

Question 1

Consider the scalar initial-value problem, for $t > 0$,

$$y^{(3)} + \frac{1}{t}y'' - 2y' + y = e^t, \quad y(1) = 2, \quad y'(1) = -1, \quad y''(1) = 0.$$

A first-order state must contain enough information to determine all of its first derivatives. A system is autonomous when its right-hand side has no explicit dependence on the independent variable.

Tasks

1. Use $X = (y, y', y'')^T$ to write $X' = A(t)X + g(t)$ and the complete initial state.
2. Classify the system as linear or nonlinear, homogeneous or nonhomogeneous, and autonomous or nonautonomous. State its dimension and the largest coefficient-continuity interval containing $t = 1$.
3. Use instead $Z = (y'', y, y')^T$. Find the transformed system and initial state, explaining why reordering state variables changes neither the underlying problem nor the number of initial values.
4. Prove equivalence in both directions between the scalar equation and your first state system. Explain what the coefficient singularity does, and does not, establish about continuation through $t = 0$.

Question 1 – Solution

Strategy. Store successive derivatives as independent state coordinates, then recover their required relations from the first rows.

Step 1: Build the derivative chain. Set $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Solving for the highest derivative gives

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -1/t \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}, \quad X(1) = (2, -1, 0)^T.$$

In particular the last row is $x_3' = -x_1 + 2x_2 - x_3/t + e^t$.

Step 2: Separate the classifications. The three-dimensional system is linear: coefficients depend on t , not on the unknown state. It is nonhomogeneous because the forcing is nonzero, and nonautonomous because $1/t$ and e^t depend explicitly on time. The largest open coefficient-continuity interval containing 1 is $(0, \infty)$.

Step 3: Reorder the coordinates consistently. Here $z_1 = x_3$, $z_2 = x_1$, $z_3 = x_2$, so

$$Z' = \begin{pmatrix} -1/t & -1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} Z + \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}, \quad Z(1) = (0, 2, -1)^T.$$

A fixed permutation is invertible. It relabels coordinates and initial data without adding or removing information; there are still three initial values.

Step 4: Check equivalence and the domain limitation. Any scalar solution supplies a state satisfying all three rows. Conversely, for a continuously differentiable state solution, the first two rows imply $x_2 = x_1'$ and $x_3 = x_1''$; the third then makes x_1 three times differentiable and gives the original equation and initial data. The standard linear-system theorem gives a unique solution on $(0, \infty)$. It does not apply through 0, where $1/t$ is undefined. This alone does not prove that a particular solution cannot have a smooth extension; any extension would require a separately specified equation at the singular point.

Question 2

Two masses move horizontally without friction. Their displacements u, v are measured to the right from equilibrium. Mass 1 has mass 1, is attached to a fixed left wall by a spring of stiffness 2, and is coupled to mass 2 by a spring of stiffness 3. Mass 2 has mass 2 and no wall spring. A force $f(t)$ acts to the right on mass 2. Use consistent mechanical units. Initially $(u, u', v, v') = (1, 0, 0, 0)$.

Tasks

1. Derive the two second-order force equations, checking the opposite signs of the coupling forces.
2. For $X = (u, u', v, v')^T$, give the first-order matrix system and initial state. Explain why (u, v) alone is insufficient.
3. When $f = 0$, differentiate $E = \frac{1}{2}(u')^2 + (v')^2 + u^2 + \frac{3}{2}(v - u)^2$ and prove conservation. For general f , find the exact power balance.
4. Find all equilibrium states for a constant force $f(t) = F$. Explain why equilibrium requires conditions on velocities as well as displacements.

Question 2 – Solution

Strategy. Apply Newton's law to each mass and use energy to check the coupling signs independently.

Step 1: Account for the spring forces. The wall spring exerts $-2u$ on mass 1. The coupling spring exerts $3(v - u)$ on mass 1 and its opposite on mass 2. Therefore

$$u'' = -5u + 3v, \quad 2v'' = 3u - 3v + f(t).$$

Dividing the second equation by its mass is essential.

Step 2: Include positions and velocities. Writing $p = u'$, $q = v'$ gives

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 3/2 & 0 & -3/2 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t)/2 \end{pmatrix}, \quad X(0) = (1, 0, 0, 0)^T.$$

Equal positions can have different velocities, so positions alone do not determine the first derivatives. The four coordinates supply the complete state.

Step 3: Verify the work-energy balance. Differentiate the stated energy before substituting the equations:

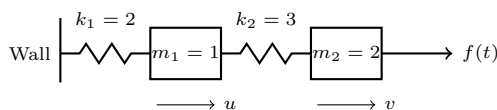
$$E' = p(-5u + 3v) + q(3u - 3v + f) + 2up + 3(v - u)(q - p).$$

Every spring term cancels, leaving $E' = f(t)v'$. Thus $f = 0$ conserves E ; for the stated initial data, $E(0) = 5/2$. The two kinetic terms reflect the different masses. The remaining terms are the nonnegative spring potential energies.

Step 4: Balance a constant force. An equilibrium has $p = q = 0$, $-5u + 3v = 0$, and $3u - 3v + F = 0$. Consequently $u = F/2$, $v = 5F/6$, and

$$X_* = (F/2, 0, 5F/6, 0)^T.$$

This is the unique equilibrium. Zero acceleration at one instant would not make a state constant if either velocity were nonzero.



Displacements from equilibrium; rightward positive.

Question 3

Two perfectly mixed tanks have constant volumes 100 L and 200 L. Fresh water enters tank 1 at 3 L/min. Tank 1 sends 5 L/min to tank 2; tank 2 returns 2 L/min to tank 1 and drains 3 L/min outside. Let x, y be the salt amounts in grams, with $x(0) = 100$, $y(0) = 0$. You may use the inward-boundary criterion: for a locally Lipschitz vector field, the nonnegative quadrant is forward invariant if its boundary derivatives point inward or are tangent.

Tasks

1. Check both volume balances and derive the first-order system for salt amounts. Identify the concentration used in each outflow term.
2. Write the amount system in matrix form. Then derive the system for concentrations $c_1 = x/100$, $c_2 = y/200$, including initial data.
3. Find the derivative of total salt and explain why the internal transfer rates cancel even though the volumes differ. Find every equilibrium.
4. Prove that nonnegative initial amounts remain nonnegative and that total salt cannot increase. Give explicit bounds for this initial state without solving the coupled system.

Question 3 – Solution

Strategy. Multiply each flow rate by the concentration in its source tank, then check the total balance.

Step 1: Balance volumes and salt separately. Tank 1 receives $3 + 2 = 5$ L/min and sends 5; tank 2 receives 5 and sends $2 + 3 = 5$. Their volumes are constant. The return concentration is $y/200$ and the forward concentration is $x/100$, giving

$$\boxed{x' = -x/20 + y/100, \quad y' = x/20 - y/40.}$$

Fresh water supplies no salt. Each coefficient has units min^{-1} .

Step 2: Change from amounts to concentrations. For $X = (x, y)^T$,

$$X' = \begin{pmatrix} -1/20 & 1/100 \\ 1/20 & -1/40 \end{pmatrix} X, \quad X(0) = (100, 0)^T.$$

Dividing each balance by its own tank volume gives

$$\boxed{c' = \begin{pmatrix} -1/20 & 1/50 \\ 1/40 & -1/40 \end{pmatrix} c, \quad c(0) = (1, 0)^T.}$$

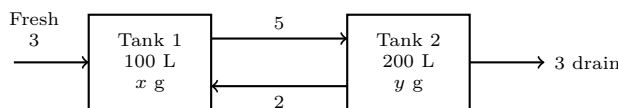
This is a constant invertible change of state; the two matrices need not match.

Step 3: Check the external balance and equilibria. Adding the amount equations gives

$$\boxed{(x + y)' = -3y/200},$$

exactly the outside drain rate times its source concentration. At an equilibrium this forces $y = 0$, and then $y' = x/20 = 0$ forces $x = 0$. The only equilibrium is the salt-free state.

Step 4: Prove positivity and bounds. The field is linear and therefore locally Lipschitz. On $x = 0$, $y \geq 0$, we have $x' = y/100 \geq 0$; on $y = 0$, $x \geq 0$, we have $y' = x/20 \geq 0$. The given inward-boundary criterion therefore proves that the nonnegative quadrant is forward invariant. The constant linear system exists for all forward times, so the conclusion holds for every $t \geq 0$. Now the total balance implies $0 \leq x + y \leq 100$. Hence $0 \leq x, y \leq 100$ g, $0 \leq c_1 \leq 1$ and $0 \leq c_2 \leq 1/2$ g/L. These are guaranteed bounds, not claims that every bound is attained.



All flow rates in L/min; arrows show actual transfer directions.

Question 4

Consider the first-order system on the real time axis

$$x' = x + y, \quad y' = 2x - y + t, \quad x(0) = 1, \quad y(0) = 2.$$

An equivalent scalar equation must preserve both the dynamics and the initial information; an eliminated coordinate must be recoverable.

Tasks

1. Eliminate y to derive a second-order equation for x . Find the two corresponding initial values.
2. Prove the converse: from any solution of your scalar IVP, reconstruct y and verify both original equations and both original initial values.
3. Instead eliminate x to obtain a scalar IVP for y and a reconstruction formula for x . Verify the forcing signs.
4. A calculation uses the correct scalar equation for x but imposes $x(0) = 1$, $x'(0) = 2$. Identify exactly which original initial condition this changes. Explain why prescribing both $x'(0) = 2$ and the original $y(0) = 2$ is inconsistent.

Question 4 – Solution

Strategy. Differentiate one row, substitute the other, and retain the algebraic formula that recovers the missing coordinate.

Step 1: Eliminate y while retaining initial information. The first row gives $y = x' - x$. Differentiating it and using the second gives $x'' = x' + 2x - y + t = 3x + t$. Thus

$$x'' - 3x = t, \quad x(0) = 1, \quad x'(0) = 3.$$

The initial slope is $x(0) + y(0)$, not the initial value of y alone.

Step 2: Reconstruct and verify. Given a twice differentiable solution of this scalar IVP, define $y = x' - x$. Then $x' = x + y$ identically, and

$$y' = x'' - x' = 3x + t - x' = 2x - (x' - x) + t = 2x - y + t.$$

Also $y(0) = 3 - 1 = 2$. This proves the converse as well as the forward implication: no extra scalar solutions remain after reconstruction and data matching.

Step 3: Eliminate in the other direction. The second row gives $x = (y' + y - t)/2$. Differentiating that row yields

$$y'' = 2x' - y' + 1 = 2x + 2y - y' + 1 = 3y + 1 - t.$$

Therefore the equivalent alternative is

$$y'' - 3y = 1 - t, \quad y(0) = 2, \quad y'(0) = 0, \quad x = \frac{y' + y - t}{2}.$$

Indeed $x' = (y'' + y' - 1)/2 = (3y + y' - t)/2 = x + y$; the reconstruction also gives $x(0) = 1$ and the second row directly.

Step 4: Diagnose the mismatched data. If $x(0) = 1$ and $x'(0) = 2$, reconstruction forces $y(0) = 2 - 1 = 1$. The calculation solves a different original IVP, with initial state $(1, 1)$. Insisting simultaneously on $y(0) = 2$ would require $x'(0) = 1 + 2 = 3$, contradicting the imposed slope. Correct differential equations alone do not make two initial-value problems equivalent.

Question 5

Let

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X(0) = (0, 1)^T,$$

and introduce a time-dependent coordinate system $X = P(t)Z$, where

$$P(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The physical state X and its coordinate column Z describe the same vector using different bases at different times.

Tasks

1. Derive the differential equation for Z by differentiating $X = PZ$. Explain why the constant-basis formula $P^{-1}AP$ is insufficient.
2. Compute the coefficient matrix and initial state for Z . Classify the transformed system as linear, homogeneous, and autonomous or nonautonomous.
3. Verify the supplied physical trajectory $X(t) = (\sin t, \cos t)^T$. Obtain $Z(t)$ and check both transformed equations directly.
4. Compare $X'(0)$ with $Z'(0)$. Explain how a zero instantaneous coordinate derivative can coexist with a changing physical vector.

Question 5 – Solution

Strategy. A moving basis contributes its own derivative; keep that contribution before multiplying by the inverse.

Step 1: Differentiate the whole change of coordinates. Since $X' = P'Z + PZ' = APZ$, invertibility of P gives

$$Z' = P^{-1}(AP - P')Z.$$

Dropping P' would treat a moving basis as fixed and generally produce incorrect coordinate derivatives. Here $\det P = 1$ for all real t .

Step 2: Compute the transformed system. The matrices are

$$P^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad AP - P' = \begin{pmatrix} 0 & 0 \\ -1 & -t \end{pmatrix}.$$

Consequently

$$Z' = \begin{pmatrix} t & t^2 \\ -1 & -t \end{pmatrix} Z, \quad Z(0) = (0, 1)^T.$$

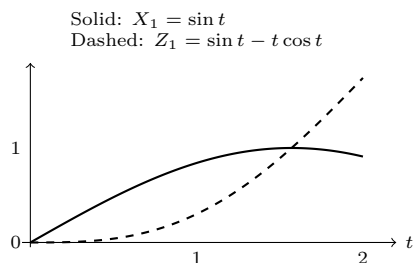
This system is linear and homogeneous, but nonautonomous. Explicit time dependence can arise entirely from a time-dependent choice of coordinates.

Step 3: Check the supplied trajectory in both descriptions. For $X = (\sin t, \cos t)^T$, $X' = (\cos t, -\sin t)^T = AX$, with the stated data. Multiplication by P^{-1} gives

$$Z = (\sin t - t \cos t, \cos t)^T.$$

Its derivative is $(t \sin t, -\sin t)^T$. The transformed right-hand side is also $(t(\sin t - t \cos t) + t^2 \cos t, -(\sin t - t \cos t) - t \cos t)^T$, which simplifies to the same vector.

Step 4: Interpret the instantaneous difference. At 0, $X' = (1, 0)^T$ but $Z' = (0, 0)^T$. The identity $X' = P'Z + PZ'$ explains this: $P'(0)Z(0) = (1, 0)^T$. The basis motion accounts for the entire instantaneous change. The figure compares the first components; the coordinate curve starts with a horizontal tangent while the physical component has slope 1.



Question 6

Consider the nonautonomous system

$$x' = tx, \quad y' = -y + \sin t, \quad x(2) = 3, \quad y(2) = -1.$$

A common way to remove explicit time is to add a clock as another state. In an autonomous system, the right-hand side depends only on the state.

Tasks

1. Write this as a two-dimensional linear system $X' = A(t)X + g(t)$ and classify homogeneity and autonomy.
2. Introduce a new independent variable τ and a clock $s(\tau)$ satisfying $s' = 1$. Give an autonomous three-dimensional system with initial data at $\tau = 0$ that reproduces the stated IVP.
3. Prove the correspondence between solutions of the two systems, specifying exactly how t and τ are related. Explain why the clock initial value is required.
4. Classify the augmented system as linear or nonlinear and find its equilibria. Explain why autonomous does not mean linear, and why adding a clock can eliminate all equilibria.

Question 6 – Solution

Strategy. Turn the old independent variable into a dependent variable whose rate is fixed and whose initial value records the time origin.

Step 1: Identify the original structure. With $X = (x, y)^T$,

$$X' = \begin{pmatrix} t & 0 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}, \quad X(2) = (3, -1)^T.$$

It is linear, nonhomogeneous and nonautonomous. Time-dependent coefficients do not violate linearity in the dependent variables.

Step 2: Add and initialize the clock. Write $U(\tau) = (s(\tau), u(\tau), v(\tau))^T$. The autonomous system is

$$\boxed{s' = 1, \quad u' = su, \quad v' = -v + \sin s, \quad U(0) = (2, 3, -1)^T.}$$

Here every prime denotes differentiation with respect to τ . There is no explicit τ on the right-hand side.

Step 3: Prove the change-of-clock correspondence. The first row forces $s(\tau) = 2 + \tau$. Given an original solution, define $u(\tau) = x(2 + \tau)$ and $v(\tau) = y(2 + \tau)$. The chain rule verifies the last two rows and the initial data. Conversely, define $x(t) = u(t - 2)$ and $y(t) = v(t - 2)$ from an augmented solution with $s(0) = 2$; substitution recovers the original system. Intervals shift by the same amount. Choosing $s(0) = 0$ would instead sample the coefficients at $t = \tau$, changing the stated starting time.

Step 4: Distinguish autonomy, linearity and equilibrium. The augmented system is nonlinear: su multiplies two dependent variables, and $\sin s$ is a nonlinear function of a dependent variable. It cannot be classified as a homogeneous linear system merely because the original equations were linear. An equilibrium would require all three right-hand sides to vanish, but $s' = 1$ never vanishes. Thus the augmented autonomous system has no equilibrium. A clock keeps moving even if some physical state components happen to be constant.

Question 7

For a real constant κ , consider the implicit derivative equations

$$x' + y' = -x, \quad x' + \kappa y' = y.$$

A classical solution is a continuously differentiable pair satisfying both relations throughout an interval. An algebraically consistent derivative at one state is not necessarily part of a solution through that state.

Tasks

1. Write $M_\kappa X' = BX$ and determine exactly when it can be solved for a unique derivative at every state. Find the explicit system in that case.
2. At $\kappa = 1$, determine the states at which the two equations are algebraically consistent for a derivative. Describe the possible derivatives at such a state.
3. Still at $\kappa = 1$, find every classical solution on an interval of positive length. Explain why most instantaneously consistent states cannot be used as initial data.
4. For $\kappa \neq 1$ and initial state $(x, y) = (1, -1)$, compute the initial derivative and the derivative of $x + y$. Does a finite, parameter-independent initial velocity imply that a solution exists through the same state at $\kappa = 1$?

Question 7 – Solution

Strategy. Separate invertibility of the derivative matrix from the extra consistency conditions imposed when its rank drops.

Step 1: Solve when the derivative matrix is invertible. Here

$$M_\kappa = \begin{pmatrix} 1 & 1 \\ 1 & \kappa \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det M_\kappa = \kappa - 1.$$

For $\kappa \neq 1$, inversion yields

$$X' = \frac{1}{\kappa - 1} \begin{pmatrix} -\kappa & -1 \\ 1 & 1 \end{pmatrix} X.$$

This constant linear system has a unique solution for every initial state.

Step 2: Test instantaneous algebraic consistency. At $\kappa = 1$, the left sides coincide, so the right sides must satisfy $y = -x$. At a state $(a, -a)$, the derivative pairs (p, q) satisfy $p + q = -a$, an entire line of possibilities. Outside $y = -x$, no derivative can satisfy both equations even at that instant.

Step 3: Enforce the constraint along a solution. A classical solution must have $y(t) = -x(t)$ at every time. Differentiating this identity gives $x' + y' = 0$. The first equation then forces $x = 0$, and the constraint forces $y = 0$. Therefore

$$X(t) \equiv (0, 0)^T \text{ is the only classical solution at } \kappa = 1.$$

It indeed satisfies both equations. Nonzero points on the consistency line admit instantaneous derivatives but no solution interval through them. Even at the origin, most algebraic derivative choices cannot persist.

Step 4: Check the limiting initial velocity. Substituting $(1, -1)$ into the explicit system gives $x' = (-\kappa + 1)/(\kappa - 1) = -1$ and $y' = 0$ for every $\kappa \neq 1$. Thus $(x + y)' = -1$, although initially $x + y = 0$. This velocity does not keep the state on the constraint line required at $\kappa = 1$. Its finiteness does not imply a limiting solution through $(1, -1)$; Step 3 proves that no such solution exists at that parameter.

Question 8

For $t \geq 0$, consider

$$x' = 2\sqrt{|x|}, \quad y' = -y, \quad x(0) = 0, \quad y(0) = 1.$$

You may use the local uniqueness theorem: continuity of a vector field and local Lipschitz continuity in the state guarantee local uniqueness. Failure of its hypotheses alone does not prove nonuniqueness.

Tasks

1. Check continuity and test the local Lipschitz condition at states with $x = 0$. Explain exactly which uniqueness guarantee is unavailable.
2. For any waiting time $a \geq 0$, construct a solution that has $x = 0$ until time a and then becomes positive. Verify the derivative at the joining time as well as both differential equations.
3. Find y and compare the initial state and initial derivative of the waiting-time solutions. Include the solution that never leaves $x = 0$.
4. If an additional observation says $x(1) = 0$, does that restore uniqueness for later times? Exhibit two distinct solutions satisfying all the data and explain what this example says about a complete initial state.

Question 8 – Solution

Strategy. Prove nonuniqueness by constructing different differentiable solutions; do not infer it solely from a theorem's failure.

Step 1: Test the regularity hypothesis. The field $F(x, y) = (2\sqrt{|x|}, -y)^T$ is continuous everywhere. For $h > 0$, the change in its first component between (h, y) and $(0, y)$, divided by the state distance, is $2\sqrt{h}/h = 2/\sqrt{h}$. This is unbounded as $h \downarrow 0$, so no local Lipschitz constant exists near $x = 0$. The quoted uniqueness theorem cannot guarantee uniqueness there.

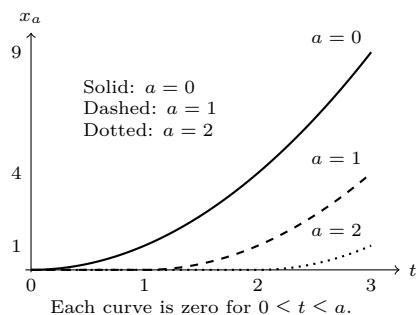
Step 2: Construct and check the waiting family. For each $a \geq 0$, define

$$x_a(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ (t-a)^2, & t > a. \end{cases}$$

Before a , both x'_a and $2\sqrt{|x_a|}$ are zero. After a , both are $2(t-a)$. At a , the left and right derivatives are zero, so the function is continuously differentiable and satisfies the equation there too. At $a = 0$, the same check uses the right derivative at the endpoint.

Step 3: Check the common data. The second scalar equation uniquely gives $y(t) = e^{-t}$. Every pair (x_a, e^{-t}) has initial state $(0, 1)$ and initial derivative $(0, -1)$. The pair $(0, e^{-t})$ is another solution, corresponding to waiting forever. Different finite waiting times produce different first components, despite identical initial states and slopes.

Step 4: Test whether one extra observation selects a solution. Every $a \geq 1$ gives $x_a(1) = 0$, as does the never-departing solution. For example, $a = 1$ and $a = 2$ satisfy all the data, but at $t = 3/2$ their first components are $1/4$ and 0 . Uniqueness is still absent. A complete state specifies all initial coordinates; unique evolution also requires appropriate properties of the differential equations. The figure shows three members of this family, with time increasing rightward.



Question 9

Three nonnegative dimensionless amounts undergo internal transfers: amount moves from compartment 1 to 2 at rate xy , and from 2 to 3 at rate $2y$. There are no external gains or losses. Their initial amounts are $(x, y, z) = (1, 1, 0)$. The rate laws are valid for nonnegative states.

Tasks

1. Derive the vector differential equation and classify it as autonomous and linear or nonlinear. Explain why a state-dependent coefficient does not make this a linear system.
2. Find the total-amount balance. Show that nonnegative initial states remain nonnegative and use this to bound every component for the stated data.
3. For the stated initial data, prove $x(t) > 0$ at every finite forward time. Derive a second conserved expression involving $x + y$ and $\ln x$.
4. Use the two conserved expressions to reduce the dynamics to one scalar equation for x , with formulas recovering y, z . State the restrictions that make the reduction meaningful; no explicit solution is required.

Question 9 – Solution

Strategy. Each internal transfer leaves one compartment and enters another at the same rate; this exposes conserved quantities before any solution is sought.

Step 1: Translate the transfers. The balances are

$$\boxed{x' = -xy, \quad y' = xy - 2y, \quad z' = 2y, \quad (x, y, z)(0) = (1, 1, 0).}$$

The system is autonomous, but nonlinear because of xy . Writing $x' = -yx$ does not make $-y$ a prescribed coefficient: y is another unknown state coordinate, not a known function of time.

Step 2: Prove conservation and nonnegativity. Adding the rows gives $(x + y + z)' = 0$, hence $x + y + z = 2$. Along any solution,

$$x(t) = x(0)e^{-\int_0^t y(s) ds}, \quad y(t) = y(0)e^{\int_0^t (x(s)-2) ds}, \quad z(t) = z(0) + 2 \int_0^t y(s) ds.$$

These formulas prove nonnegativity for nonnegative data on every forward existence interval. Thus $0 \leq x, y, z \leq 2$. The polynomial field is locally Lipschitz; boundedness prevents finite-time escape, so the solution continues for all $t \geq 0$.

Step 3: Obtain a second conserved quantity. With $x(0) = 1$ and $0 \leq y \leq 2$, the exponential formula gives $x(t) \geq e^{-2t} > 0$ at finite times. Therefore $\ln x$ is legitimate, and

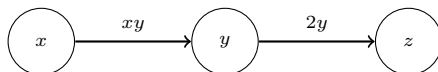
$$(x + y - 2 \ln x)' = -2y - 2(-xy)/x = 0.$$

Its initial value is 2, so $\boxed{x + y - 2 \ln x = 2}$.

Step 4: Reduce and recover the state. The two conservation laws give

$$\boxed{y = 2 - x + 2 \ln x, \quad z = -2 \ln x, \quad x' = -x(2 - x + 2 \ln x), \quad x(0) = 1.}$$

The physical branch has $x > 0$, $y \geq 0$, $z \geq 0$, in particular $x \leq 1$. Conversely, on this branch differentiation gives $y' = (x - 2)y$ and $z' = 2y$, proving recovery of the original system. Taking logarithms at $x = 0$ would be invalid; that boundary is excluded from this reduction.



Internal transfers only: $x + y + z = 2$ for the given data.

Question 10

An unknown autonomous affine system has the form $X' = AX + b$ in $\mathbb{R}^2$, where A, b are constant. Independent measurements of its velocity field F give

$$F(0, 0) = (1, -1)^T, \quad F(1, 0) = (-1, 2)^T, \quad F(0, 1) = (2, -3)^T.$$

An affine system is written as a nonhomogeneous linear differential system; its constant forcing need not vanish.

Tasks

1. Recover A and b , and prove uniqueness within the class of affine systems. Explain why the measured inputs are states rather than initial derivatives of a single trajectory.
2. Find every equilibrium and translate it to the origin. Write the differential equation in the translated state and explain what happens to the constant forcing.
3. A fourth measurement reports $F(1, 1) = (0, 1)^T$. Is it compatible with the first three under the affine assumption? Quantify the discrepancy.
4. Construct a smooth nonlinear vector field fitting all four measurements. Explain why the first three determine an affine model uniquely but do not determine an unrestricted vector field.

Question 10 – Solution

Strategy. Subtract the velocity at the origin to isolate the matrix columns, then test the model assumption with new data.

Step 1: Recover the affine coefficients. At the origin, $b = F(0, 0) = (1, -1)^T$. The differences $F(1, 0) - b = (-2, 3)^T$ and $F(0, 1) - b = (1, -2)^T$ are the first and second columns of A . Therefore

$$A = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \quad b = (1, -1)^T.$$

These columns and b are forced, proving uniqueness within the affine class. The measurements prescribe the field at three different states; they are not three independently assignable derivatives at one initial state.

Step 2: Translate the unique equilibrium. The equation $AX_* + b = 0$ reads $-2x + y + 1 = 0$ and $3x - 2y - 1 = 0$. It gives $X_* = (1, 1)^T$; uniqueness also follows from $\det A = 1$. For $Z = X - X_*$, the translation is constant, and

$$Z' = A(Z + X_*) + b = AZ, \quad \boxed{Z' = AZ.}$$

The forcing vanishes because the translation is by an equilibrium, not because every translation removes every forcing term.

Step 3: Check the fourth report. The recovered model predicts $F(1, 1) = (0, 0)^T$, while the report is $(0, 1)^T$. The reported-minus-predicted discrepancy is $(0, 1)^T$. No affine system fits all four measurements, since the first three already fix its coefficients. This conclusion tests the affine model assumption.

Step 4: Exhibit a nonlinear alternative. Define

$$\boxed{\tilde{F}(x, y) = (-2x + y + 1, 3x - 2y - 1 + xy)^T.}$$

The extra term xy vanishes at the first three states and equals 1 at $(1, 1)$, so all four measurements are met. This polynomial field is smooth and nonlinear. In it, $(1, 1)$ is not an equilibrium. More generally, fields obtained by adding $(0, cxy)^T$ all fit the first three data for any real c . Finite measurements uniquely identify this affine class, but leave unrestricted nonlinear models undetermined.