

Question 1

Let

$$X' = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Use a complex eigenvector to construct real solutions. Angles in the original (x, y) plane are measured counterclockwise from the positive x -axis.

Tasks

1. Find the eigenvalues and an eigenvector for the eigenvalue with positive imaginary part. Separate its exponential solution into real and imaginary parts.
2. Prove that the resulting two real solutions are independent and solve the IVP.
3. Find the radius, an unwrapped angle, and the first positive time at which this trajectory reaches the positive x -axis.
4. Find the radius multiplier after one full revolution. Is the nonzero solution periodic? Explain why the conjugate eigenpair does not supply two additional independent real solutions.

Question 1 – Solution

Strategy. Take real and imaginary parts with their signs intact, then use polar coordinates to separate decay from rotation.

Step 1: Extract real solutions. The eigenvalues are $-1 \pm 2i$. For $\lambda = -1 + 2i$, choose $v = (1, -i)^T$, which satisfies $Av = \lambda v$. Writing $e^{\lambda t}v = U + iV$ gives

$$U = e^{-t} \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}, \quad V = e^{-t} \begin{pmatrix} \sin 2t \\ -\cos 2t \end{pmatrix}.$$

Because A is real, both parts solve the original real system.

Step 2: Resolve the initial state. The determinant of the columns (U, V) is $-e^{-2t} \neq 0$. At zero they are $(1, 0)^T$ and $(0, -1)^T$, so $X = U - 2V$:

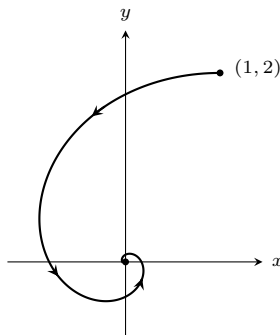
$$X = e^{-t} \begin{pmatrix} \cos 2t - 2 \sin 2t \\ \sin 2t + 2 \cos 2t \end{pmatrix}.$$

The nonzero determinant and linear-system uniqueness give the complete real family as arbitrary real combinations of U, V .

Step 3: Locate the first positive-axis crossing. Let $\theta_0 = \arctan 2$, which lies in $(0, \pi/2)$. Here $r = \sqrt{5}e^{-t}$ and a continuous angle is $\theta = \theta_0 + 2t$. Rotation is counterclockwise. The first positive time on the positive x -axis requires $\theta = 2\pi$, hence $t_* = (2\pi - \arctan 2)/2$. The earlier negative-axis crossing is not the requested event.

Step 4: Separate revolution from repetition. One revolution takes π , reducing the radius by $e^{-\pi}$. Since the radius strictly decreases, no nonzero solution is periodic; the origin is a stable spiral. Conjugating the exponential solution gives $U - iV$, whose parts are $U, -V$, already in the same real span. Thus a conjugate pair supplies two real degrees of freedom, not four. The figure shows a finite forward segment of the stated IVP.

Counterclockwise decay; $r = \sqrt{5}e^{-t}$



Question 2

Consider

$$x' = 4y, \quad y' = -x, \quad (x(0), y(0)) = (2, 0).$$

A student sees eigenvalues $\pm 2i$ and claims that the trajectory must be a circle rotating with constant Euclidean angular speed 2.

Tasks

1. Find the eigenvalues and solve the IVP in real form.
2. Find a conserved positive quadratic expression and determine the complete phase curve. Locate its intercepts.
3. Find the least positive period and the direction of traversal. Identify coordinates in which the motion is a uniform circular rotation.
4. Compute the original Euclidean angular velocity $\theta' = (xy' - yx')/(x^2 + y^2)$. Find its range on the orbit and evaluate the student's claim.

Question 2 – Solution

Strategy. A linear coordinate change can turn a circle into an ellipse while preserving elapsed time and period.

Step 1: Solve the oscillatory IVP. The characteristic polynomial is $\lambda^2 + 4$, giving $\pm 2i$. From $x'' = -4x$ and $x'(0) = 0$,

$$x = 2 \cos 2t, \quad y = -\sin 2t.$$

Differentiation verifies both first-order equations and the initial data.

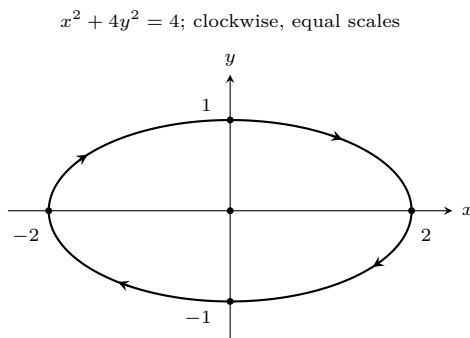
Step 2: Recover the invariant ellipse. The derivative of $H = x^2 + 4y^2$ is $8xy - 8xy = 0$. Thus the orbit is $x^2 + 4y^2 = 4$, with intercepts $(\pm 2, 0)$ and $(0, \pm 1)$. The parametrization covers the whole ellipse. Its unequal semiaxes must remain unequal in an equal-scale drawing; it is not a circle in the original Euclidean coordinates.

Step 3: Determine period and orientation. The least positive period is π . At $(2, 0)$ the velocity is $(0, -2)$, so traversal is clockwise. The coordinates $u = x$, $v = 2y$ satisfy $u' = 2v$, $v' = -2u$ and $u^2 + v^2 = 4$. In that transformed plane the angular velocity is constantly -2 . The invertible transformation preserves the least period.

Step 4: Compute the actual angular speed. In the original plane,

$$\theta' = -\frac{x^2 + 4y^2}{x^2 + y^2} = -\frac{4}{x^2 + y^2}.$$

On the ellipse, $1 \leq x^2 + y^2 \leq 4$, so $-4 \leq \theta' \leq -1$. It equals -1 at the horizontal intercepts and -4 at the vertical ones. Thus imaginary eigenvalues determine a temporal frequency, but do not force a Euclidean circle or constant Euclidean angular speed. The average signed angular velocity over a complete traversal is nevertheless -2 .



Question 3

Let

$$X' = \begin{pmatrix} -1 & 8 \\ -1/2 & -1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Use $\|X\| = \sqrt{x^2 + y^2}$. A stable spiral means that all nearby states approach the origin with continuing rotation, not necessarily that their ordinary distance decreases at every instant.

Tasks

1. Find the eigenvalues and solve the IVP. Determine its rotation direction.
2. Evaluate the norm at $t = \pi/4$ and compare it with its initial value. Compute $(\|X\|^2)'$ at a general state.
3. Find a positive quadratic expression $H = x^2 + cy^2$ with $H' = -2H$. Use it to establish decay of every solution.
4. For the given IVP and $0 < \rho < 1$, give an explicit time after which the state is guaranteed to remain in $\|X\| \leq \rho$. Explain why this guarantee need not be the first entry time.

Question 3 – Solution

Strategy. Use coordinates that balance the oscillatory coupling, then compare the adapted quadratic size with Euclidean distance.

Step 1: Solve the damped rotation. The characteristic polynomial is $(\lambda + 1)^2 + 4$, so the rates are $-1 \pm 2i$. Direct substitution verifies

$$x = 4e^{-t} \sin 2t, \quad y = e^{-t} \cos 2t.$$

For any nonzero state, $xy' - yx' = -x^2/2 - 8y^2 < 0$; rotation is clockwise. The transformed coordinates $(x, 4y)$ rotate uniformly with decay.

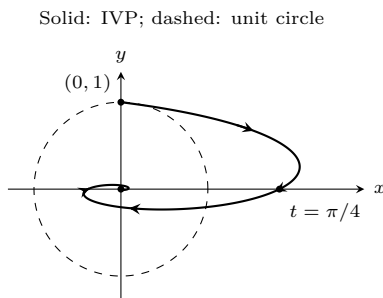
Step 2: Exhibit temporary growth. Initially the norm is one, whereas $\|X(\pi/4)\| = 4e^{-\pi/4} > 1$. More generally,

$$(\|X\|^2)' = -2x^2 + 15xy - 2y^2,$$

which equals $11 > 0$ at $(1, 1)$. Its value at the given initial state is -2 . Hence even this trajectory can first shrink and subsequently exceed its initial distance. The negative real parts guarantee eventual decay, not a sign for this derivative at every state.

Step 3: Find a decreasing quadratic size. Differentiating $x^2 + cy^2$ cancels the mixed term when $c = 16$. Then $H = x^2 + 16y^2$, $H' = -2H$, so $H(t) = H(0)e^{-2t}$. Since $\|X\|^2 \leq H$, every initial state tends to zero. Together with the strict rotation sign this identifies a stable spiral, even though the drawn orbit can move outside the unit circle.

Step 4: Give a permanent-entry bound. For the specified data, $H(0) = 16$ and $\|X(t)\| \leq 4e^{-t}$. Thus every $t \geq \ln(4/\rho)$ satisfies $\|X(t)\| \leq \rho$. This bound controls all later times. It need not be the first entry: H overestimates the ordinary squared norm except where $y = 0$. Transient contraction and expansion can occur before the guaranteed time; no claim of a sharp first-entry time is made.



Question 4

For $a \in \mathbb{R}$, consider

$$X' = \begin{pmatrix} a & -3 \\ 3 & a \end{pmatrix} X.$$

For $r > 0$, define the positive-axis return map $P_a(r)$ as the radius the solution starting at $(r, 0)$ has on its first later return to the positive x -axis. A return to this ray need not be a return to the same point.

Tasks

1. Find the eigenvalues and the solution from an arbitrary nonzero initial state in polar form.
2. Classify the origin for $a < 0$, $a = 0$, and $a > 0$. Determine exactly when a nonzero solution is periodic.
3. Find the first return time, the map P_a , and all its fixed radii in $r > 0$. Explain their relation to periodic trajectories.
4. An experiment finds $P_a(r) = r/2$ for one nonzero radius. Recover a , and find the radius after n complete turns for every integer $n \geq 0$.

Question 4 – Solution

Strategy. The real part sets radial growth and the imaginary part sets turning time; a ray return becomes periodic only when the radius is restored.

Step 1: Read radial and angular evolution. The eigenvalues are $a \pm 3i$. For initial radius $r_0 > 0$ and angle θ_0 , the solution has

$$r(t) = r_0 e^{at}, \quad \theta(t) = \theta_0 + 3t.$$

Equivalently $X(t) = e^{at} R(3t) X(0)$, where $R(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$. The only equilibrium is zero because the determinant is $a^2 + 9 > 0$.

Step 2: Classify all parameter cases. For $a < 0$ the origin is a stable spiral; for $a > 0$ it is an unstable spiral. At $a = 0$ it is a center: every nonzero state follows a circle with least period $2\pi/3$. For $a \neq 0$, the radius is strictly monotone, so no nonzero solution is periodic. Zero remains constant and has no least positive period.

Step 3: Compute the return map. The first return to the positive ray takes $T = 2\pi/3$, independent of r and a . Consequently

$$P_a(r) = e^{2\pi a/3} r.$$

For $r > 0$, the fixed-point equation holds for every radius if $a = 0$ and for none if $a \neq 0$. These fixed radii are exactly the nonzero periodic trajectories. Returning to an angle alone is insufficient.

Step 4: Recover damping from one revolution. The measurement gives $e^{2\pi a/3} = 1/2$, hence

$$a = -\frac{3 \ln 2}{2\pi}, \quad r_n = r_0 2^{-n}.$$

The coefficient a is unique because the real exponential is one-to-one. The same multiplier applies to every radius by linearity; a single nonzero measurement therefore determines the damping rate in this specified family.

Question 5

An unknown real matrix A has eigenvalues $-1 \pm 3i$. Every solution of $X' = AX$ satisfies

$$\frac{d}{dt}(x^2 + 4y^2) = -2(x^2 + 4y^2).$$

At the state $(2, 0)$, the velocity has a negative second component. Use all three observations to reconstruct the system.

Tasks

1. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and extract the restrictions imposed by the quadratic identity at arbitrary states.
2. Use the eigenvalues to find every matrix consistent with that identity before applying the direction observation.
3. Select the correct matrix and solve the IVP starting at $(2, 0)$. Verify its quadratic size and rotation direction.
4. Find the time and contraction factor for one complete turn. Explain precisely which ambiguity would remain if the direction observation were omitted.

Question 5 – Solution

Strategy. An identity valid for all initial states determines quadratic coefficients; the remaining sign is geometric orientation.

Step 1: Compare polynomial coefficients. Differentiation gives $2ax^2 + 2(b+4c)xy + 8dy^2 = -2x^2 - 8y^2$. Every point can be an initial state, so coefficients must agree:

$$a = d = -1, \quad b = -4c.$$

Step 2: Use the spectral determinant. The eigenvalues require trace -2 and determinant 10 . The trace is already correct, while $\det A = 1 - bc = 1 + 4c^2 = 10$ gives $c = \pm 3/2$. Thus precisely two matrices remain:

$$\begin{pmatrix} -1 & 6 \\ -3/2 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -6 \\ 3/2 & -1 \end{pmatrix}.$$

Both have the required quadratic decay and the required eigenvalues.

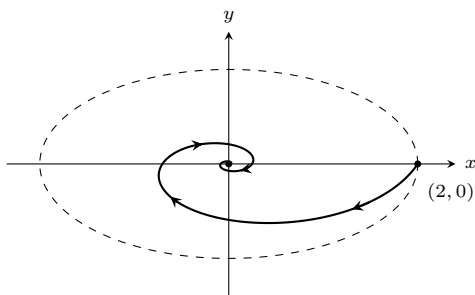
Step 3: Use orientation and solve. At $(2, 0)$, $y' = 2c < 0$, so select $c = -3/2$, $b = 6$. The IVP is

$$x = 2e^{-t} \cos 3t, \quad y = -e^{-t} \sin 3t.$$

It satisfies the selected equations, initial data and $x^2 + 4y^2 = 4e^{-2t}$. Moreover $xy' - yx' = -3x^2/2 - 6y^2 < 0$ off zero, confirming clockwise rotation. The shrinking ellipses are size contours, not individual orbits.

Step 4: Separate timing, contraction and sign. The coordinates $(x, 2y)$ make one clockwise revolution in $2\pi/3$. An invertible positive diagonal rescaling preserves a full turn, so this is also the time to return to the initial positive ray. The whole state then equals $e^{-2\pi/3}X(0)$; this is not a periodic return. Without the direction observation, the second matrix would remain equally possible, with counterclockwise rotation and the same contraction factor. The unordered conjugate eigenvalues do not determine orientation.

Clockwise spiral; dashed initial-size contour



Question 6

Let

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 0 \end{pmatrix}, \quad B = A - I.$$

Find the normalized evolution matrix for $X' = AX$ without selecting or normalizing complex eigenvectors. A normalized evolution matrix F satisfies $F' = AF$ and $F(0) = I$.

Tasks

1. Find the eigenvalues and prove a simple identity for B^2 .
2. Construct a real formula for $F(t)$ using that identity, and verify both defining conditions.
3. Solve the IVP $X(0) = (1, 1)^T$. Compute $F(\pi)$ and explain why a complete oscillation does not produce a periodic nonzero solution.
4. Prove the multiplication law $F(t + s) = F(t)F(s)$, find $F(t)^{-1}$, and compute its determinant. Does the determinant alone determine how every vector length changes?

Question 6 – Solution

Strategy. Remove the real part of the spectrum; the remaining matrix acts algebraically like multiplication by an imaginary number.

Step 1: Isolate the oscillatory matrix. The characteristic polynomial is $\lambda^2 - 2\lambda + 5$, with roots $1 \pm 2i$. Here $B = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}$ and direct multiplication gives $B^2 = -4I$.

Step 2: Construct a real evolution formula. Set

$$F(t) = e^t \left(I \cos 2t + \frac{1}{2} B \sin 2t \right).$$

Its value at zero is I . Differentiating the bracket gives $-2I \sin 2t + B \cos 2t$, which equals B times that bracket because $B^2 = -4I$. The product rule therefore yields $F' = (I + B)F = AF$. Uniqueness verifies that this formula evolves every initial state.

Step 3: Solve and inspect one oscillation. Since $B(1, 1)^T = (-4, 0)^T$,

$$X = e^t \begin{pmatrix} \cos 2t - 2 \sin 2t \\ \cos 2t \end{pmatrix}.$$

The initial state is $(1, 1)^T$ and initial derivative $(-3, 1)^T = A(1, 1)^T$. Also $F(\pi) = e^\pi I$: a full trigonometric cycle multiplies the state by e^π . More generally the periodic bracket and its inverse are bounded, so every nonzero solution grows without bound forward in time. Thus there is no nonzero periodic solution.

Step 4: Verify the evolution algebra. Multiplying two brackets and using $B^2 = -4I$ gives the sine and cosine addition formulas, hence $F(t + s) = F(t)F(s)$. Therefore $F(t)^{-1} = F(-t)$. For scalars c, s , $\det(cI + sB/2) = c^2 + s^2$, so $\det F(t) = e^{2t}$. This controls area, not every vector length: at the state $(1, 1)$ the squared-length derivative is $2(1, 1) \cdot (-3, 1) = -4 < 0$, despite increasing area. Shape distortion and eventual growth can coexist.

Question 7

A damped oscillator is written as a state system

$$x' = y, \quad y' = -5x - 2y, \quad (x(0), y(0)) = (1, 0).$$

Here x is displacement and y is velocity. A turning point means a time when the displacement has a local maximum or minimum.

Tasks

1. Find the complex eigenvalues and the real IVP solution.
2. Find the first positive zero of the displacement and the velocity there, retaining its sign.
3. Find the first positive turning time and classify that turning point. Determine the ratio of the magnitudes of consecutive displacement extrema.
4. Compute the derivative of the mechanical energy $E = (y^2 + 5x^2)/2$. Explain why $E'(0) = 0$ does not make the motion constant or periodic.

Question 7 – Solution

Strategy. The state components have different phases: displacement zeros and velocity zeros mark different events on the spiral.

Step 1: Solve the state IVP. The characteristic polynomial is $\lambda^2 + 2\lambda + 5$, giving $-1 \pm 2i$. The initial conditions yield

$$x = e^{-t}(\cos 2t + \tfrac{1}{2} \sin 2t), \quad y = -\tfrac{5}{2}e^{-t} \sin 2t.$$

Differentiation gives $x' = y$ and $y' = -5x - 2y$, including $y'(0) = -5$.

Step 2: Locate the first displacement crossing. Write $\alpha = \arctan 2 \in (0, \pi/2)$. The first positive root of $\cos 2t + \frac{1}{2} \sin 2t = 0$ occurs at

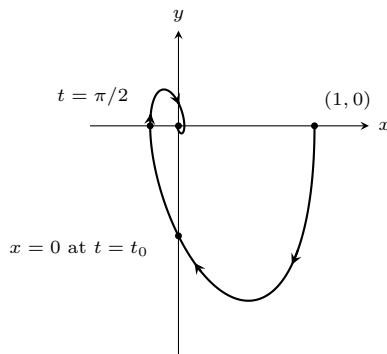
$$t_0 = (\pi - \alpha)/2, \quad y(t_0) = -\sqrt{5}e^{-t_0}.$$

Indeed $\sin(\pi - \alpha) = 2/\sqrt{5}$ and $\cos(\pi - \alpha) = -1/\sqrt{5}$. The negative velocity shows a crossing from positive to negative displacement, not a turning point.

Step 3: Separate the turning events. Velocity vanishes exactly at $t = n\pi/2$, for integers n . The first positive turning time is $\pi/2$, with displacement $-e^{-\pi/2}$. There $x'' = y' = 5e^{-\pi/2} > 0$, so it is a minimum. Successive extrema have $x(n\pi/2) = (-1)^n e^{-n\pi/2}$; their absolute values have ratio $e^{-\pi/2}$ for consecutive forward extrema. Successive positive maxima are one full oscillation apart and have ratio $e^{-\pi}$.

Step 4: Interpret dissipation at a zero derivative. The energy derivative is $E' = yy' + 5xx' = -2y^2 \leq 0$. At zero, $y = 0$ so $E' = 0$, but the acceleration is -5 and the state immediately changes. On any nontrivial time interval this solution has $y \neq 0$ somewhere, so its energy strictly drops over that interval. A nonconstant periodic solution would instead require zero total energy loss over a period, forcing $y \equiv 0$ and then $x \equiv 0$, a contradiction. The phase arrows distinguish the crossing and turning points.

Displacement–velocity phase curve



Question 8

For a nonzero real frequency ω , consider

$$X' = \begin{pmatrix} -1 & -\omega \\ \omega & -1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

An instrument records only selected times. At every integer $n \geq 0$ it reports $X(n) = e^{-n}(1, 0)^T$. Assume the measurements are exact.

Tasks

1. Solve the IVP and determine every nonzero frequency consistent with all the integer-time measurements.
2. Can these measurements determine rotation direction or the number of revolutions per unit time? Explain why the observed points do not imply a straight-line continuous trajectory.
3. An additional measurement is $X(1/4) = e^{-1/4}(0, 1)^T$. Find every frequency still allowed, including negative frequencies.
4. If one also knows $0 < \omega < 4\pi$, recover the frequency uniquely. Exhibit two different frequencies that fit both sets of measurements when that extra bound is absent.

Question 8 – Solution

Strategy. Sampling observes angles only modulo complete revolutions; additional information is needed to recover the continuous frequency.

Step 1: Solve and impose integer samples. The eigenvalues are $-1 \pm i\omega$, and

$$X(t) = e^{-t} \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix}.$$

The measurement at $n = 1$ requires $\omega = 2\pi k$, $k \in \mathbb{Z}$. Every such value satisfies all integer samples; the nonzero-frequency assumption excludes $k = 0$.

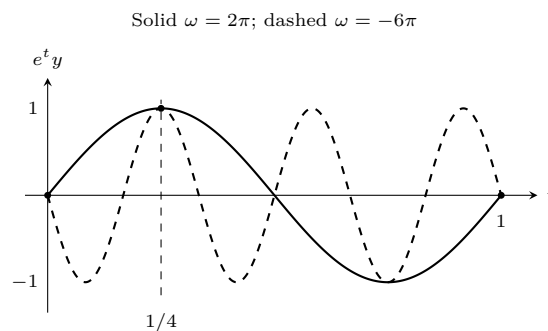
Step 2: Identify what the samples conceal. The sign of k gives counterclockwise or clockwise rotation, and $|k|$ gives revolutions per unit time. Neither is fixed by the integer samples. For every permitted nonzero frequency the continuous angle changes and the radius decays, tracing a spiral. The sampled points all lie on one ray because whole turns occur between them, not because that ray is an invariant trajectory.

Step 3: Use the quarter-time observation. The additional angle condition is $\omega/4 = \pi/2 + 2\pi m$. Thus

$$\omega = 2\pi(1 + 4m), \quad m \in \mathbb{Z}.$$

These values already satisfy the integer constraints. Negative possibilities remain, for example $m = -1$ gives $\omega = -6\pi$. No zero frequency occurs in this family.

Step 4: Use the external frequency bound. Under $0 < \omega < 4\pi$, only $m = 0$ remains, giving $\omega = 2\pi$. Without that bound, 2π and -6π both fit all stated observations, despite opposite directions and different speeds. The figure plots $e^t y(t)$ for these two solutions: they agree at the recorded times but differ between them. The vertical variable is the rescaled component, not the radius or a phase-plane coordinate.



Question 9

Consider

$$X' = \begin{pmatrix} -1 & -2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Use Euclidean boundedness on $[0, \infty)$ or $(-\infty, 0]$ as appropriate. For a nonzero state its direction means the normalized vector $X/\|X\|$.

Tasks

1. Find the eigenvalues and the complete real solution for arbitrary initial data.
2. Classify the initial states giving bounded solutions forward in time, backward in time, or for all real time.
3. When $r \neq 0$, determine the limiting direction as $t \rightarrow \infty$ and justify why the rotating components do not affect it.
4. When $r = 0$ but $(p, q) \neq (0, 0)$, determine the state limit and decide whether its direction has a limit. Give a sequence-based justification.

Question 9 – Solution

Strategy. Decouple the rotating plane from the real exponential axis, and distinguish convergence of a state from convergence of its direction.

Step 1: Separate the blocks. The eigenvalues are $-1 \pm 2i$ and 1. The complete solution is

$$\begin{aligned} x &= e^{-t}(p \cos 2t - q \sin 2t), \\ y &= e^{-t}(p \sin 2t + q \cos 2t), \\ z &= r e^t. \end{aligned}$$

The planar rotation is invertible at every time, so the arbitrary three initial components are represented without loss.

Step 2: Classify boundedness in both time directions. Exactly $r = 0$ gives forward boundedness. For backward boundedness, $x^2 + y^2 = (p^2 + q^2)e^{-2t}$ forces $p = q = 0$; the remaining z component then tends to zero backward. The intersection of this axis with the forward-bounded plane is zero, so $\boxed{X \equiv 0}$ is the only solution bounded on the entire real line. Rotation cannot cancel the planar radius.

Step 3: Find the direction with an unstable component. If $r \neq 0$, the ratio of planar radius to $|z|$ is $\sqrt{p^2 + q^2} e^{-2t} / |r| \rightarrow 0$. Consequently $\boxed{X / \|X\| \rightarrow (0, 0, \operatorname{sgn} r)^T}$. The state itself grows without bound; its direction can still converge because the real growing mode dominates both decaying components.

Step 4: Test direction on the stable plane. If $r = 0$ and $s = \sqrt{p^2 + q^2} > 0$, then $X \rightarrow 0$, but its normalized planar state is $R(2t)(p, q)^T / s$. At $t_n = n\pi$ it equals $(p, q)^T / s$; at $t_n = n\pi + \pi/2$ it equals its negative. Both sequences tend to infinity and the two unit vectors are distinct, so no directional limit exists. At $n\pi + \pi/4$ the direction is perpendicular to its value at $n\pi$, so even the unoriented line of approach has no limit. A convergent state need not behave like a nonzero real dominant eigenmode.

Question 10

For $\gamma > 0$, consider the four-dimensional system

$$x'_1 = -x_2, \quad x'_2 = x_1, \quad x'_3 = -\gamma x_4, \quad x'_4 = \gamma x_3.$$

A state is periodic if there is $T > 0$ with $X(t+T) = X(t)$ for all real t . A constant solution has no least positive period.

Tasks

1. Find the eigenvalues and solve the IVP $X(0) = (1, 0, 1, 0)^T$. Describe its projections onto the two coordinate planes.
2. For $\gamma = \sqrt{2}$, prove that this full state is not periodic, even though both planar projections are periodic.
3. Still with $\gamma = \sqrt{2}$, classify exactly which arbitrary initial states give periodic solutions, and give their least periods when nonconstant.
4. Replace γ by $3/2$. Find the least period for the stated IVP and explain why purely imaginary eigenvalues alone do not guarantee a common period in higher dimensions.

Question 10 – Solution

Strategy. A full state repeats only when every active rotating block repeats at the same time.

Step 1: Solve each oscillatory block. The eigenvalues are $\pm i$ and $\pm i\gamma$. For the stated initial data,

$$X(t) = (\cos t, \sin t, \cos \gamma t, \sin \gamma t)^T.$$

The two projections are unit circles with respective least periods 2π and $2\pi/\gamma$. Their closure as plane curves alone does not settle repetition of the combined state.

Step 2: Exclude a common period for irrational frequency ratio. At $\gamma = \sqrt{2}$, any period must satisfy $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$ for positive integers m, n . This would imply $\sqrt{2} = n/m$, impossible. Hence the four-dimensional state is not periodic. Its norm nevertheless remains $\sqrt{2}$, so the obstruction is incompatible timing, not growth or damping.

Step 3: Classify all initial states in the irrational case. Each initially zero two-component block remains zero. A nonzero first block repeats exactly at integer multiples of 2π ; a nonzero second block repeats exactly at integer multiples of $2\pi/\sqrt{2}$. Thus precisely the states with at least one block zero give periodic solutions. If only the first block is nonzero, the least period is 2π ; if only the second is nonzero, it is $\sqrt{2}\pi$. If both vanish, the solution is constant. If both are nonzero, the same irrationality argument rules out every positive period, regardless of their initial phases.

Step 4: Recover a common period for rational frequency ratio. With $\gamma = 3/2$ and both blocks active, $T = 2\pi m$ must also satisfy $3m/2 \in \mathbb{Z}$. The least positive choice is $m = 2$, giving $T = 4\pi$. In these independent rotation systems all states are bounded, but common periodicity requires commensurate active frequencies. Two planar centers therefore need not combine into a periodic full state; purely imaginary eigenvalues alone do not provide the missing frequency ratio.