

Question 1

Let $X = (x, y)^T$ solve

$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

For geometric interpretation use modal coordinates $u = (x + y)/2$ and $v = (x - y)/2$. A stable node means that all nearby states tend to the origin forward in time, with two real negative eigenvalues.

Tasks

1. Find both eigenpairs and solve the IVP. Explain why the two modes give every solution.
2. Eliminate time for this IVP in the (u, v) coordinates, retaining the domain. Translate the result to the original phase plane.
3. Determine the time direction and the limiting tangent line at the origin. Find $\lim_{t \rightarrow \infty} y(t)/x(t)$.
4. For arbitrary $X(0) = (p, q)^T$, identify precisely the initial states that approach along the faster eigendirection instead. Classify the origin.

Question 1 – Solution

Strategy. Decompose into independent eigendirections, then distinguish the dominant slow mode from initial data that cancel it.

Step 1: Resolve the modes. The characteristic polynomial is $(\lambda+1)(\lambda+3)$. Eigenvectors are $v_1 = (1, 1)^T$ for -1 and $v_2 = (1, -1)^T$ for -3 . They form a basis, so modal coordinates satisfy $u' = -u$, $v' = -3v$. Initially $u = v = 1$, giving

$$X = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This decoupling also proves completeness for arbitrary initial data.

Step 2: Retain the orbit domain. Here $u = e^{-t} > 0$ and $v = e^{-3t} = u^3$ for all real t . Thus the orbit is

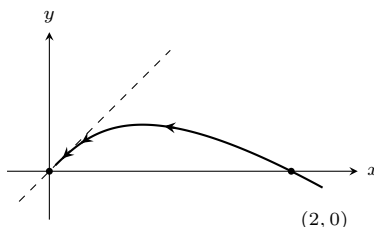
$$\frac{x-y}{2} = \left(\frac{x+y}{2} \right)^3, \quad x+y > 0.$$

The origin is a limiting point, not a point attained at finite time.

Step 3: Read the approach direction. As time increases, u decreases toward zero. Moreover $y/x = (1 - e^{-2t})/(1 + e^{-2t}) \rightarrow 1$. Both the state direction and the velocity direction approach the line $y = x$; indeed $dy/dx = (1 - 3e^{-2t})/(1 + 3e^{-2t}) \rightarrow 1$. At $(2, 0)$ the velocity is $(-4, 2)$, consistent with the arrows.

Step 4: Keep the exceptional line. For initial (p, q) , the slow and fast coefficients are $(p+q)/2$ and $(p-q)/2$. Every solution tends to zero. A nonzero solution approaches along the fast line $y = -x$ exactly when $p+q = 0$; otherwise its limiting line is $y = x$. The zero state remains fixed and has no approach direction. Both eigenvalues are negative, so the origin is a stable node.

Stable node: slow line $y = x$ (dashed)



Question 2

Consider

$$X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X.$$

Use the Euclidean distance to the line $y = -x$. In the perturbed initial data below assume $0 < \varepsilon < 1/\sqrt{2}$.

Tasks

1. Find the eigenpairs and general solution. Identify every initial state producing a bounded solution for $t \geq 0$.
2. Explain the saddle geometry and the forward directions on its two eigenlines. Can a nonconstant trajectory cross either eigenline?
3. For $X(0) = (1 + \varepsilon, -1 + \varepsilon)^T$, find when the magnitudes of its stable and unstable modal coefficients first become equal.
4. Find the first $t \geq 0$ at which this solution has distance 1 from $y = -x$. Explain why small initial error can eventually cause a large departure.

Question 2 – Solution

Strategy. Track stable and unstable coefficients separately; perpendicular distance measures departure from the stable line.

Step 1: Separate decay from growth. Eigenpairs are $3, (1, 1)^T$ and $-1, (1, -1)^T$. For data (p, q) ,

$$X = \frac{p+q}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{p-q}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

These independent vectors give the full family. A forward solution is bounded exactly when $p+q=0$, and then it tends to zero.

Step 2: Interpret the invariant lines. The unstable line $y=x$ points away from the origin; the stable line $y=-x$ points toward it. The opposite signs make the origin a saddle. Each modal coefficient preserves its sign unless identically zero. Thus off-line trajectories cannot cross either eigenline at finite time. Each equilibrium or nonzero half-line is a separate trajectory.

Step 3: Compute modal balance. The perturbed solution is $X = \varepsilon e^{3t}(1, 1)^T + e^{-t}(1, -1)^T$. Equality of modal magnitudes requires $\varepsilon e^{3t} = e^{-t}$, hence

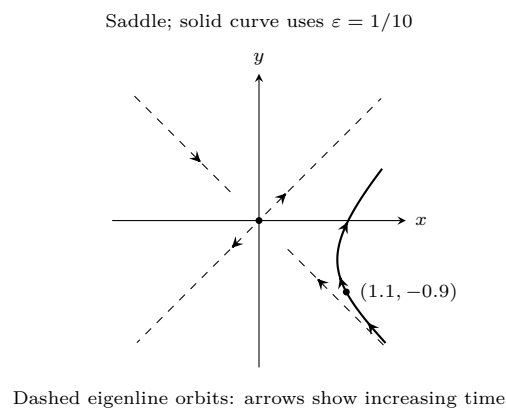
$$t_{\text{balance}} = \frac{1}{4} \ln(1/\varepsilon).$$

The eigenvectors have equal length, so the same time balances their Euclidean contributions. It is nonnegative under the stated assumption.

Step 4: Measure the delayed departure. Distance to $y=-x$ is $|x+y|/\sqrt{2} = \sqrt{2}\varepsilon e^{3t}$, strictly increasing from a value below one. Its first value one occurs at

$$t_{\text{exit}} = \frac{1}{3} \ln\left(\frac{1}{\sqrt{2}\varepsilon}\right).$$

The delay diverges as $\varepsilon \downarrow 0$, but every positive error has a growing unstable component. Only exact cancellation stays on the stable line. The figure illustrates $\varepsilon = 1/10$.



Question 3

For

$$X' = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

a student claims that positive eigenvalues force both coordinates to increase for all real time. Investigate the entire trajectory, including $t < 0$.

Tasks

1. Find the eigenpairs and the solution, and classify the equilibrium.
2. Eliminate time to find the exact phase curve and its domain. Determine its time direction.
3. Find every stationary point of $x(t)$ and classify it. Determine when $x(t)$ is negative and when it vanishes.
4. Identify the limiting eigenline as $t \rightarrow -\infty$ and the limiting direction as $t \rightarrow \infty$. Explain what is wrong with the student's claim.

Question 3 – Solution

Strategy. Growing modal amplitudes can combine with opposite signs; compute coordinate derivatives before making a monotonicity claim.

Step 1: Solve in an eigenbasis. Eigenpairs are $2, (1, 0)^T$ and $1, (-1, 1)^T$. The initial vector is their sum, so

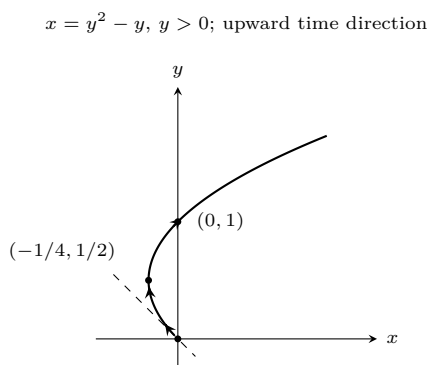
$$x = e^{2t} - e^t, \quad y = e^t.$$

Both eigenvalues are positive, making the origin an unstable node (a source). Every solution tends to zero backward in time, not forward.

Step 2: Recover the geometric branch. Since $y = e^t > 0$, the phase curve is $x = y^2 - y, y > 0$. It is traversed toward increasing y . The limiting origin is excluded; the finite-time point with $x = 0$ is $(0, 1)$. The curve's leftmost point has a vertical tangent, not an equilibrium.

Step 3: Locate coordinate reversal. We have $x' = e^t(2e^t - 1)$, zero only at $t = -\ln 2$. It is negative before this time and positive after it. Therefore the unique global minimum is $x = -1/4$, at $y = 1/2$. For $t < 0$, $x = e^t(e^t - 1) < 0$; it vanishes only at $t = 0$ and is positive for $t > 0$. The limit $x \rightarrow 0$ as $t \rightarrow -\infty$ is not another zero.

Step 4: Distinguish modal and coordinate behavior. Backward, $X/e^t \rightarrow (-1, 1)^T$, so the approach line is $y = -x$. Forward, $X/e^{2t} \rightarrow (1, 0)^T$, giving the positive horizontal direction. Positive eigenvalues describe growth of modes with fixed coefficients; they do not force each original coordinate to increase. Here the two contributions to x have opposite signs, producing a genuine reversal. The faster eigenmode eventually dominates because its coefficient is nonzero.



Question 4

Let

$$X' = \begin{pmatrix} -1 & 6 \\ 0 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Write $\|X\| = \sqrt{x^2 + y^2}$. Investigate whether negative eigenvalues alone force this ordinary distance from the origin to decrease at every time.

Tasks

1. Find both eigenpairs and solve the IVP. Show that all initial states decay as $t \rightarrow \infty$.
2. Find the maximum of $x(t)$ on $t \geq 0$. Evaluate $\|X\|$ there and compare it with $\|X(0)\|$.
3. Compute $(\|X\|^2)'$ for a general state. Give a state at which it is positive, and determine its sign at the stated initial point.
4. Construct a positive quadratic quantity using $u = x + 6y$, $v = y$ that strictly decreases along every nonzero solution. Explain how this reconciles decay with transient growth.

Question 4 – Solution

Strategy. Euclidean distance mixes nonorthogonal modes; a quadratic quantity in modal coordinates separates their decay.

Step 1: Decouple the eigenmodes. The eigenpairs are $-1, (1, 0)^T$ and $-2, (-6, 1)^T$. Since $(0, 1)^T = 6(1, 0)^T + (-6, 1)^T$,

$$x = 6(e^{-t} - e^{-2t}), \quad y = e^{-2t}.$$

Every initial vector has this two-mode form with suitable coefficients, so all solutions tend to zero.

Step 2: Exhibit transient amplification. The derivative $x' = 6(-e^{-t} + 2e^{-2t})$ changes from positive to negative at $t = \ln 2$. Thus $x_{\max} = 3/2$, while $y = 1/4$ there. Consequently

$$\|X(\ln 2)\| = \sqrt{37}/4 > 1 = \|X(0)\|.$$

This is the maximum of x , not a claim about the time of maximum norm.

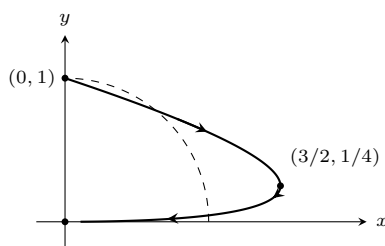
Step 3: Test instantaneous norm growth. Direct differentiation gives

$$(\|X\|^2)' = -2x^2 + 12xy - 4y^2.$$

At $(1, 1)$ it equals $6 > 0$, so the norm can increase. At the given initial state $(0, 1)$ it equals -4 : this trajectory first moves closer, but later exceeds its initial distance. Negative eigenvalues do not make this quadratic expression negative at every state.

Step 4: Use a norm adapted to the modes. The invertible coordinates satisfy $u' = -u$, $v' = -2v$. Thus $E = (x + 6y)^2 + y^2 = u^2 + v^2$ is positive except at zero and $E' = -2u^2 - 4v^2 < 0$ for every nonzero state. The Euclidean circles and the level ellipses of E measure different sizes. Decrease of the latter is compatible with temporary Euclidean growth, and the explicit modes still establish eventual decay.

Solid: IVP; dashed: unit-circle arc



Question 5

For a real parameter $-1 < a < 1$, consider

$$X' = A_a X, \quad A_a = \frac{1}{2} \begin{pmatrix} a-2 & a+2 \\ a+2 & a-2 \end{pmatrix}.$$

Treat the zero-eigenvalue case explicitly; do not assume that every equilibrium is isolated.

Tasks

1. Find the eigenvalues, an eigenbasis independent of a , and the solution with arbitrary initial state $(p, q)^T$.
2. Classify the origin for $a < 0$ and $a > 0$, and identify the initial states that converge to it in each case.
3. At $a = 0$, find every equilibrium and the limiting state of every solution. Is the origin attracting all nearby states?
4. For initial state $(2, 0)^T$, compare $\lim_{a \uparrow 0} \lim_{t \rightarrow \infty} X_a(t)$ with $\lim_{t \rightarrow \infty} \lim_{a \uparrow 0} X_a(t)$. Explain the difference.

Question 5 – Solution

Strategy. Keep the parameter-dependent mode visible; its decay disappears at the zero eigenvalue.

Step 1: Use fixed eigendirections. The vectors $(1, 1)^T$ and $(1, -1)^T$ have eigenvalues a and -2 . They remain independent, and their eigenvalues are distinct throughout the stated parameter interval. Setting $m = (p + q)/2$, $d = (p - q)/2$ gives

$$X_a(t) = me^{at} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + de^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This formula satisfies the system and the initial data for every allowed a .

Step 2: Classify the nonzero-parameter cases. For $a < 0$, both modes decay, so the origin is a stable node and all initial states converge to it. For $a > 0$, the origin is a saddle; convergence occurs exactly when $m = 0$, that is, on $y = -x$. The $y = x$ mode then grows. There are no other equilibria in these two cases because $\det A_a = -2a \neq 0$.

Step 3: Resolve the equilibrium line. At $a = 0$, every point on $y = x$ is an equilibrium and $X_0(t) \rightarrow (m, m)^T$. This is the orthogonal projection of the initial state onto that line. The origin does not attract all nearby states: nearby nonzero points of the equilibrium line remain fixed. Nevertheless it is stable in the distance sense, since the orthogonal constant and decaying modes never increase the Euclidean norm.

Step 4: Compare the orders of limiting. For the specified data, $m = d = 1$. At each fixed $a < 0$ the time limit is zero; hence the first iterated limit is $\boxed{(0, 0)^T}$. At each fixed finite t , taking $a \uparrow 0$ instead gives $(1, 1)^T + e^{-2t}(1, -1)^T$, whose time limit is $\boxed{(1, 1)^T}$. The decay e^{at} becomes arbitrarily slow as a approaches zero from below. It is not uniform over such parameters on an unbounded time interval, so these two operations need not commute.

Question 6

An unknown constant real 2×2 matrix A is known to satisfy

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

These are exact eigenvector observations, not merely measured slopes.

Tasks

1. Reconstruct A and justify its uniqueness.
2. Solve $X' = AX$ with $X(0) = (3, 3)^T$. Verify the initial state and derivative directly.
3. Find the entire set of initial states that decay forward in time. Determine the limiting direction for the stated IVP.
4. Would knowing only $\text{tr } A = 1$ and $\det A = -2$ determine that decaying line? Give a concrete comparison matrix and explain the effect of rescaling either supplied eigenvector.

Question 6 – Solution

Strategy. An eigenbasis fixes the linear map, while eigenvalues alone do not fix its directions.

Step 1: Reconstruct the operator. Let $P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $D = \text{diag}(-1, 2)$. Since $\det P = -3 \neq 0$,

$$A = PDP^{-1} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

A linear map is uniquely determined by its values on a basis, so there is no other matrix satisfying both given observations.

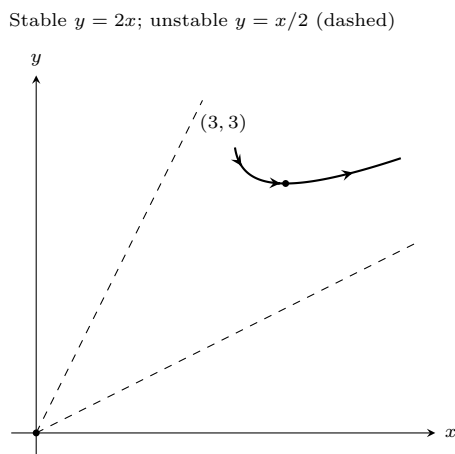
Step 2: Solve and check the IVP. The initial vector is $(1, 2)^T + (2, 1)^T$, giving

$$X = e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

At zero this is $(3, 3)^T$, and its derivative is $(-1, -2)^T + (4, 2)^T = (3, 0)^T = A(3, 3)^T$. More generally each mode differentiates to its eigenvalue times itself, verifying the ODE for every time.

Step 3: Separate stable data from generic growth. For initial (p, q) , the unstable coefficient is $(2p - q)/3$. Decay therefore occurs exactly on $q = 2p$, the line spanned by $(1, 2)^T$. For the stated IVP the unstable coefficient is one, so $X/e^{2t} \rightarrow (2, 1)^T$ and $X/\|X\| \rightarrow (2, 1)^T/\sqrt{5}$. The drawn eigenlines are invariant; the off-line curve cannot cross them.

Step 4: Distinguish rates from directions. The matrix $B = \text{diag}(-1, 2)$ has the same trace and determinant, hence the same eigenvalues, but its decaying line is the horizontal axis. Trace and determinant alone do not determine the stable line. Multiplying either supplied eigenvector by any nonzero scalar changes its coordinate coefficient but leaves PDP^{-1} unchanged: the diagonal rescaling commutes with D . Zero rescaling is not allowed.



Question 7

Consider the three-dimensional homogeneous system

$$X' = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

A solution is called bounded forward (backward) if its Euclidean norm is bounded on $[0, \infty)$ (respectively $(-\infty, 0]$).

Tasks

1. Find an eigenbasis and use it to write the complete solution for arbitrary (p, q, r) .
2. Find exactly the initial states giving bounded forward solutions. Within that set, distinguish the slow and fast nonzero decay directions.
3. Find exactly the initial states giving bounded backward solutions, and describe their forward behavior.
4. Determine every solution bounded on the entire real line. Explain why cancellation between distinct growing modes cannot produce additional cases.

Question 7 – Solution

Strategy. Express the initial state in a basis of three real modes, then read boundedness from individual modal coordinates.

Step 1: Resolve all three components. The triangular matrix has distinct eigenvalues $-2, -1, 1$. Corresponding vectors are $v_1 = (1, 0, 0)^T$, $v_2 = (1, 1, 0)^T$, $v_3 = (1, 1, 1)^T$, forming an invertible triangular basis. Solving for coefficients yields

$$X = (p - q)e^{-2t}v_1 + (q - r)e^{-t}v_2 + re^tv_3.$$

Each term satisfies the ODE; their sum at zero is $(p, q, r)^T$. Equivalently the independent coordinates $x - y, y - z, z$ solve scalar equations with rates $-2, -1, 1$, proving completeness.

Step 2: Find the stable plane and its exception. Forward boundedness requires and is implied by $r = 0$; this is the plane $z = 0$ of initial states. If $q \neq 0$, then $X/e^{-t} \rightarrow qv_2$, so the slow limiting line is $\text{span}(v_2)$. If $q = 0$ and $p \neq 0$, only the faster mode $pe^{-2t}v_1$ remains. The zero state is separate. All states in this plane tend to zero.

Step 3: Find the backward-bounded line. As $t \rightarrow -\infty$, the negative-rate exponentials grow. Their coefficients must vanish: $p - q = 0$ and $q - r = 0$. Thus the exact set is the line $\boxed{p = q = r}$. On it $X = re^tv_3$, tending to zero backward and, for $r \neq 0$, growing without bound forward. At $r = 0$ it is the equilibrium.

Step 4: Intersect the two conditions. A solution bounded for all real time lies both in $r = 0$ and in $p = q = r$, so $\boxed{X \equiv 0}$ is the only one. This is not merely an inference from the norm of a sum. The inverse basis map is fixed and linear; bounded X forces all its modal coordinates $x - y, y - z, z$ to be bounded. Each is one scalar exponential, so a forbidden growing coefficient cannot be hidden by cancellation with another mode.

Question 8

Let

$$X' = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} X, \quad x(0) = 1, \quad y(T) = b, \quad T > 0.$$

Only the initial value $q = y(0)$ is unknown. These conditions concern different times; they are not a complete initial state at one time.

Tasks

1. Find the real eigenpairs and write the solution in terms of q .
2. Derive the equation that q must satisfy. Identify every positive time at which its coefficient vanishes.
3. At the exceptional time, classify all values of b by whether there are zero, one, or infinitely many solutions. Explain why uniqueness for an IVP is not contradicted.
4. For other times, give q explicitly and its change under a measurement error $b \mapsto b + \delta$. Describe what happens near the exceptional time.

Question 8 – Solution

Strategy. Two real exponential modes can cancel in an endpoint observation even when the full state remains uniquely determined by its initial data.

Step 1: Find the two modes. The characteristic polynomial is $\lambda^2 - 1$. Eigenvectors are $(1, 1)^T$ for 1 and $(1, 2)^T$ for -1 . The data $(1, q)^T$ give coefficients $2 - q$ and $q - 1$, so

$$X = (2 - q)e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (q - 1)e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Step 2: Locate the singular observation time. The endpoint condition becomes

$$b = 2e^T - 2e^{-T} + q(2e^{-T} - e^T).$$

The coefficient of q vanishes exactly when $e^{2T} = 2$, hence $T_* = \frac{1}{2} \ln 2$, the unique positive exceptional time.

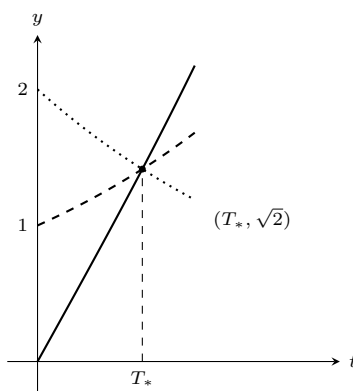
Step 3: Classify compatibility at that time. At T_* , the equation reduces to $b = \sqrt{2}$. If $b = \sqrt{2}$, every real q gives a solution; otherwise none does. There is never exactly one at this time. Distinct q specify distinct initial states. In fact $x(T_*) = (3 - q)/\sqrt{2}$, so the full states remain different even though their second components agree. Each individual IVP still has its unique solution. The figure shows three such second components.

Step 4: Quantify sensitivity away from the exception. For $T \neq T_*$ there is exactly one solution, with

$$q = \frac{b - 2e^T + 2e^{-T}}{2e^{-T} - e^T}, \quad \Delta q = \frac{\delta}{2e^{-T} - e^T}.$$

The denominator tends to zero as $T \rightarrow T_*$, so the absolute amplification factor diverges. This is a poorly conditioned recovery of missing initial data, not loss of uniqueness for a fully specified IVP. For a fixed nonzero δ , the recovered initial error can become arbitrarily large.

Solid $q = 0$; dashed $q = 1$; dotted $q = 2$



Question 9

Let

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}, \quad P = A + 2I, \quad Q = -(A + I).$$

Here a projection means a matrix R with $R^2 = R$; it need not be an orthogonal projection. Develop a solution formula without choosing normalized eigenvectors.

Tasks

1. Find the eigenvalues and verify $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$.
2. Identify the range and kernel of each projection. Are these projections orthogonal in the Euclidean plane?
3. Prove that $F(t) = e^{-t}P + e^{-2t}Q$ satisfies $F' = AF$ and $F(0) = I$. Use it to solve the IVP $X(0) = (1, 0)^T$.
4. Prove $F(t+s) = F(t)F(s)$ and find its inverse. Explain why these identities hold for every pair of real times here.

Question 9 – Solution

Strategy. Polynomial projections isolate the two real eigenspaces and make the evolution law a scalar calculation on each one.

Step 1: Verify complementary projections. The characteristic polynomial is $(\lambda+1)(\lambda+2)$, and direct multiplication gives $A^2 + 3A + 2I = 0$. Here $P = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}$. Thus $P + Q = I$, $P^2 = P$, $Q^2 = Q$, and $PQ = QP = 0$. For example $PQ = -(A + 2I)(A + I) = 0$; the other identities follow by multiplication or from $Q = I - P$.

Step 2: Identify the oblique directions. The range of P is $\text{span}(2, 3)^T$, the -1 eigenspace; its kernel is $\text{span}(1, 1)^T$, the -2 eigenspace. For Q the range and kernel are interchanged. These two directions have nonzero dot product 5, so neither projection is orthogonal. The matrices are also visibly nonsymmetric.

Step 3: Construct and verify evolution. The same polynomial identity gives $AP = -P$, $AQ = -2Q$. Consequently $F' = AF$ and $F(0) = P + Q = I$. For arbitrary initial vector X_0 , $F(t)X_0$ solves the IVP; constant linear-system uniqueness gives the complete solution. In particular

$$X = e^{-t} \begin{pmatrix} -2 \\ -3 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Its initial state is $(1, 0)^T$ and its initial derivative $(-4, -3)^T = A(1, 0)^T$.

Step 4: Multiply by modes. All mixed products vanish, so

$$F(t)F(s) = e^{-(t+s)}P + e^{-2(t+s)}Q = F(t+s).$$

Taking $s = -t$ yields $F(t)^{-1} = F(-t) = e^tP + e^{2t}Q$. No finite real time makes either scalar exponential vanish. This is the normalized evolution of one constant matrix; the fixed complementary projections justify the multiplication rule for every $s, t \in \mathbb{R}$. No orthogonality or eigenvector normalization was used.

Question 10

An unknown real matrix has eigenvectors $(1, 1)^T$ and $(1, -1)^T$ with distinct negative eigenvalues λ and μ , respectively. For the solution with $X(0) = (2, 0)^T$, define $u = (x + y)/2$, $v = (x - y)/2$. The observed entire phase curve is $v = u^2$, $u > 0$, with direction toward the origin. A clock measurement gives $u(\ln 2) = 1/2$.

Tasks

1. Determine what the phase curve alone implies about λ and μ . Explain why its shape and arrows do not determine their absolute values.
2. Use the clock measurement to recover both eigenvalues and the matrix.
3. Write and verify the full IVP solution, including the observed curve, its domain, and the clock measurement.
4. Find the limiting tangent at the origin and the maximum of $y(t)$ on $t \geq 0$. Explain how uniformly speeding up this system would affect the curve, maximum value, and time of that maximum.

Question 10 – Solution

Strategy. Geometry determines a ratio of modal rates; one elapsed-time observation fixes their common scale.

Step 1: Extract a rate ratio. Initially $u = v = 1$, so $u = e^{\lambda t}$ and $v = e^{\mu t}$. The identity $v = u^2$ for every time implies $\mu = 2\lambda$. Any $\lambda < 0$ with this ratio gives the same oriented orbit as t ranges over the real line. Thus the curve and its arrows alone leave one positive speed factor undetermined.

Step 2: Calibrate the clock. The measurement yields $e^{\lambda \ln 2} = 1/2$, so $\lambda = -1$ and $\mu = -2$. Using the specified eigenbasis reconstructs

$$A = \frac{1}{2} \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}.$$

Step 3: Verify the reconstructed trajectory. The solution is

$$x = e^{-t} + e^{-2t}, \quad y = e^{-t} - e^{-2t}.$$

Differentiating and multiplying by the displayed matrix agree component by component, and the initial state is $(2, 0)^T$. Moreover $u = e^{-t} > 0$, $v = e^{-2t} = u^2$, and $u(\ln 2) = 1/2$. Every $u > 0$ occurs at a finite real time; the limiting origin is excluded from the orbit.

Step 4: Separate geometric and temporal conclusions. As $t \rightarrow \infty$, $dy/dx = (1 - 2e^{-t})/(1 + 2e^{-t}) \rightarrow 1$, giving the tangent line $y = x$. On $t \geq 0$, $y' = e^{-t}(2e^{-t} - 1)$ changes from positive to negative at $t = \ln 2$. Hence $y_{\max} = 1/4$, attained at $x = 3/4$. Replacing A by kA , $k > 0$, changes the solution to $X(kt)$: the oriented curve and maximum value remain the same, but the maximum occurs at $(\ln 2)/k$. Except for $k = 1$, it fails the given clock measurement.

Recovered orbit; dashed tangent $y = x$

