

Question 1

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad u = (1, 1)^T, \quad v = (1, -1)^T,$$

and write a real starting vector as $x_0 = \alpha u + \beta v$.

Tasks

1. Find the eigenvalues, their full eigenspaces and both multiplicities. Explain why arbitrary nonzero multiples of an eigenvector are allowed but the zero vector is not.
2. Express $w = (3, 1)^T$ in the eigenvector basis and decide whether w itself is an eigenvector.
3. Find $A^m x_0$ for every nonnegative integer m , and specialize to w . Verify the result without multiplying out m matrices.
4. For every nonzero real x_0 , determine the limit of $A^m x_0 / \|A^m x_0\|$. Include the cases $\alpha > 0$, $\alpha < 0$ and $\alpha = 0$.

Question 1 – Solution

Strategy. Resolve the starting vector into invariant directions before analyzing repeated multiplication.

Step 1: Compute the eigenspaces. The characteristic polynomial is $(\lambda - 2)^2 - 1 = (\lambda - 3)(\lambda - 1)$. Solving the two singular systems gives

$$E_3 = \text{span}\{u\}, \quad E_1 = \text{span}\{v\}.$$

Both algebraic and geometric multiplicities are 1. Scaling a nonzero eigenvector preserves its eigenvalue equation. Zero solves every homogeneous eigenvalue equation and supplies no direction, so it is excluded as an eigenvector.

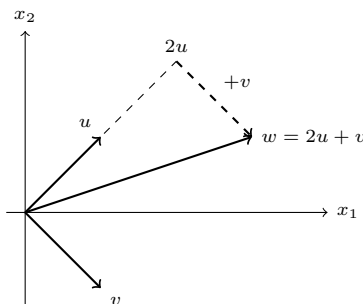
Step 2: Decompose and test the given vector. The equations $\alpha + \beta = 3$, $\alpha - \beta = 1$ give $w = 2u + v$. Both components are nonzero and their eigenvalues differ. Directly, $Aw = (7, 5)^T$ is not a scalar multiple of $(3, 1)^T$, so w is not an eigenvector.

Step 3: Compute all iterates. Linearity and $Au = 3u$, $Av = v$ give, by induction,

$$A^m x_0 = 3^m \alpha u + \beta v, \quad A^m w = (2 \cdot 3^m + 1, 2 \cdot 3^m - 1)^T.$$

The formula holds at $m = 0$, and one multiplication advances each coefficient by its eigenvalue, proving the next case.

Step 4: Classify normalized direction limits. If $\alpha \neq 0$, divide the iterate by $3^m |\alpha|$ before normalizing. The remaining v coefficient tends to zero, so the limit is $\boxed{u/\sqrt{2}}$ for $\alpha > 0$ and $\boxed{-u/\sqrt{2}}$ for $\alpha < 0$. If $\alpha = 0$, then $\beta \neq 0$ and the iterate is the unchanged vector βv , giving limit $\boxed{\text{sgn}(\beta)v/\sqrt{2}}$. The dominant eigenvalue determines the limit only when its component is present. The diagram shows the two directions and the decomposition $w = 2u + v$.



Question 2

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For real a , consider

$$B_a = \begin{pmatrix} 2 & a \\ 0 & 2 \end{pmatrix}.$$

A student claims that a repeated eigenvalue automatically provides two independent eigenvectors.

Tasks

1. Find the eigenvalue and both multiplicities for every a , treating $a = 0$ separately.
2. Classify diagonalizability over $\mathbb{R}$ and over $\mathbb{C}$. Explain whether allowing complex eigenvectors repairs any failure.
3. Find an exact formula for B_a^m for every nonnegative integer m . Derive it rather than assuming diagonalization.
4. For a fixed vector $x = (x_1, x_2)^T$, classify when $2^{-m} B_a^m x$ stays bounded as $m \rightarrow \infty$. Compare this result with what the repeated eigenvalue alone suggests.

Question 2 – Solution

Strategy. Root multiplicity and the number of independent eigenvectors answer different questions.

Step 1: Solve the eigenspace equation. The characteristic polynomial is $(\lambda - 2)^2$ for every a . The equation $(B_a - 2I)x = 0$ reduces to $ax_2 = 0$. Thus the algebraic multiplicity is always 2, while

$$a = 0 : E_2 = \mathbb{R}^2, \dim E_2 = 2; \quad a \neq 0 : E_2 = \text{span}\{(1, 0)^T\}, \dim E_2 = 1.$$

A repeated root does not itself produce a second independent vector.

Step 2: Classify diagonalization in both fields. At $a = 0$, $B_a = 2I$ is already diagonal. At $a \neq 0$, there is only one independent eigenvector, so no eigenvector basis exists. The same equation $ax_2 = 0$ holds over $\mathbb{C}$, with the same dimension count there. Hence B_a is diagonalizable exactly when $a = 0$ over either field.

Step 3: Derive the powers from a terminating product. Let $N = B_a - 2I = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, so $N^2 = 0$. The commuting binomial expansion gives

$$B_a^m = 2^m \begin{pmatrix} 1 & ma/2 \\ 0 & 1 \end{pmatrix}.$$

For $m = 0$ this is I . Multiplication by B_a adds $a/2$ to the scaled upper-right entry, confirming the formula inductively without an eigenvector basis.

Step 4: Identify the remaining growth. The scaled iterate is

$$2^{-m} B_a^m x = (x_1 + max_2/2, x_2)^T.$$

It is bounded exactly when $ax_2 = 0$, and then is constant. For every $a \neq 0$ and $x_2 \neq 0$, its first coordinate grows in magnitude linearly. Dividing out the common exponential factor 2^m does not remove this extra growth. The missing eigenvector permits behavior that the list of eigenvalues, even with multiplicity, does not describe by itself.

Question 3

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Allow complex eigenvalues and eigenvectors for the real matrix

$$C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Write $i^2 = -1$, and let $e_1 = (1, 0)^T$.

Tasks

1. Find both complex eigenvalues and their eigenspaces, and verify a representative eigenvector for each directly.
2. Prove that there is no nonzero real eigenvector, despite the existence of a complex eigenvector basis.
3. Write an eigenvector for $1 + 2i$ as $p + iq$ with p, q real, and derive the two real equations obtained from its eigenvalue equation.
4. Express C as a positive scale times a real rotation. Find $C^m e_1$ and its length for every nonnegative integer m , with the rotation angle specified unambiguously.

Question 3 – Solution

Strategy. Complex eigenvectors encode a real invariant plane; they need not describe a real line fixed in direction.

Step 1: Find and verify the complex pairs. The polynomial is $(\lambda - 1)^2 + 4$, so

$$\lambda_{\pm} = 1 \pm 2i, \quad E_{1+2i} = \text{span}_{\mathbb{C}}\{(1, -i)^T\}, \quad E_{1-2i} = \text{span}_{\mathbb{C}}\{(1, i)^T\}.$$

Indeed $C(1, -i)^T = (1 + 2i, 2 - i)^T = (1 + 2i)(1, -i)^T$. Conjugation verifies the other pair. The distinct eigenvalues give independent complex eigenvectors.

Step 2: Exclude a real eigenvector. If a real nonzero x satisfied $Cx = \lambda x$, a nonzero coordinate of x would make λ the ratio of two real numbers. Thus λ would be real, but the characteristic polynomial has no real root. No such vector exists.

Step 3: Extract the real plane relations. For $(1, -i)^T = p + iq$, take $p = (1, 0)^T$, $q = (0, -1)^T$. Equating real and imaginary parts of $C(p + iq) = (1 + 2i)(p + iq)$ gives

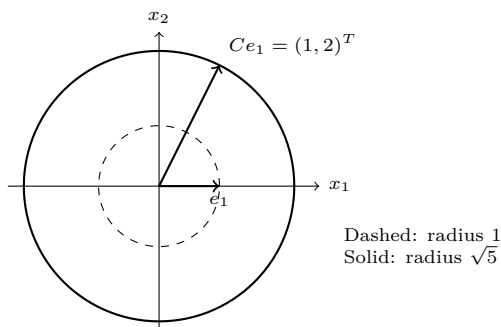
$$Cp = p - 2q, \quad Cq = 2p + q.$$

Direct multiplication verifies both. These independent real vectors span the plane even though neither is a real eigenvector.

Step 4: Recover the rotation and scale. Let $\theta \in (0, \pi/2)$ satisfy $\cos \theta = 1/\sqrt{5}$ and $\sin \theta = 2/\sqrt{5}$; equivalently $\theta = \arctan(2)$. Then

$$C = \sqrt{5} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad C^m e_1 = 5^{m/2} (\cos m\theta, \sin m\theta)^T.$$

Its length is $5^{m/2}$. The figure shows e_1 and its image $(1, 2)^T$, with unit and image circles drawn on equal scales.



Question 4

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Consider the real symmetric matrix

$$S = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Tasks

1. Find an orthonormal eigenvector basis and an orthogonal diagonalization $S = QDQ^T$. Verify the columns directly.
2. Prove that eigenvectors of a real symmetric matrix belonging to distinct real eigenvalues are perpendicular, and connect the proof to your basis.
3. Find the sharp minimum and maximum of $x^T S x$ over real vectors with $\|x\| = 1$, giving every equality case.
4. Use the spectral decomposition to find S^{-1} and verify it. Explain why a negative eigenvalue does not prevent invertibility but does prevent positive definiteness.

Question 4 – Solution

Strategy. An orthonormal eigenvector basis turns both a quadratic form and an inverse into scalar calculations.

Step 1: Construct and check the basis. Take

$$q_1 = \frac{1}{\sqrt{2}}(1, 1, 0)^T, \quad q_2 = \frac{1}{\sqrt{2}}(1, -1, 0)^T, \quad q_3 = (0, 0, 1)^T.$$

Direct multiplication gives eigenvalues 3, 1, -1, respectively. The columns are orthonormal, so $Q = (q_1 \ q_2 \ q_3)$ has $Q^T Q = I$. Thus $SQ = QD$ with $D = \text{diag}(3, 1, -1)$, and $S = QDQ^T$.

Step 2: Prove perpendicularity. For a real symmetric matrix, if $Sa = \lambda a$ and $Sb = \mu b$, then

$$\lambda a^T b = (Sa)^T b = a^T S^T b = a^T S b = \mu a^T b.$$

When $\lambda \neq \mu$, this forces $a^T b = 0$. Normalizing the three perpendicular directions gives exactly the orthonormal basis above.

Step 3: Derive the sharp quadratic-form bounds. Write $x = c_1 q_1 + c_2 q_2 + c_3 q_3$. For a unit vector, $c_1^2 + c_2^2 + c_3^2 = 1$, and

$$x^T S x = 3c_1^2 + c_2^2 - c_3^2.$$

This is a weighted average of 3, 1, -1, so $-1 \leq x^T S x \leq 3$. The minimum occurs exactly at $x = \pm q_3$; the maximum exactly at $x = \pm q_1$. The coefficients of all other directions must vanish at an endpoint.

Step 4: Invert the nonzero eigenvalues. Since no eigenvalue is zero,

$$S^{-1} = Q \text{diag}(1/3, 1, -1) Q^T = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Multiplying by S gives I_3 . Invertibility excludes zero eigenvalues, not negative ones. But $q_3^T S q_3 = -1 < 0$, so S is not positive definite; in fact its quadratic form takes both positive and negative values.

Question 5

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A real 2×2 matrix A is required to have

$$Au = 2u, \quad Av = -v, \quad u = (1, 1)^T, \quad v = (2, -1)^T.$$

Tasks

1. Construct the unique matrix satisfying these specifications and verify both eigenpairs.
2. Explain the effect of rescaling the two eigenvectors or swapping their order together with the corresponding eigenvalues. Contrast this with assigning the eigenvalues to the opposite directions.
3. Can the same matrix also have $(1, 0)^T$ as an eigenvector for some eigenvalue? Prove your answer.
4. Keeping the two prescribed directions, replace their eigenvalues by arbitrary real r, s . Find the full matrix formula and explain how its eigenspaces change when $r = s$.

Question 5 – Solution

Strategy. Independent eigenvectors form an input basis, so their prescribed images determine every column of the transformation.

Step 1: Build the spectral representation. The basis matrix and inverse are

$$P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Therefore

$$A = P \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Its products with u, v are $(2, 2)^T = 2u$ and $(-2, 1)^T = -v$. Since u, v span the plane, these images determine A uniquely.

Step 2: Separate notation changes from changed assignments. Multiplying either basis column by a nonzero scalar leaves its eigenvalue relation unchanged, so the reconstructed matrix stays the same. Reordering columns and the matching diagonal entries also leaves the map unchanged. Assigning -1 to u and 2 to v instead produces $I - A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$, a different map. The correspondence between each direction and its scalar matters.

Step 3: Test a proposed third direction. Directly, $A(1, 0)^T = (0, 1)^T$, not a multiple of $(1, 0)^T$. Thus no choice of eigenvalue makes the additional request possible. Equivalently, $(1, 0)^T = (u + v)/3$ combines both distinct eigendirections.

Step 4: Recover the general two-scalar family. The same basis computation yields

$$A_{r,s} = \frac{1}{3} \begin{pmatrix} r + 2s & 2r - 2s \\ r - s & 2r + s \end{pmatrix}.$$

For $r \neq s$, the eigenspaces are exactly the two specified lines. For $r = s$, the matrix becomes rI_2 and every nonzero vector is an eigenvector; the eigenspace is the whole plane. Repeating an eigenvalue here enlarges the eigenspace, unlike a defective repeated-root example.

Question 6

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Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Two matrices commute when $AB = BA$.

Tasks

1. Find the full family of real matrices B commuting with A , by equating entries of the two products.
2. Find a common eigenvector basis for every matrix in this family. Explain why simultaneous diagonalization always implies commutation.
3. For $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, find the eigenvalues of AB and $A + B$ using the common basis. Explain how two invertible matrices can have a singular sum.
4. Within the commuting family, classify every invertible B for which $A + B$ is singular. Identify when the sum has a one-dimensional kernel and when it is the zero matrix.

Question 6 – Solution

Strategy. Commutation forces preservation of this matrix's two distinct eigendirections, allowing scalar tests in a common basis.

Step 1: Equate the products. Subtracting the products gives

$$AB - BA = \begin{pmatrix} r - q & s - p \\ p - s & q - r \end{pmatrix}.$$

Thus $r = q$ and $s = p$, with no further restrictions. Thus the entire commuting family is $B = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$.

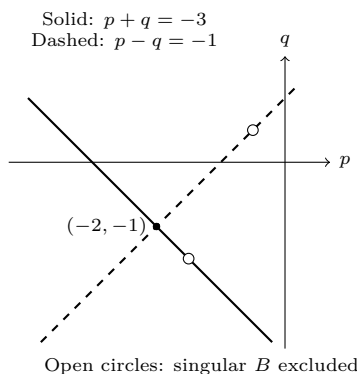
Step 2: Use the common directions. For $u = (1, 1)^T$ and $v = (1, -1)^T$, the eigenvalues of A are 3, 1, and those of B are $p + q, p - q$, respectively. These independent vectors form a common eigenvector basis, even if the two eigenvalues of B coincide. Conversely, if $A = PDP^{-1}$ and $B = PEP^{-1}$ with D, E diagonal, then $AB = PDEP^{-1} = PEDP^{-1} = BA$.

Step 3: Analyze the concrete sum and product. For $p = 1, q = 2$, the eigenvalues of B on u, v are 3, -1 . Therefore the eigenvalues of AB are 9, -1 and those of $A + B$ are 6, 0. Both A and B are invertible, but their actions on v cancel in the sum. The sum has kernel $\text{span}\{v\}$.

Step 4: Classify all cancellation cases. Invertibility of B requires $(p + q)(p - q) \neq 0$. Singularity of the sum requires $(p + q + 3)(p - q + 1) = 0$. Equivalently the full allowed set is

$$\{p + q = -3, p - q \neq 0\} \cup \{p - q = -1, p + q \neq 0\}.$$

On the first line only, the kernel is the line through u ; on the second only, it is the line through v . Their intersection is $(p, q) = (-2, -1)$, where $B = -A$ and $A + B = 0$, so the kernel is the entire plane. The excluded points are $(-3/2, -3/2)$ and $(-1/2, 1/2)$, where B itself is singular. The diagram marks these exclusions with open circles.



Question 7

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For a real parameter K , let

$$T_K = \begin{pmatrix} 1/2 & K \\ 0 & 1/4 \end{pmatrix}.$$

Both eigenvalues have magnitude less than one. We compare long-term iteration with the stronger requirement $\|T_K x\| \leq \|x\|$ for every real vector x .

Tasks

1. Find an eigenvector basis for every K and determine whether the matrix is diagonalizable.
2. Derive T_K^m for every nonnegative integer m and prove that every fixed starting vector tends to zero under iteration.
3. For the starting vector $e_2 = (0, 1)^T$, find the first-step length. Determine exactly when that first step increases length, and explain why this does not contradict the long-term limit.
4. Find the necessary and sufficient condition on K for $\|T_K x\| \leq \|x\|$ to hold for every real x . Prove it by completing a square, rather than testing only e_2 .

Question 7 – Solution

Strategy. Decaying eigencomponents need not make Euclidean length decrease at every step when the eigenvectors are not orthogonal.

Step 1: Find the distinct-eigenvalue basis. The eigenvalues are $1/2$ and $1/4$, with eigenvectors $u = (1, 0)^T$ and $v = (-4K, 1)^T$. These are independent for every K , so T_K is always diagonalizable over $\mathbb{R}$.

Step 2: Compute all powers. For $m \geq 1$, the upper-right entry in the product is

$$K \sum_{j=0}^{m-1} (1/2)^{m-1-j} (1/4)^j = 4K(2^{-m} - 4^{-m}).$$

This finite geometric sum yields

$$T_K^m = \begin{pmatrix} 2^{-m} & 4K(2^{-m} - 4^{-m}) \\ 0 & 4^{-m} \end{pmatrix}.$$

The formula is I at $m = 0$. Every entry tends to zero for each fixed K , so $T_K^m x \rightarrow 0$ for every fixed starting vector x .

Step 3: Exhibit a transient length increase. The first image of e_2 is $(K, 1/4)^T$, of length $\sqrt{K^2 + 1/16}$. It exceeds 1 exactly when $|K| > \sqrt{15}/4$. An early increase and eventual decay concern different times, so there is no contradiction. The plot uses $K = 2$ and shows the actual lengths at integer iteration counts; the initial increase is followed by decay.

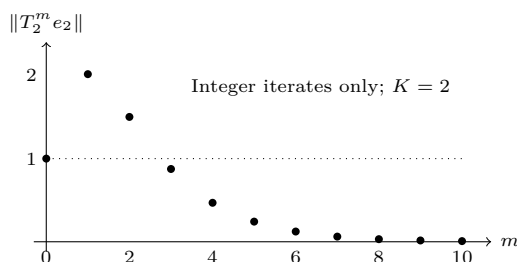
Step 4: Test all directions, not just one. For $x = (a, b)^T$, direct expansion and completion of the square give

$$\|T_K x\|^2 - \|x\|^2 = -\frac{3}{4} \left(a - \frac{2K}{3}b \right)^2 + \left(\frac{4K^2}{3} - \frac{15}{16} \right) b^2.$$

This is nonpositive for every a, b exactly when the last coefficient is nonpositive: necessity follows by choosing $a = 2Kb/3$ with $b \neq 0$. Hence the exact condition is

$$|K| \leq \frac{3\sqrt{5}}{8}.$$

This stricter threshold cannot be recovered by checking only e_2 or only the eigenvalues. At equality, nonzero vectors on $a = 2Kb/3$ retain their length in one step; outside that line they decrease.



Question 8

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For real τ , consider

$$A_\tau = \begin{pmatrix} 0 & 1 \\ -\tau & 2 \end{pmatrix}.$$

Investigate diagonalization over both $\mathbb{R}$ and $\mathbb{C}$.

Tasks

1. Find the characteristic polynomial and classify its roots for $\tau < 1$, $\tau = 1$ and $\tau > 1$. Check trace and determinant against the roots.
2. Find every eigenspace and classify diagonalizability in both fields, including the repeated-root case.
3. For $\tau > 1$, write a complex eigenvector as $p + iq$ and construct a real basis in which A_τ has a real 2×2 block representing its complex pair.
4. As $\tau \rightarrow 1^-$, examine a real eigenvector basis and its inverse. Explain why the limiting matrix can fail to be diagonalizable even though every matrix before the limit is diagonalizable.

Question 8 – Solution

Strategy. Follow the eigenvectors as well as the roots. Distinct roots can merge into one direction at the transition.

Step 1: Classify the roots. The polynomial is $\lambda^2 - 2\lambda + \tau$. For $\tau < 1$, set $d = \sqrt{1 - \tau} > 0$; the roots are $1 \pm d$. At $\tau = 1$ the root is 1 twice. For $\tau > 1$, set $b = \sqrt{\tau - 1} > 0$; the roots are $1 \pm ib$. Their sum is $2 = \text{tr } A_\tau$ and product is $\tau = \det A_\tau$ in all cases.

Step 2: Find the eigenspaces and field restrictions. The first row of $A_\tau x = \lambda x$ forces $x_2 = \lambda x_1$. The second row is then exactly the characteristic equation. Thus every root has complex eigenspace spanned by $(1, \lambda)^T$; for real roots, the real eigenspace is the real span of the same vector. At $\tau = 1$ this is only one-dimensional, despite algebraic multiplicity 2. Consequently

	$\mathbb{R}$ -diagonalizable	$\mathbb{C}$ -diagonalizable
$\tau < 1$	yes	yes
$\tau = 1$	no	no
$\tau > 1$	no	yes

The real failure above 1 comes from nonreal eigenvalues, not a shortage of complex eigenvectors.

Step 3: Construct a real representation for the complex pair. For $\lambda = 1 + ib$, write $(1, 1 + ib)^T = p + iq$ with $p = (1, 1)^T$, $q = (0, b)^T$. Equating parts gives $A_\tau p = p - bq$ and $A_\tau q = bp + q$. Since $\det(p \ q) = b \neq 0$, this is a real basis, with representation

$$(p \ q)^{-1} A_\tau (p \ q) = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}.$$

It is a real block, not a real diagonalization.

Step 4: Inspect the collapsing eigenvector basis. Below 1, use $P = \begin{pmatrix} 1 & 1 \\ 1 + d & 1 - d \end{pmatrix}$. Its determinant is $-2d$, and

$$P^{-1} = \frac{1}{2d} \begin{pmatrix} d - 1 & 1 \\ 1 + d & -1 \end{pmatrix}.$$

In particular, the second column is unbounded as $d \rightarrow 0^+$. Both columns tend to $(1, 1)^T$ as $d \rightarrow 0^+$. The limiting basis is singular and cannot diagonalize the limiting matrix. Diagonalizability need not persist when distinct eigendirections coalesce.

Question 9

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Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = A^2.$$

For a polynomial p , define $p(A)$ by replacing each scalar power with the corresponding matrix power and each constant with that constant times I_3 .

Tasks

1. Find the eigenvalues and eigenspaces of A and B , and explain why squaring merges some eigenspaces.
2. Give a nonzero eigenvector of B that is not an eigenvector of A . Prove the general forward implication from an eigenpair of A to an eigenpair of $p(A)$, and explain why its converse can fail.
3. Find the monic polynomial of least degree satisfying $p(A) = 0$. Prove both the identity and the minimality of its degree.
4. Use that polynomial identity to express A^{-1} as a polynomial in A . Compute the inverse explicitly and verify it.

Question 9 – Solution

Strategy. A polynomial changes eigenvalues without changing each original eigenvector, but different eigenvalues can acquire the same image.

Step 1: Compare the two spectra. For A , the eigenspaces are

$$E_1 = \text{span}\{(1, 1, 0)^T\}, \quad E_{-1} = \text{span}\{(1, -1, 0)^T\}, \quad E_2 = \text{span}\{(0, 0, 1)^T\}.$$

These vectors form a basis. Since $B = \text{diag}(1, 1, 4)$, its eigenspace for 1 is the entire plane $x_3 = 0$, and its eigenspace for 4 is the third coordinate axis. Squaring sends both 1 and -1 to 1.

Step 2: Prove the forward rule and refute the converse. The vector $e_1 = (1, 0, 0)^T$ satisfies $Be_1 = e_1$ but $Ae_1 = (0, 1, 0)^T$, not a multiple of e_1 . In general, $Av = \lambda v$ implies $A^k v = \lambda^k v$ by induction; linearity then gives $p(A)v = p(\lambda)v$. The merged eigenspace permits new combinations that are eigenvectors of $p(A)$ without being eigenvectors of A .

Step 3: Find the least-degree annihilating polynomial. Any polynomial with $p(A) = 0$ must vanish at $1, -1, 2$, by applying the identity to the three nonzero eigenvectors. A nonzero polynomial therefore needs degree at least three. The monic candidate is

$$p(z) = (z - 1)(z + 1)(z - 2) = z^3 - 2z^2 - z + 2.$$

It annihilates all three basis eigenvectors, hence every vector, proving $p(A) = 0$ and minimality. It is the unique monic cubic with those roots.

Step 4: Recover the inverse from the identity. Rearrange $A^3 - 2A^2 - A + 2I = 0$ to get $A(-A^2 + 2A + I)/2 = I$. The factors commute, so the reverse product is also I . Thus

$$A^{-1} = \frac{-A^2 + 2A + I}{2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

Direct multiplication confirms the inverse without division by a matrix expression.

Question 10

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

The symmetric matrix

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

measures differences along the path 1-2-3. Let $\mathbf{1} = (1, 1, 1)^T$. A vector has zero mean when $x_1 + x_2 + x_3 = 0$.

Tasks

1. Find all eigenvalues and an orthogonal eigenvector basis. Verify each pair and show that the list is complete.
2. Express $x^T Lx$ as a sum of squares. Determine all vectors for which this expression is zero, and explain the zero eigenvalue.
3. Give a necessary and sufficient condition for $Lx = b$ to have a real solution. For $b = (2, -1, -1)^T$, find all solutions and the unique one with zero mean.
4. On the zero-mean subspace, prove the sharp bounds $\|x\|^2 \leq x^T Lx \leq 3\|x\|^2$ and give every equality case. Explain why the lower bound fails without the zero-mean restriction.

Question 10 – Solution

Strategy. Separate the constant direction, which the matrix cannot detect, from the two directions of variation.

Step 1: Verify a complete orthogonal basis. Direct multiplication gives

$$L\mathbf{1} = 0, \quad Lv = v, \quad v = (1, 0, -1)^T, \quad Lw = 3w, \quad w = (1, -2, 1)^T.$$

The three vectors are nonzero and mutually perpendicular, so they form a basis of $\mathbb{R}^3$. Thus the complete eigenvalue list is $[0, 1, 3]$, with the corresponding one-dimensional eigenspaces.

Step 2: Interpret the nonnegative quadratic form. Expanding gives

$$x^T Lx = (x_1 - x_2)^2 + (x_2 - x_3)^2 \geq 0.$$

It vanishes exactly when $x_1 = x_2 = x_3$, the constant direction. The zero eigenvalue reflects the fact that adding the same constant to all three entries changes no adjacent difference.

Step 3: Solve subject to the compatibility condition. Since $\mathbf{1}^T L = 0$, solvability requires $\mathbf{1}^T b = 0$. Conversely, every such b can be written $b = \alpha v + \beta w$, and $x = \alpha v + (\beta/3)w + c\mathbf{1}$ solves the system for every real c . For the stated b , dot products give $\alpha = 3/2$ and $\beta = 1/2$. Therefore all solutions are

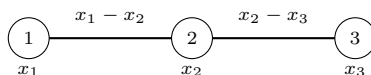
$$x = (5/3, -1/3, -4/3)^T + c\mathbf{1}, \quad c \in \mathbb{R}.$$

The displayed particular vector has zero mean; the total sum is $3c$, so zero mean uniquely selects $c = 0$.

Step 4: Prove the sharp restricted bounds. A zero-mean vector has form $x = av + bw$. Orthogonality and $\|v\|^2 = 2$, $\|w\|^2 = 6$ give

$$\|x\|^2 = 2a^2 + 6b^2, \quad x^T Lx = 2a^2 + 18b^2.$$

The lower equality holds exactly on the line through v ; the upper exactly on the line through w , with zero included in both. This proves both sharp bounds. A nonzero constant vector has positive length but zero quadratic form, so the lower bound cannot hold on all of $\mathbb{R}^3$.



Two edges of weight 1; energy is the sum of squared differences.