temporary Euclidean growth when $k > 2$; the two quantities measure different families of ellipses. The spectral decay is unchanged by this choice of measurement.

Question 9

The critically damped equation

$$x'' + 2x' + x = 0, \quad x(0) = 1, \quad x'(0) = -a, \quad a \geq 0$$

has a repeated negative characteristic root. A student claims that critical damping prevents the displacement from crossing its equilibrium for every choice of initial velocity.

Tasks

1. Write the equation as a state system with $y = x'$, find its eigenspace, and solve the stated IVP.
2. Classify exactly the values of a for which $x(t)$ crosses zero at a positive time. Give that time when it exists.
3. For the crossing case, find the subsequent minimum displacement and its time. For the other cases, describe forward monotonicity.
4. If the first displacement crossing is observed at a prescribed $T > 0$, recover the initial velocity. Explain why a single crossing does not imply oscillatory motion.

Question 9 – Solution

Strategy. The linear polynomial multiplying the repeated-root exponential determines whether the initial velocity produces an overshoot.

Step 1: Find the defective state solution. The state matrix is $\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$, with double eigenvalue -1 and eigenspace $\text{span}(1, -1)^T$. The initial data give

$$x = e^{-t}[1 + (1-a)t], \quad y = e^{-t}[-a - (1-a)t].$$

These formulas verify $x' = y$, $y' = -x - 2y$, and $(x(0), y(0)) = (1, -a)$.

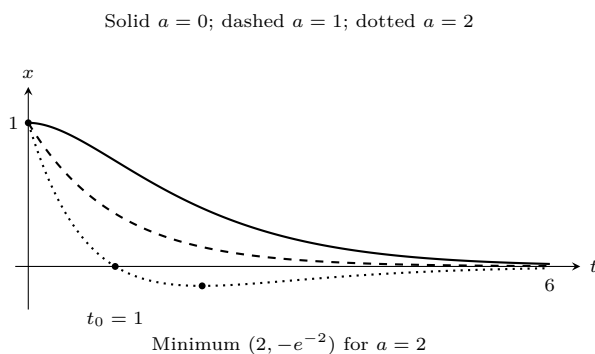
Step 2: Classify positive-time crossings. Because $e^{-t} > 0$, a positive-time zero occurs exactly when $a > 1$. It is unique and equals $t_0 = 1/(a-1)$. The polynomial has negative slope, so this is a crossing from positive to negative displacement. For $0 \leq a \leq 1$, the displacement remains positive at every finite forward time. Thus critical damping alone does not prevent overshoot.

Step 3: Locate the minimum or prove monotonicity. If $a > 1$, $y = e^{-t}[(a-1)t - a]$ changes from negative to positive at

$$t_{\min} = \frac{a}{a-1} = t_0 + 1, \quad x_{\min} = -(a-1)e^{-a/(a-1)}.$$

The displacement then increases toward zero from below. For $0 \leq a \leq 1$, $y < 0$ for all $t > 0$, so the positive displacement strictly decreases to zero; when $a = 0$, only the initial derivative vanishes.

Step 4: Infer the velocity from the crossing. Solving $T = 1/(a-1)$ gives $x'(0) = -a = -(1 + 1/T)$. The solution has only one zero because its exponential factor never vanishes and its polynomial factor is linear. After the single minimum it approaches zero from below without further crossings. This is an initial-data-dependent overshoot, not a repeating oscillation. The time plot compares $a = 0, 1, 2$; only the $a = 2$ curve crosses in those examples.



Question 10

For $\varepsilon \geq 0$, consider

$$A_\varepsilon = \begin{pmatrix} -1 & \varepsilon & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad X' = A_\varepsilon X, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A matrix is diagonalizable if it has a basis of eigenvectors. For nonzero solutions, compare their normalized directions $X/\|X\|$.

Tasks

1. Find the eigenvalues and eigenspaces. For which ε is the matrix diagonalizable?
2. Find the complete solution for arbitrary initial data, including the polynomial term when present.
3. Identify all forward-bounded initial states. Within that set, classify the limiting forward directions for every nonzero initial state.
4. For initial $(0, 1, 0)^T$, compare the iterated limits of $X_\varepsilon/\|X_\varepsilon\|$ as $t \rightarrow \infty$ and $\varepsilon \downarrow 0$, in both orders. Explain what changes even though the eigenvalues do not.

Question 10 – Solution

Strategy. A repeated stable eigenvalue can have a two-dimensional eigenspace or a defective chain; the limiting direction detects that difference.

Step 1: Determine the eigenspaces. The eigenvalues are -1 (multiplicity two) and 2 . For -1 , $z = 0$ and $\varepsilon y = 0$. Thus its eigenspace is the whole plane $z = 0$ when $\varepsilon = 0$, but only the x -axis when $\varepsilon > 0$. The 2 eigenspace is always $\text{span}(1, 0, 1)^T$. There is a full eigenbasis exactly when $\boxed{\varepsilon = 0}$; this diagonalizable three-dimensional matrix is not a scalar matrix.

Step 2: Solve the coupled components. First $z = re^{2t}$ and $y = qe^{-t}$. Then $(e^t x)' = \varepsilon q + 3re^{3t}$, giving

$$X = e^{-t} \begin{pmatrix} p - r + \varepsilon q t \\ q \\ 0 \end{pmatrix} + re^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

This satisfies every component and recovers $(p, q, r)^T$ at zero.

Step 3: Classify the stable-plane directions. Forward boundedness holds exactly when $r = 0$. On that plane all states decay. For $\varepsilon = 0$, the nonzero direction stays constantly $(p, q, 0)^T / \sqrt{p^2 + q^2}$. For $\varepsilon > 0$ and $q \neq 0$, the normalized direction tends to $(\text{sgn } q, 0, 0)^T$. If $q = 0$, $p \neq 0$, it is constantly $(\text{sgn } p, 0, 0)^T$. The zero solution has no normalized direction.

Step 4: Compare the two orders of limits. For the specified initial state, the normalized vector is $(\varepsilon t, 1, 0)^T / \sqrt{\varepsilon^2 t^2 + 1}$. Taking $t \rightarrow \infty$ first with $\varepsilon > 0$, then $\varepsilon \downarrow 0$, gives $\boxed{(1, 0, 0)^T}$. Reversing the operations gives $\boxed{(0, 1, 0)^T}$. At every fixed finite time the solution depends continuously on ε , but long-time direction is not uniform near zero. The eigenvalues stay fixed while the eigenspace dimension changes.