

Question 1

Consider $x' = y$, $y' = -4x$, with initial state $(x, y) = (1, 0)$ at $t = 0$. A phase trajectory is the curve traced by $(x(t), y(t))$, with its direction as time increases. Nullclines are sets where one coordinate derivative is zero.

Tasks

1. Find the equilibrium and both nullclines. Determine the directions of the vector field on each coordinate axis away from the origin.
2. Find a conserved quadratic expression and the phase curve through the initial state. Explain why the curve must be drawn with equal coordinate scales.
3. Find a parametrization solving the IVP, its least positive period, and the direction in which the phase curve is traversed.
4. Locate every horizontal and vertical tangent on this phase curve. Explain why writing it globally as a single-valued function $y(x)$ would lose information.

Question 1 – Solution

Strategy. Use a conserved quantity to identify the geometric curve, then use the original equations to recover direction and timing.

Step 1: Locate zero derivatives. The x -nullcline is $y = 0$, and the y -nullcline is $x = 0$. Their only common point, $(0, 0)$, is the equilibrium. At $(x, 0)$ the field is $(0, -4x)$: downward for $x > 0$, upward for $x < 0$. At $(0, y)$ it is $(y, 0)$: rightward for $y > 0$, leftward for $y < 0$.

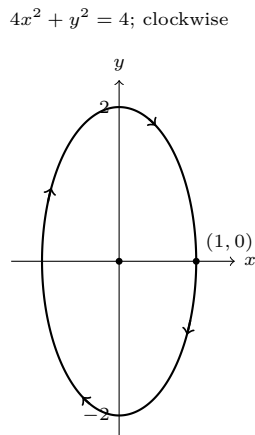
Step 2: Determine the conserved curve. Differentiating gives $(4x^2 + y^2)' = 8xy - 8xy = 0$. The initial state therefore lies on

$$4x^2 + y^2 = 4,$$

an ellipse with horizontal semiaxis 1 and vertical semiaxis 2. Equal coordinate scales preserve this actual geometric aspect ratio.

Step 3: Restore orientation and time. Direct substitution verifies $x = \cos(2t)$, $y = -2\sin(2t)$ and the initial data. The least positive period is π . At the initial point $(1, 0)$ the velocity is $(0, -4)$, so the ellipse is traversed clockwise. The conserved equation alone does not specify this orientation or period.

Step 4: Identify all tangent directions. Horizontal tangents occur where $y' = 0$, namely $(0, 2)$ and $(0, -2)$, where $x' \neq 0$. Vertical tangents occur where $x' = 0$, namely $(1, 0)$ and $(-1, 0)$, where $y' \neq 0$. For $-1 < x < 1$, the same x has two possible values $y = \pm 2\sqrt{1 - x^2}$ on this orbit. A single branch cannot represent the full closed trajectory. The origin is a separate equilibrium, not part of this ellipse. Arrows in the figure indicate increasing time.



Question 2

For $x' = x$, $y' = -y$, a student divides the equations to obtain $dy/dx = -y/x$ and concludes that every trajectory is a hyperbola $xy = C$. Investigate both the usefulness and the limitations of this calculation.

Tasks

1. Derive the conserved quantity without dividing by x or y . Find all equilibria.
2. Classify the trajectories lying on the coordinate axes, including their time directions. Explain what division can omit.
3. For $xy \neq 0$, identify the separate trajectory branches and their directions in all four quadrants. Explain why an entire two-branch hyperbola is not one trajectory.
4. Solve the IVP $(x(0), y(0)) = (2, 1/2)$ and describe both time limits. Determine all initial states whose solutions tend to the origin as $t \rightarrow \infty$.

Question 2 – Solution

Strategy. Keep zero-coordinate cases before separating variables, and use explicit signs to orient each connected branch.

Step 1: Find a division-free invariant. The product rule gives $(xy)' = x'y + xy' = xy - xy = 0$. Thus $xy = C$ along every solution, including axis solutions. The only equilibrium is $(0, 0)$.

Step 2: Recover the axis trajectories. If $y = 0$, then $x = x_0 e^t$; each nonzero horizontal half-axis is a trajectory directed away from the origin. If $x = 0$, then $y = y_0 e^{-t}$; each nonzero vertical half-axis is directed toward the origin. The origin itself is a constant trajectory. Division by x loses the vertical axis; subsequent division by y can also lose the horizontal axis.

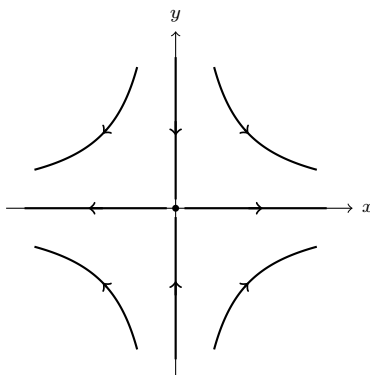
Step 3: Separate the hyperbola branches. For nonzero coordinates, $x = x_0 e^t$ and $y = y_0 e^{-t}$ preserve both signs. In quadrants I, II, III, IV the directions are respectively right/down, left/down, left/up, right/up. Each fixed nonzero C has two disconnected branches; no continuous solution switches between them. The equation $xy = 0$ likewise combines several distinct trajectories rather than describing one orbit through the origin.

Step 4: Apply the initial data and characterize attraction. For the stated IVP,

$$x = 2e^t, \quad y = \frac{1}{2}e^{-t}, \quad xy = 1.$$

As $t \rightarrow \infty$, $x \rightarrow \infty$, $y \rightarrow 0$; as $t \rightarrow -\infty$, $x \rightarrow 0$, $y \rightarrow \infty$. Neither limit is the origin. For general initial data, convergence to the origin in forward time occurs exactly when $x_0 = 0$, including the zero state. The figure distinguishes the axis directions from representative hyperbola branches.

$xy = \pm 1$ and the four axis half-orbits



Question 3

Consider the autonomous nonlinear system

$$x' = y - x^2, \quad y' = 1 - x.$$

A nullcline describes a zero coordinate velocity. It need not be a trajectory. No explicit solution of the full system is required.

Tasks

1. Find the equilibrium and both nullclines.
2. Use inequalities relative to the nullclines to determine the signs of x' , y' in every region. Give the actual velocity at $(0, 0)$, $(0, 1)$ and $(1, 0)$.
3. Determine tangent directions along each nullcline away from the equilibrium. Prove that neither whole nullcline is an invariant trajectory.
4. For a solution starting at $(0, 0)$, calculate $x'(0)$, $y'(0)$, $x''(0)$, $y''(0)$. Explain how it can have a vertical tangent while immediately leaving the x -nullcline.

Question 3 – Solution

Strategy. Read component signs from the two defining expressions, then distinguish instantaneous tangency from remaining on a curve.

Step 1: Locate the nullclines and their intersection. The x -nullcline is $y = x^2$, and the y -nullcline is $x = 1$. Their sole intersection is the equilibrium $(1, 1)$.

Step 2: Determine the regional directions. Above the parabola $x' > 0$; below it $x' < 0$. To the left of $x = 1$, $y' > 0$; to its right, $y' < 0$. Combining these independent signs gives right/up, left/up, right/down or left/down wherever the corresponding inequalities hold. The requested velocities are $F(0, 0) = (0, 1)$, $F(0, 1) = (1, 1)$ and $F(1, 0) = (-1, 0)$.

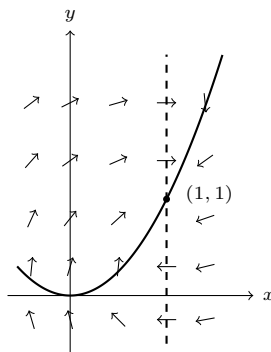
Step 3: Check nullcline tangency correctly. On $y = x^2$, away from $(1, 1)$, the nonzero velocity is vertical. The parabola has finite-slope tangent $(1, 2x)$, so this velocity is not tangent to that parabola. On $x = 1$, away from the equilibrium, the velocity is horizontal and not tangent to the vertical line. Thus neither complete nullcline is invariant. Each is crossed by typical solutions; the equilibrium is the exception where both velocities vanish.

Step 4: Examine departure from the origin. Differentiating the equations gives $x'' = y' - 2xx'$ and $y'' = -x'$. At the initial state,

$$x' = 0, \quad y' = 1, \quad x'' = 1, \quad y'' = 0.$$

Hence $x(t) = t^2/2 + o(t^2)$ and $y(t) = t + o(t^2)$ as $t \rightarrow 0$. For small positive t , $y - x^2 = t + o(t) > 0$, so the solution leaves the nullcline and acquires positive horizontal velocity. A vertical tangent at one instant does not mean that x stays constant. The figure shows nullclines and local velocity arrows, not guessed solution curves.

Solid: $y = x^2$
Dashed: $x = 1$; small arrows: velocities



Question 4

Compare the autonomous vector fields

$$F(x, y) = (-y, x), \quad G(x, y) = (1 + x^2 + y^2)(-y, x), \quad H(x, y) = (y, -x).$$

All three have the origin as an equilibrium. A geometric orbit and the time needed to traverse it are different pieces of information.

Tasks

1. Prove that every nonzero trajectory of each field lies on a circle centered at the origin.
2. On a circle of radius $R > 0$, compute the angular velocity for each field and determine its time direction.
3. Find the least positive period on that circle for each field. Give explicit solutions starting at $(2, 0)$ for F and G .
4. Explain precisely which information is shared by the three phase pictures and which requires arrows or time parametrization. Does a positive scalar multiple of a field reverse its directions?

Question 4 – Solution

Strategy. Compute radial and angular rates separately; multiplication of a field can change timing without changing its curves.

Step 1: Verify the common invariant circles. For F , $(x^2 + y^2)' = 2x(-y) + 2yx = 0$. Multiplication by $1 + x^2 + y^2$ preserves this cancellation for G , and changing the sign preserves it for H . A nonzero initial radius stays fixed and each field is nonzero on that circle.

Step 2: Compute orientation from angular velocity. For $R^2 = x^2 + y^2 > 0$,

$$\theta' = \frac{xy' - yx'}{R^2}.$$

Substitution gives $\theta'_F = 1$, $\theta'_G = 1 + R^2$, $\theta'_H = -1$. Thus F, G traverse circles counterclockwise, while H traverses them clockwise.

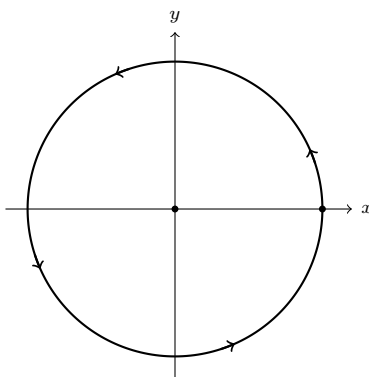
Step 3: Calculate periods and the given trajectories. A full revolution requires angular change of magnitude 2π , so

$$T_F = T_H = 2\pi, \quad T_G = \frac{2\pi}{1 + R^2}.$$

At radius 2, the requested solutions are $X_F = (2 \cos t, 2 \sin t)$ and $X_G = (2 \cos 5t, 2 \sin 5t)$. They start at the same state and cover the same circle at different rates. The zero equilibrium is not assigned a least positive revolution period.

Step 4: Distinguish geometry, direction and speed. All three fields have the same nonzero orbit sets and the same equilibrium. Arrows separate H from F, G ; the time parametrization separates F from G . A strictly positive multiplier preserves velocity direction wherever the field is nonzero, so it cannot reverse an orbit. A negative multiplier does reverse it. The plotted circle is shared by F, G ; its arrows show their common orientation, and the caption records their different periods at radius 2.

Same radius-2 orbit for F, G : counterclockwise



$T_F = 2\pi$, $T_G = 2\pi/5$; H reverses the arrows.

Question 5

The time-dependent system $x' = 1$, $y' = 2t$ has two proposed solutions

$$U(t) = (t, t^2), \quad V(t) = (t + 1, t^2), \quad t \in \mathbb{R}.$$

Investigate whether their projected curves in the (x, y) plane can cross without contradicting uniqueness of the original initial-value problem.

Tasks

1. Verify the candidates, derive the full solution family, and find the equations of their two projected curves.
2. Find the intersection of those curves, the time at which each solution visits it, and the velocity of each at its visit.
3. Decide whether $U(t)$ and $V(t)$ ever agree at the same time. Explain why the usual no-crossing conclusion for a locally Lipschitz autonomous plane field does not apply to these projections.
4. Add time as a state to obtain an autonomous three-dimensional system. Show that the two lifted trajectories do not intersect, including above their projected intersection.

Question 5 – Solution

Strategy. Record the time attached to each projected point; the vector field here is not determined by the plane coordinates alone.

Step 1: Verify and eliminate time. Both derivatives are $(1, 2t)$, so both candidates solve the system. Integration gives the full family $(x, y) = (t + a, t^2 + b)$. The candidate plane curves are respectively $y = x^2$ and $y = (x - 1)^2$, each directed toward increasing x .

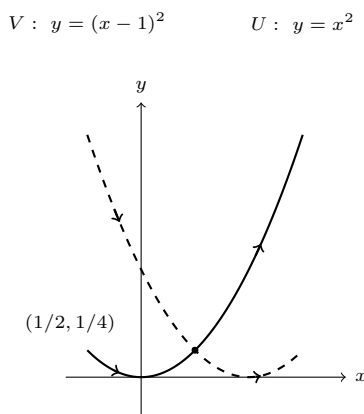
Step 2: Locate the projected crossing and its times. Solving $x^2 = (x - 1)^2$ gives $(x, y) = (1/2, 1/4)$. The first solution reaches it at $t = 1/2$, with velocity $(1, 1)$; the second reaches it at $t = -1/2$, with velocity $(1, -1)$. The different tangent directions reflect different evaluation times.

Step 3: Check what uniqueness prohibits. At any common time, $V(t) - U(t) = (1, 0)$, so the full states never agree. The IVP is unique, but the projected point (x, y) alone does not fix $y' = 2t$. There is no single autonomous plane velocity at the projected crossing. Thus the autonomous no-crossing rule is inapplicable. Geometric intersection at different times is not an intersection of these solutions at the same specified time and state.

Step 4: Restore the clock in the state. With new independent variable τ , use

$$s' = 1, \quad x' = 1, \quad y' = 2s.$$

The lifted candidates are $(s, x, y) = (t, t, t^2)$ and $(t, t + 1, t^2)$. An intersection of lifted curves would first require equal s , hence equal t , and would then contradict the fixed horizontal difference 1. Above the plane crossing their clock coordinates are $1/2$ and $-1/2$, so the lifted states are distinct. The figure deliberately labels only projected plane curves and shows their separate rightward directions.



Projected curves; intersection occurs at different times.

Question 6

For $r^2 = x^2 + y^2$, consider

$$x' = x(1 - r^2) - y, \quad y' = y(1 - r^2) + x.$$

For $r > 0$, use polar coordinates with a continuously followed angle along each trajectory. A periodic orbit is a nonconstant closed trajectory.

Tasks

1. Find every equilibrium. Derive differential equations for r and the angular coordinate.
2. Identify the invariant unit circle, its orientation and least positive period. Determine whether nearby trajectories move toward it from each side.
3. For any initial radius $r_0 > 0$, solve the equation for $u = r^2$ and prove convergence to the unit circle in forward time. State separately what happens at $r_0 = 0$.
4. Does a nonzero solution converge to one particular point on the unit circle? Prove your answer and distinguish approaching an orbit from approaching an equilibrium.

Question 6 – Solution

Strategy. Separate the radial attraction from the angular motion; approaching a circle does not require stopping on it.

Step 1: Find the radial and angular equations. For a nonzero state, the equilibrium coefficient matrix has determinant $(1 - r^2)^2 + 1 > 0$, so it cannot annihilate that state. The only equilibrium is the origin. Direct differentiation gives

$$r' = r(1 - r^2), \quad \theta' = 1 \quad (r > 0).$$

The formulas follow from $rr' = xx' + yy'$ and $r^2\theta' = xy' - yx'$.

Step 2: Identify the periodic orbit and radial directions. At $r = 1$, the radius is constant and the angle increases at unit rate. This circle is a counterclockwise periodic orbit of least period 2π . For $0 < r < 1$, $r' > 0$; for $r > 1$, $r' < 0$. Both sides move radially toward the circle rather than away from it.

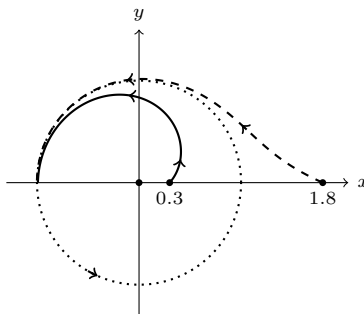
Step 3: Prove the forward radial limit. The squared radius satisfies $u' = 2u(1 - u)$, with solution

$$u(t) = \frac{1}{1 + (r_0^{-2} - 1)e^{-2t}} \quad (t \geq 0).$$

The denominator is positive for every $t \geq 0$, and direct differentiation and evaluation at zero verify the formula. It tends to 1, hence $r \rightarrow 1$. With $\theta = \theta_0 + t$, this also supplies a forward-global solution. At $r_0 = 0$ the solution remains at the equilibrium; the reciprocal formula is not used there.

Step 4: Rule out convergence to a single point. Along times $t = 2\pi n$ the states tend to $(\cos \theta_0, \sin \theta_0)$. Along $t = 2\pi n + \pi$ they tend to its negative. These are distinct, so no nonzero solution has a single limiting point. Its distance to the unit circle, $|r - 1|$, tends to zero while it continues rotating. The figure shows the unit orbit and sample inward/outward spirals; all arrows represent increasing time.

Solid: $r_0 = 0.3$; dashed: $r_0 = 1.8$
Dotted: invariant unit orbit



Question 7

Consider the damped nonlinear system

$$x' = y, \quad y' = -x - x^3 - cy, \quad c > 0,$$

with initial state $(1, 0)$. Define $E(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4$. A level curve of E need not itself be a solution curve.

Tasks

1. Find the equilibrium and derive the exact change of E along a solution.
2. Use the initial energy to prove bounds on both coordinates for every forward time. Explain why the smooth system then has a global forward solution.
3. At the initial state, $E' = 0$. Does this mean that the solution is stationary or remains on its original energy level? Use the original equations to decide.
4. Prove that no nonconstant periodic orbit exists, by integrating the energy balance over a hypothetical period. Explain what the enclosing energy curve represents in the phase picture.

Question 7 – Solution

Strategy. A decreasing energy supplies a containing region and rules out a closed return without requiring an explicit trajectory.

Step 1: Compute the equilibrium and dissipation. Equilibrium requires $y = 0$ and $x + x^3 = 0$, hence only $(0, 0)$. Along solutions,

$$E' = y(-x - x^3 - cy) + (x + x^3)y = \boxed{-cy^2 \leq 0}.$$

Thus energy decreases except at instants when $y = 0$.

Step 2: Bound the forward solution. Initially $E = 3/4$. Since all energy terms are nonnegative, $y^2/2 \leq 3/4$ gives $|y| \leq \sqrt{3/2}$. Also $x^4 + 2x^2 \leq 3$, so $(x^2 + 1)^2 \leq 4$ and $|x| \leq 1$. Therefore

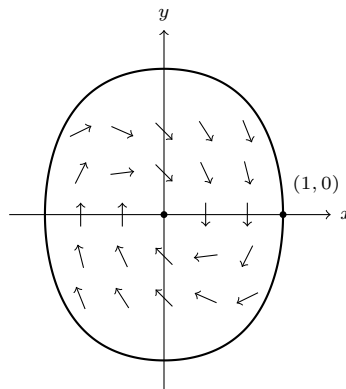
$$\boxed{|x(t)| \leq 1, \quad |y(t)| \leq \sqrt{3/2} \quad (t \geq 0).}$$

The polynomial field is locally Lipschitz. These bounds keep the state in a compact region, preventing finite-time escape and allowing continuation for all forward times.

Step 3: Interpret zero instantaneous energy loss. At $(1, 0)$, the velocity is $(0, -2)$, not zero. Moreover $y(t) = -2t + o(t)$ for small positive t , so y becomes nonzero and $E' = -cy^2 < 0$. The solution enters lower energy levels. A horizontal zero of E' at one instant does not make energy constant on an interval or make its level curve a trajectory.

Step 4: Exclude nonconstant periodic motion. If a solution had period $T > 0$, integrating gives $0 = E(T) - E(0) = -c \int_0^T y(t)^2 dt$. Continuity and $c > 0$ force $y \equiv 0$, then $x' = 0$ and $x + x^3 = 0$, so the solution is the zero equilibrium. Thus no nonconstant periodic orbit exists. The drawn $E = 3/4$ curve bounds a forward-invariant region; it is not a closed orbit. Interior arrows show velocities for the illustrative choice $c = 1$.

$E = 3/4$ is a containing boundary, not an orbit
Interior velocity arrows use $c = 1$



Question 8

Start with $x' = -y$, $y' = x$, and introduce the nonlinear change of coordinates $u = x$, $v = y + x^2$. The original unit-circle trajectory is $(x, y) = (\cos t, \sin t)$. Both coordinate planes should be drawn with equal scales when interpreting their geometry.

Tasks

1. Find the inverse coordinate change and derive the autonomous system for (u, v) .
2. Find the image of the unit circle as an implicit equation and a time parametrization. Verify the parametrization in the transformed differential equations.
3. Determine the equilibrium, the least positive period of the transformed unit orbit, and its velocity at the image of $(1, 0)$.
4. Can this invertible coordinate change turn the unit circle into a self-intersecting trajectory? Prove your conclusion and explain which geometric features can change even though the time parametrization is preserved.

Question 8 – Solution

Strategy. Transform both the curve and its velocity by the chain rule; invertibility preserves distinct states even when shapes change.

Step 1: Derive the inverse and new field. The inverse is $x = u$, $y = v - u^2$, valid on the whole plane. Therefore

$$u' = u^2 - v, \quad v' = u + 2u(u^2 - v).$$

The term $2xx'$ in $v' = y' + 2xx'$ is essential. This is a nonlinear autonomous system, despite the linear original field.

Step 2: Transform and verify the orbit. Substituting into $x^2 + y^2 = 1$ gives

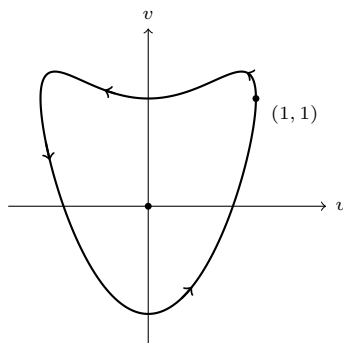
$$u^2 + (v - u^2)^2 = 1, \quad (u, v) = (\cos t, \sin t + \cos^2 t).$$

Here $u' = -\sin t = u^2 - v$, and $v' = \cos t - 2\cos t \sin t = u + 2u(u^2 - v)$. Thus this is the actual transformed trajectory, not merely a guessed level curve.

Step 3: Check equilibrium, period and orientation locally. From $u' = 0$ we have $v = u^2$; then $v' = u$ forces $u = v = 0$. The only equilibrium is the image of the original origin. The transformed orbit has period 2π . A smaller period would, through the inverse map, give a smaller period of $(\cos t, \sin t)$, which is impossible. The image of $(1, 0)$ is $(1, 1)$, with velocity $(0, 1)$: upward.

Step 4: Exclude a self-intersection. If two transformed states agree, the inverse gives equal original states. On the unit circle this occurs only for times differing by an integer multiple of 2π . No new self-intersection can appear within one traversal. The circle becomes a distorted closed curve, so Euclidean lengths, angles and curvature can change. The same time variable and invertible state map preserve the period and prevent distinct states from being identified. The figure plots the transformed orbit in equal u, v scales.

$$u^2 + (v - u^2)^2 = 1; \text{ equal coordinate scales}$$



Question 9

For $x' = 1 - x^2$, $y' = -y$, focus on the strip $-1 < x < 1$ and its boundaries. You may use $(\tanh t)' = 1 - \tanh^2 t$ and $\operatorname{artanh} x = \frac{1}{2} \ln((1+x)/(1-x))$ on this strip.

Tasks

1. Find the equilibria and verify the trajectory $(x, y) = (\tanh t, 0)$, including its direction and its limits as $t \rightarrow \pm\infty$.
2. Explain why the closed segment joining the two equilibria is not a single trajectory, even though it is the closure of the connecting orbit.
3. For initial state $(0, b)$, find a parametrization and eliminate time on the strip. Describe the forward and backward limits, treating $b = 0$ separately.
4. Determine trajectories on $x = -1$ and $x = 1$. Can a nonconstant one reach its equilibrium in finite time? Distinguish approaching an endpoint from attaining it.

Question 9 – Solution

Strategy. Keep equilibria separate from nearby nonconstant orbits, and retain the open domain when eliminating time.

Step 1: Identify the connecting trajectory. The equilibria are $(-1, 0), (1, 0)$. The function $(\tanh t, 0)$ solves both equations, has $x' > 0$, and tends to $(-1, 0)$ backward in time and $(1, 0)$ forward in time. Its orbit is the open horizontal segment $-1 < x < 1$, directed rightward.

Step 2: Separate the orbit from its closure. For every finite t , $-1 < \tanh t < 1$, so neither endpoint is reached. Each endpoint is its own constant trajectory. Uniqueness for the smooth field also prevents a solution from reaching an equilibrium and then leaving it. The closed segment is the union of three trajectories, not one.

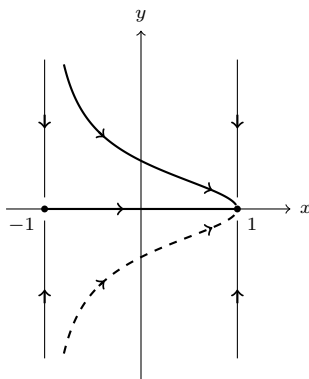
Step 3: Recover the whole one-parameter family. For initial $(0, b)$,

$$x = \tanh t, \quad y = be^{-t}, \quad y = b\sqrt{\frac{1-x}{1+x}}, \quad -1 < x < 1.$$

The inverse hyperbolic tangent formula proves the eliminated equation. Every member tends to $(1, 0)$ as $t \rightarrow \infty$. If $b = 0$, the backward limit is $(-1, 0)$. If $b \neq 0$, then $x \rightarrow -1$ while $y \rightarrow +\infty$ for $b > 0$ or $-\infty$ for $b < 0$; there is no finite backward limiting state.

Step 4: Check the invariant boundary lines. On either $x = \pm 1$, the first coordinate stays fixed and $y = y_0 e^{-(t-t_0)}$. For $y_0 \neq 0$, each vertical half-line is an orbit directed toward $(\pm 1, 0)$, reached only as $t \rightarrow \infty$. At $y_0 = 0$ the state is already the equilibrium. The figure marks equilibria as separate filled points; connecting arrows and nearby trajectories approach them without asserting finite-time arrival.

$b = 1/2$ (solid), $b = -1/2$ (dashed), $b = 0$
Filled equilibria are not part of the connecting orbit.



Question 10

On the positive quadrant $x > 0$, $y > 0$, consider

$$x' = x(1 - y), \quad y' = y(x - 1), \quad (x(0), y(0)) = (2, 1).$$

You may use the elementary geometric fact that a compact regular level of a smooth strictly convex function in the plane, enclosing its unique minimum, is a simple closed curve.

Tasks

1. Find the positive equilibrium and nullclines. Determine the velocity and direction at the stated initial point.
2. Prove that $H = x - \ln x + y - \ln y$ is conserved, and find its level for the initial data.
3. Use convexity, boundary behavior and the vector field to show that this nonconstant orbit is closed and traversed in a finite period. No period formula is required.
4. Find exact equations for the minimum and maximum of each population on this orbit, identify where those extrema occur, and explain why their maxima are not simultaneous.

Question 10 – Solution

Strategy. A proper conserved function confines the positive orbit, and nonzero tangential velocity turns its level curve into a periodic trajectory.

Step 1: Read the positive nullclines. In the positive quadrant, $x' = 0$ exactly at $y = 1$, and $y' = 0$ exactly at $x = 1$. The only positive equilibrium is $(1, 1)$. At $(2, 1)$ the velocity is $(0, 1)$, so the orbit initially moves upward. The positive axes are not included in the stated domain.

Step 2: Derive the conserved level. Differentiating yields

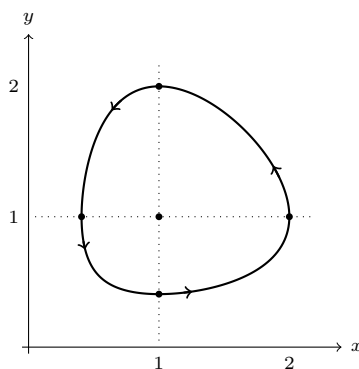
$$H' = (1 - 1/x)x(1 - y) + (1 - 1/y)y(x - 1) = 0.$$

For the stated data, $\boxed{H = 3 - \ln 2}$. The Hessian is diagonal with entries $1/x^2, 1/y^2 > 0$, so H is strictly convex with unique minimum $H(1, 1) = 2$.

Step 3: Justify a closed periodic orbit. The function $z - \ln z$ tends to infinity as $z \downarrow 0$ or $z \rightarrow \infty$. Thus this level is compact within the positive quadrant. Its gradient vanishes only at $(1, 1)$, below the stated level; it is regular. The given convex-level fact makes it a simple closed curve. The vector field is tangent by conservation and never zero on this curve. Its continuous speed therefore has a positive minimum, so traversal of the finite-length smooth closed curve takes finite time. Uniqueness then gives a periodic orbit. It is counterclockwise, as its velocity at the rightmost point $(2, 1)$ is upward.

Step 4: Locate and compare the extrema. Extrema of x occur at $y = 1$, hence $x - \ln x = 2 - \ln 2$. This has roots 2 and a unique $a \in (0, 1)$, since $z - \ln z$ decreases on $(0, 1)$ and increases on $(1, \infty)$. By symmetry, $\boxed{x_{\min} = y_{\min} = a, \quad x_{\max} = y_{\max} = 2}$. The four extreme points are $(a, 1), (2, 1), (1, a), (1, 2)$. In particular the two maxima occur at different states, so they are not simultaneous. The plotted level uses the exact conserved equation.

$$x - \ln x + y - \ln y = 3 - \ln 2$$



Counterclockwise; dotted nullclines $x = 1, y = 1$.