

Question 1

For $X' = AX$, with $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, consider

$$U(t) = e^t(1, 0)^T, \quad V(t) = e^t(t, 1)^T, \quad W(t) = e^t(1, t)^T.$$

A vector solution must satisfy both rows at every time in its interval. A fundamental matrix has solution columns that form a basis at each time.

Tasks

1. Compute the residual $F' - AF$ for each candidate $F = U, V, W$. Decide which are solutions on a nonempty open interval.
2. Use the valid candidates to form a fundamental matrix. Prove that every solution is their constant linear combination, rather than merely checking that combinations work.
3. Solve the IVP $X(0) = (2, -1)^T$ and verify both the equations and initial data.
4. For this IVP, find the first zero of its first component for $t \geq 0$. Determine the full state and derivative there, and explain why a zero component does not mean a zero solution.

Question 1 – Solution

Strategy. Check vector residuals first; then solve the triangular equations to establish completeness.

Step 1: Test all candidates. Direct differentiation gives $U' - AU = 0$ and $V' - AV = 0$. For the third candidate,

$$W' = e^t(1, t+1)^T, \quad AW = e^t(1+t, t)^T, \quad \boxed{W' - AW = e^t(-t, 1)^T}.$$

Its second residual component never vanishes, so W is not a solution on any nonempty open interval.

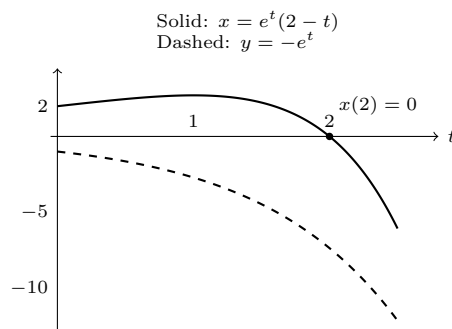
Step 2: Prove that the valid family is complete. The matrix $\Phi = (U \ V) = e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ has determinant $e^{2t} > 0$. More explicitly, the second equation gives $y = c_2 e^t$. Then $(e^{-t}x)' = c_2$, so

$$\boxed{X = e^t(c_1 + c_2 t, c_2)^T = c_1 U + c_2 V}.$$

This derivation reaches every solution and introduces exactly two arbitrary constants. The family is valid on the whole real axis.

Step 3: Select and verify the initial-value solution. At 0, the constants are $c_1 = 2$, $c_2 = -1$, hence $\boxed{X = e^t(2-t, -1)^T}$. Its derivative is $e^t(1-t, -1)^T$, exactly AX , and its initial state is $(2, -1)^T$ as required.

Step 4: Interpret a component zero. Since $e^t \neq 0$, the first component first vanishes at $t = 2$. There $X(2) = (0, -e^2)^T$ and $X'(2) = (-e^2, -e^2)^T$. The state is nonzero and continues moving; the first component crosses zero with negative slope. For a homogeneous linear system with continuous coefficients, only a full zero state at a time would force the identically zero solution by uniqueness. The plot displays both components.



Question 2

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}.$$

A principal fundamental matrix at 0 is a fundamental matrix Y satisfying $Y(0) = I$. For real a , also define $R_a = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ and $\Psi_a = \Phi R_a$.

Tasks

1. Verify $\Phi' = A\Phi$ and calculate its determinant. Explain why its columns describe a complete homogeneous solution family.
2. Construct the principal fundamental matrix $Y(t)$ and verify $Y(0) = I$. Express its entries using hyperbolic functions or exponentials.
3. For arbitrary initial data $X(0) = (p, q)^T$, find the constants in $X = \Phi c$ and the same solution in the form $X = YX(0)$.
4. Classify when Ψ_a is a fundamental matrix. At $a = 1$ and $a = -1$, determine exactly which initial states can still be represented by $X = \Psi_a d$, even though every column remains a solution.

Question 2 – Solution

Strategy. Distinguish solution columns from a complete basis, and normalize by the matrix at the initial time.

Step 1: Verify the columns and independence. Differentiating each exponential column gives $\Phi' = A\Phi$. Its determinant is -2 , never zero. Every initial vector can therefore be represented by $\Phi(0)c$; uniqueness of the linear IVP proves that the resulting constant combination supplies every solution.

Step 2: Normalize at the initial time. Since $\Phi(0)^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

$$Y(t) = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

Here $\cosh t = (e^t + e^{-t})/2$, $\sinh t = (e^t - e^{-t})/2$. Thus $Y(0) = I$ and $Y' = AY$ by differentiation or constant right multiplication.

Step 3: Recover arbitrary data. Solving $\Phi(0)c = (p, q)^T$ gives

$$c_1 = (p + q)/2, \quad c_2 = (p - q)/2, \quad X = (p \cosh t + q \sinh t, \quad p \sinh t + q \cosh t)^T.$$

This equals $Y(t)(p, q)^T$. The constants in a nonnormalized basis are not generally the initial state coordinates themselves.

Step 4: Detect a collapsed family. Constant combinations remain solutions, but $\det \Psi_a = -2(1 - a^2)$, so the family is fundamental exactly when $a \neq \pm 1$. At $a = 1$, both columns are $U + V$, whose value at 0 is $(2, 0)^T$; exactly the initial states $(p, 0)^T$ are representable. At $a = -1$, the columns are $U - V$ and its negative, with initial value $(0, 2)^T$; exactly $(0, q)^T$ are representable. Here U, V denote the columns of Φ . Neither singular case supplies all initial states, despite having two displayed solution columns.

Question 3

Consider the continuous time-dependent linear system

$$X' = A(t)X, \quad A(t) = \begin{pmatrix} 2t & 1 \\ 0 & -t \end{pmatrix}.$$

For two solution columns $U = (u_1, u_2)^T$, $V = (v_1, v_2)^T$, write $W = u_1v_2 - u_2v_1$. Do not assume a determinant evolution formula without deriving it.

Tasks

1. Differentiate W and use the original system to derive its scalar differential equation.
2. Find $W(t)$ when the matrix of columns equals I at $t = 0$. Is the claim $W = e^{t^2}$ consistent with the system?
3. Construct the principal fundamental matrix at 0. A definite integral is an acceptable exact entry; verify the initial matrix and both column equations.
4. Find the solution with initial state $(0, 1)^T$. Explain why growth of the fundamental determinant for $t > 0$ does not imply growth of every solution component.

Question 3 – Solution

Strategy. Derive the determinant law by the product rule, and construct the columns by solving the triangular equations.

Step 1: Differentiate the determinant. Substitution gives

$$W' = (2tu_1 + u_2)v_2 + u_1(-tv_2) - (-tu_2)v_1 - u_2(2tv_1 + v_2) = tW.$$

The off-diagonal terms cancel. This is the trace law here, since $\text{tr } A = t$, obtained directly rather than assumed.

Step 2: Apply the determinant's initial value. At 0, $W(0) = 1$, and solving $W' = tW$ gives $W(t) = e^{t^2/2}$. The proposed e^{t^2} has derivative $2te^{t^2}$ and fails the required equation except at the isolated time 0; matching the initial value is insufficient.

Step 3: Construct and verify the matrix. Put $J(t) = \int_0^t e^{-3s^2/2} ds$. The second equation gives $y = c_2 e^{-t^2/2}$, and the first gives $(e^{-t^2} x)' = c_2 e^{-3t^2/2}$. Therefore

$$\Phi(t) = \begin{pmatrix} e^{t^2} & e^{t^2} J(t) \\ 0 & e^{-t^2/2} \end{pmatrix}.$$

Since $J(0) = 0$, $\Phi(0) = I$. The derivative of its upper-right entry is $2te^{t^2} J + e^{-t^2/2}$, precisely the required first-row expression. The remaining entries satisfy their rows directly. Its determinant agrees with Step 2 and never vanishes, so this is a principal fundamental matrix.

Step 4: Interpret the selected solution. The initial vector $(0, 1)^T$ selects the second column: $X = (e^{t^2} J(t), e^{-t^2/2})^T$. Its second component strictly decreases for $t > 0$, even while W grows. A determinant concerns a pair of independent columns, not the separate monotonicity of each component. The nonzero determinant guarantees a complete independent family throughout the real axis.

Question 4

For the nonautonomous system

$$x' = 2ty, \quad y' = 0,$$

define the transition matrix $T(t, s)$ by $X(t) = T(t, s)X(s)$ for arbitrary real starting time s . It must satisfy $T(s, s) = I$.

Tasks

1. Derive $T(t, s)$ from the component equations and verify its differential equation and normalization.
2. Prove $T(t, r)T(r, s) = T(t, s)$ and determine the inverse of $T(t, s)$. Explain the role of the intermediate time.
3. Let $Y(t) = T(t, 0)$. Determine whether $Y(t + s) = Y(t)Y(s)$ holds for all real t, s , and give a concrete counterexample if it fails.
4. Solve the IVP $X(-1) = (2, 3)^T$. Compare $X(1)$ with $X(-1)$ and compute $X(0)$. Does either equality imply that the solution is constant?

Question 4 – Solution

Strategy. Keep both endpoint times: a time-dependent equation need not evolve according only to elapsed time.

Step 1: Integrate from the actual starting time. The second component is $y(t) = y(s)$, and $x(t) = x(s) + y(s) \int_s^t 2u \, du$. Hence

$$T(t, s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.$$

Its derivative with respect to t is $\begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}$, which equals $A(t)T(t, s)$, and $T(s, s) = I$.

Step 2: Check composition and inversion. Multiplying two such shears adds their upper-right entries: $(t^2 - s^2) + (s^2 - r^2) = t^2 - r^2$. Thus

$$T(t, r)T(r, s) = T(t, s), \quad T(t, s)^{-1} = T(s, t).$$

The intermediate time cancels only because it is the end of the first journey and the start of the second, in the correct order.

Step 3: Test the one-parameter shortcut. The upper-right entry of $Y(t)Y(s)$ is $t^2 + s^2$, whereas that of $Y(t+s)$ is $(t+s)^2 = t^2 + s^2 + 2ts$. They agree only when $ts = 0$, not for all times. For example, $Y(1)Y(1)$ has upper-right entry 2, while $Y(2)$ has entry 4. This does not contradict the two-time composition identity.

Step 4: Apply the transition to the IVP. The solution is

$$X(t) = T(t, -1)(2, 3)^T = (3t^2 - 1, 3)^T.$$

It returns to $(2, 3)^T$ at $t = 1$, and $X'(0) = (0, 0)^T$. Nevertheless $X' = (6t, 0)^T$ is nonzero away from 0, so the solution is not constant. In a nonautonomous system, a repeated state at different times or a zero derivative at one instant does not force stationary evolution.

Question 5

Consider

$$X' = AX + f(t), \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad f(t) = (1, 2e^{-t})^T.$$

The proposed particular solution is $P(t) = (1, 2e^{-t})^T$. Distinguish combinations of homogeneous solutions from combinations of solutions of this fixed nonhomogeneous system.

Tasks

1. Verify P , and prove that the full solution family is $P + c_1(e^{-t}, 0)^T + c_2(0, e^{-2t})^T$.
2. If U, V both solve the fixed forced system, determine exactly when $\alpha U + \beta V$ also solves it. Prove the condition for real constants α, β .
3. Solve the IVP $X(0) = (0, 0)^T$ and check its initial derivative against the original right-hand side.
4. For $t \geq 0$, find the maximum of the second component of this IVP and its time. Find the limiting state and decide whether that limiting vector, held constant, is itself a solution of the forced system.

Question 5 – Solution

Strategy. Subtract one particular solution to expose the homogeneous family; keep the forcing when testing superposition.

Step 1: Verify and complete the affine family. Here $P' = (0, -2e^{-t})^T = AP + f$. For any other solution X , the difference $H = X - P$ satisfies $H' = AH$, so $H = (c_1e^{-t}, c_2e^{-2t})^T$. Conversely each such sum solves the forced equation. Thus the proposed family is complete.

Step 2: Test a weighted combination. For $Z = \alpha U + \beta V$,

$$Z' - AZ = (\alpha + \beta)f(t).$$

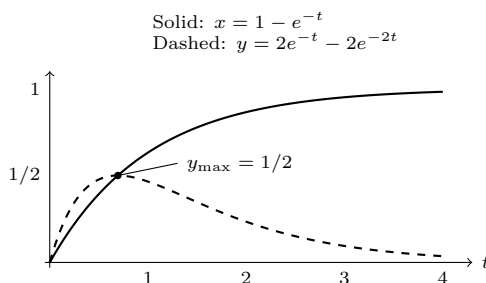
Because the first component of f is always 1, equality to f holds exactly when $\boxed{\alpha + \beta = 1}$. In particular an average is a solution, while a sum generally is not. The difference of two forced solutions solves the homogeneous equation.

Step 3: Impose the zero initial state. The constants are $c_1 = -1$, $c_2 = -2$, yielding

$$\boxed{X(t) = (1 - e^{-t}, 2e^{-t} - 2e^{-2t})^T}.$$

At 0 this is zero, but $X'(0) = (1, 2)^T = f(0)$. A zero initial state therefore does not imply a zero solution in this nonhomogeneous problem.

Step 4: Locate the peak and interpret the limit. For $y = 2e^{-t}(1 - e^{-t})$, $y' = 2e^{-t}(2e^{-t} - 1)$. It is positive before $\ln 2$ and negative after, so the unique maximum on $t \geq 0$ is $\boxed{y(\ln 2) = 1/2}$. The state tends to $(1, 0)^T$. However the constant vector $(1, 0)^T$ has zero derivative while $A(1, 0)^T + f(t) = (0, 2e^{-t})^T \neq 0$ at finite times. A limiting state need not itself be a constant solution of a time-dependent forced equation. The plot marks the actual second-component peak.



Question 6

For $X' = AX$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, consider

$$U(t) = (\cos t, -\sin t)^T, \quad V(t) = (\sin t, \cos t)^T.$$

You may use uniqueness of linear initial-value problems with continuous coefficients. The independent variable is real time.

Tasks

1. Verify both solutions and state their initial vectors.
2. Find all real times when their first components agree. At those times compare their second components and decide whether the full states coincide.
3. Prove that two solutions of the same continuous linear system agreeing in their full state at one time must agree throughout their common interval. Apply this to the zero solution of a homogeneous system.
4. Find a constant time shift relating V to U . Explain why identical states at different times are compatible with your uniqueness statement.

Question 6 – Solution

Strategy. Keep equality of one coordinate, equality of full states at the same time, and a time-shift relation separate.

Step 1: Verify the two vector equations. Differentiation gives $U' = (-\sin t, -\cos t)^T = AU$ and $V' = (\cos t, -\sin t)^T = AV$. Their initial vectors are $U(0) = (1, 0)^T$ and $V(0) = (0, 1)^T$.

Step 2: Find the coordinate coincidences. The equation $\cos t = \sin t$ holds exactly at

$$t = \pi/4 + k\pi, \quad k \in \mathbb{Z}.$$

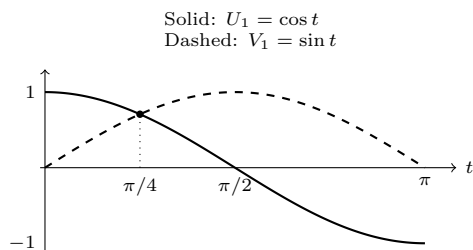
At such a time the common first component is $(-1)^k/\sqrt{2}$. The second components are its negative for U and its positive for V , so they differ. More generally,

$$\|U(t) - V(t)\|^2 = (\cos t - \sin t)^2 + (-\sin t - \cos t)^2 = 2.$$

The full states never agree at the same time. The figure displays first components only, and marks one coordinate coincidence.

Step 3: State what uniqueness actually implies. Suppose two solutions share the full vector $X(t_0) = x_0$. They solve the same IVP at t_0 , so uniqueness makes them identical on their common interval. Equivalently their difference solves the homogeneous equation with zero initial data and must vanish throughout that interval. Taking one solution to be zero proves that a homogeneous solution reaching the full zero state cannot later depart from it.

Step 4: Check the shifted identity. Trigonometric identities give $V(t) = U(t - \pi/2)$. This compares states at different time arguments; it does not assert $U(t) = V(t)$. An autonomous system permits translating a solution in time, and the shifted function normally has different initial data at time 0. Thus the shift creates no conflict with same-time IVP uniqueness.



Question 7

For a real parameter a , solve the nonlinear IVP

$$x' = x^2, \quad y' = -y, \quad x(0) = a, \quad y(0) = 1.$$

A maximal interval means the largest connected open time interval containing 0 on which the classical solution exists. The vector field is smooth.

Tasks

1. Derive the solution for $a \neq 0$, and handle $a = 0$ without dividing by the zero solution.
2. Find the maximal interval for each sign of a . Explain why smoothness of the field guarantees local uniqueness but not necessarily existence for all real times.
3. At $a = 1$, the rational formula is finite at $t = 2$. Does that value belong to a continuation of the stated IVP through $t = 1$? Justify your answer.
4. Compare $a = 0$ with $a = \varepsilon > 0$. Prove a small-data bound on any fixed $[0, T]$ when $\varepsilon T \leq 1/2$, and evaluate the difference at $t = 1/\varepsilon - 1$ for $0 < \varepsilon < 1$. Explain why the two conclusions are consistent.

Question 7 – Solution

Strategy. Derive explicit solutions, then distinguish the algebraic formula's domain from the connected interval of an initial-value solution.

Step 1: Solve and verify, retaining the zero case. For $a \neq 0$, separation gives $1/x = 1/a - t$, hence

$$X_a(t) = \left(\frac{a}{1-at}, e^{-t} \right)^T.$$

The first derivative is $a^2/(1-at)^2 = x^2$, and the data match at 0. For $a = 0$, the solution is $X_0 = (0, e^{-t})^T$; local uniqueness of the smooth field excludes other departures from $x = 0$.

Step 2: Find the maximal connected intervals. The first component blows up at $1/a$ when $a \neq 0$. Thus

$$a > 0 : (-\infty, 1/a); \quad a = 0 : \mathbb{R}; \quad a < 0 : (1/a, \infty).$$

The exponential component introduces no further restriction. A finite-valued continuous extension across the pole is impossible. Smoothness is a local regularity condition and does not prevent unbounded growth in finite time.

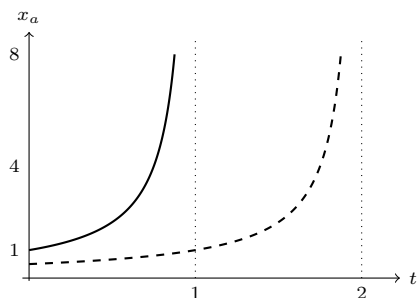
Step 3: Reject a disconnected continuation. At $a = 1$, the formula gives $x(2) = -1$, but the IVP through 0 exists only for $t < 1$. Its first component tends to $+\infty$ as $t \uparrow 1$. The branch for $t > 1$ solves the differential equation separately; it cannot be joined into a classical solution through the intervening pole.

Step 4: Compare finite-time and long-time sensitivity. For $0 \leq t \leq T$ with $\varepsilon T \leq 1/2$,

$$\|X_\varepsilon(t) - X_0(t)\| = \frac{\varepsilon}{1 - \varepsilon t} \leq 2\varepsilon.$$

But at $t = 1/\varepsilon - 1$, still before the pole, the difference equals 1. That comparison time tends to infinity as $\varepsilon \downarrow 0$. There is no contradiction: closeness on each fixed finite interval does not give a uniform bound over arbitrarily long, parameter-dependent times. The plot shows only the initial-value branches before their poles.

Solid: $a = 1$; dashed: $a = 1/2$
Dotted verticals: poles; $a = 0$ stays on the time axis.



Positive branches shown up to $x = 8$; no continuation across a pole.

Question 8

Let $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $e_1 = (1, 0)^T$. Consider the periodically forced system

$$X' = AX + R(t)e_1.$$

You may use $R' = AR$, $R^{-1} = R(-t)$ and $R(t+2\pi) = R(t)$. A 2π -periodic solution satisfies $X(t+2\pi) = X(t)$ for all real t .

Tasks

1. Verify that $X_c(t) = R(t)(c + te_1)$ is a solution for every constant vector c . Determine its initial state.
2. Prove that this family contains every solution, by differentiating $R(-t)X(t)$.
3. Compute the change in state over one forcing period and decide whether any choice of initial data produces a 2π -periodic solution.
4. Replace e_1 in the forcing by an arbitrary constant vector $b = (\alpha, \beta)^T$. Find the full solution family and classify exactly when a 2π -periodic solution exists.

Question 8 – Solution

Strategy. Factor out the supplied rotating matrix and test periodicity on the full solution, not just on its coefficients and forcing.

Step 1: Verify the proposed family directly. The product rule gives

$$X'_c = AR(t)(c + te_1) + R(t)e_1 = AX_c + R(t)e_1.$$

Also $X_c(0) = c$. Thus every initial state has a member of this family.

Step 2: Prove completeness independently. Set $Z = R(-t)X$. Differentiating the inverse identity gives $(R^{-1})' = -R^{-1}A$. Consequently

$$Z' = -R^{-1}AX + R^{-1}(AX + Re_1) = e_1.$$

It follows that $Z = c + te_1$, so $X = R(t)(c + te_1)$ is the entire family, valid on the real axis.

Step 3: Examine one full forcing period. Periodicity of R gives

$$X(t + 2\pi) - X(t) = 2\pi R(t)e_1.$$

This difference has length 2π and never vanishes, independently of c . In particular $X(2\pi) = X(0) + (2\pi, 0)^T$. No initial state gives a 2π -periodic solution. A periodic forcing can produce a response with an accumulating change rather than a periodic response.

Step 4: Classify the general constant-amplitude forcing. The same calculation gives

$$X = R(t)(c + tb), \quad X(t + 2\pi) - X(t) = 2\pi R(t)b.$$

Since $R(t)$ is invertible, a periodic solution exists exactly when $b = 0$, that is, $\alpha = \beta = 0$. In this case every member $X = R(t)c$ is 2π -periodic, including the zero solution. If $b \neq 0$, no choice of c can cancel the per-period difference. The classification concerns this specified forcing family; it does not claim that every nonzero periodic forcing precludes a periodic response.

Question 9

For

$$x' = y, \quad y' = -x,$$

only the measurements $x(0) = 1$ and $x(T) = b$ are given, where $T > 0$. The second initial component is unknown. You may verify solutions using sine and cosine identities; no phase-plane analysis is required.

Tasks

1. Find every solution consistent with $x(0) = 1$, using one arbitrary constant, and identify what that constant means.
2. Classify existence and uniqueness for all $T > 0$ and $b \in \mathbb{R}$, including the exceptional times $T = k\pi$.
3. Apply the classification to $(T, b) = (\pi, -1)$ and $(\pi/2, 2)$. Give the full solution set in each case.
4. When $\sin T \neq 0$, suppose the reported value of b has an error δ . Find the exact resulting error in the recovered second initial component. Explain what happens as $T \rightarrow \pi$ and why this is not failure of uniqueness for a fully specified IVP.

Question 9 – Solution

Strategy. Treat the unknown initial velocity as a parameter and ask whether the observations actually determine it.

Step 1: Parametrize all compatible initial states. Since $x'' = -x$ and $x(0) = 1$, the complete family is

$$x = \cos t + c \sin t, \quad y = -\sin t + c \cos t.$$

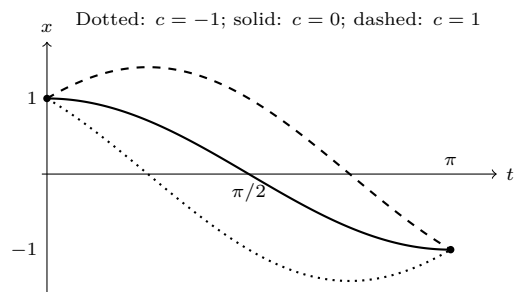
Direct differentiation verifies both rows. The arbitrary constant is $c = y(0) = x'(0)$, the missing initial component.

Step 2: Test the observation equation. The second measurement requires $\cos T + c \sin T = b$. If $\sin T \neq 0$, there is exactly one solution, with $c = (b - \cos T)/\sin T$. If $T = k\pi$ for a positive integer k , the measurement is independent of c : there are infinitely many solutions when $b = (-1)^k$, and none when $b \neq (-1)^k$. These cases exhaust all positive times.

Step 3: Evaluate the two specified experiments. At $(\pi, -1)$ every real c is allowed, giving the full family in Step 1. The figure shows three of these indistinguishable endpoint measurements. At $(\pi/2, 2)$, $c = 2$, so the unique solution is

$$X = (\cos t + 2 \sin t, -\sin t + 2 \cos t)^T.$$

Step 4: Quantify sensitivity and interpret it. Replacing b by $b + \delta$ changes c by exactly $\Delta c = \delta / \sin T$. Thus the absolute amplification factor is $1/|\sin T|$, which is unbounded as $T \rightarrow \pi$ through nonexceptional times. A fixed nonzero measurement error can then create a large recovered initial-velocity error. At $T = \pi$, that component is unobservable from these two measurements. Each fully specified initial vector still has a unique solution; the difficulty lies in recovering it from partial observations.



All share $x(0) = 1$ and $x(\pi) = -1$; initial velocities differ.

Question 10

An unknown continuous homogeneous linear system has the two specified solution columns

$$\Phi(t) = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}.$$

We ask whether these columns determine a unique system $X' = A(t)X$, rather than assuming that a displayed pair automatically comes from a chosen matrix.

Tasks

1. Find $A(t)$ and verify $\Phi' = A\Phi$. Prove uniqueness of the recovered coefficient matrix at every real time.
2. Write the complete solution family and solve the IVP $X(1) = (2, 3)^T$. Verify the recovered initial state.
3. Could both columns instead solve an uncoupled system $x' = a(t)x$, $y' = b(t)y$ on any nonempty open interval? Give a direct contradiction using their first components.
4. Compute the determinant and trace relevant to this fundamental matrix. Does its constant determinant imply that every solution has constant Euclidean length? Test the second column and explain the conclusion.

Question 10 – Solution

Strategy. An invertible matrix of solution columns determines its coefficient matrix through its derivative, while its determinant alone does not determine lengths.

Step 1: Recover and verify the matrix. The determinant of Φ is 1, so

$$\Phi^{-1} = \begin{pmatrix} 1+t^2 & -t \\ -t & 1 \end{pmatrix}, \quad \boxed{A = \Phi' \Phi^{-1} = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}.}$$

Multiplication gives $A\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 2t \end{pmatrix} = \Phi'$. Any coefficient matrix satisfying the column equations must equal $\Phi' \Phi^{-1}$, proving uniqueness at every real time.

Step 2: Select the constants for the IVP. The full solution is $\boxed{X = \Phi(t)c}$ with constant $c \in \mathbb{R}^2$. Indeed $(\Phi^{-1}X)' = 0$ for any solution, proving completeness. At time 1,

$$c = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} (2, 3)^T = (1, 1)^T.$$

Thus $\boxed{X = (1+t, 1+t+t^2)^T}$, which equals $(2, 3)^T$ at 1. Its derivative follows either directly or from the verified column identity.

Step 3: Rule out an uncoupled alternative. The first component of the first column is the constant 1. Its equation would force $a(t) = 0$ throughout the interval. The first component of the second column is t , whose derivative is 1; with $a = 0$ its proposed equation would instead give derivative 0. This is impossible on any nonempty interval. The coupling is essential.

Step 4: Separate independence from length preservation. Here $\det \Phi = 1$ and $\text{tr } A = -t + t = 0$, consistent with the constant determinant. But the second solution column has squared length

$$\boxed{t^2 + (1+t^2)^2 = 1 + 3t^2 + t^4,}$$

which is not constant. A nonzero determinant ensures independence of the columns; even a constant determinant does not force individual solution lengths to stay fixed. The explicit column supplies a counterexample.