

Question 1

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Consider the rectangular matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Tasks

1. Check the dimensions of AB and BA , compute both products, and identify which identity matrix occurs.
2. For an arbitrary $b = (b_1, b_2)^T$, describe every solution of $Ax = b$ and explain which solution Bb selects.
3. Prove that no matrix C can satisfy $CA = I_3$. Use a nonzero vector sent to zero by A , rather than only counting equations.
4. For $P = BA$, prove $P^2 = P$, describe all vectors fixed by P , and compute Pv for $v = (2, -1, 3)^T$. Explain what information this operation discards.

Question 1 – Solution

Strategy. A rectangular matrix can have a right inverse without having a left inverse. Check what happens to vectors, not just products.

Step 1: Multiply with dimensions attached. The sizes are $A : 2 \times 3$ and $B : 3 \times 2$. Thus

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \quad BA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3.$$

The two products even act on spaces of different dimensions.

Step 2: Recover the whole solution family. The equation $Ax = b$ says $x_1 + x_3 = b_1$ and $x_2 + x_3 = b_2$. Hence

$$x = (b_1 - t, b_2 - t, t)^T, \quad t \in \mathbb{R}.$$

Every such vector works, and every solution has this form. The vector $Bb = (b_1, b_2, 0)^T$ selects $t = 0$; it is a choice, not the only inverse image.

Step 3: Rule out a left inverse. The nonzero vector $n = (-1, -1, 1)^T$ satisfies $An = 0$. If $CA = I_3$, then $n = I_3n = CAn = C0 = 0$, a contradiction. Thus no left inverse exists, even though $AB = I_2$.

Step 4: Interpret the repeated operation. Associativity gives $P^2 = B(AB)A = BI_2A = P$. For $x = (x_1, x_2, x_3)^T$, $Px = (x_1 + x_3, x_2 + x_3, 0)^T$. Consequently $Px = x$ exactly when $x_3 = 0$, and

$$P(2, -1, 3)^T = (5, 2, 0)^T.$$

The discarded difference is $x - Px = x_3(-1, -1, 1)^T$, invisible to A . Indeed $APx = Ax$. This is an idempotent projection onto the plane $x_3 = 0$, but it is not the perpendicular projection: it changes the first two coordinates.

Question 2

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In the plane, apply the shear and quarter-turn

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Composition acts on column vectors from right to left.

Tasks

1. Compute RS and SR . Apply each to $v = (1, 1)^T$ and explain the difference in operation order.
2. Find every vector w for which $RSw = SRw$. Prove that your list is complete.
3. Find $(RS)^{-1}$ by reversing the operations, and verify the inverse by multiplication.
4. Map the four vertices of the unit square in their cyclic order under both products. Compare areas and orientation, and explain why equal determinants do not imply equal transformations.

Question 2 – Solution

Strategy. Matrix multiplication records the order of geometric operations. Determinants retain only part of that information.

Step 1: Compute both compositions. Direct multiplication gives

$$RS = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad SR = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus $RSv = (-1, 3)^T$ and $SRv = (1, 1)^T$. The first shears then rotates; the second rotates then shears.

Step 2: Test agreement on an arbitrary vector. Subtracting gives

$$(RS - SR)w = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

This is zero exactly when $w_1 = w_2 = 0$. Therefore the two compositions agree only on the zero vector, despite each being invertible.

Step 3: Undo the operations in reverse order. Since $S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $R^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

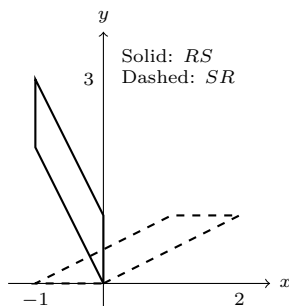
$$(RS)^{-1} = S^{-1}R^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Multiplication with RS on either side gives I_2 .

Step 4: Compare the images of the square. For vertices $(0, 0), (1, 0), (1, 1), (0, 1)$ in that order, the images are

$$\begin{array}{l|cccc} RS & (0, 0) & (0, 1) & (-1, 3) & (-1, 2) \\ SR & (0, 0) & (2, 1) & (1, 1) & (-1, 0) \end{array}$$

Both determinants are 1, so both images have area 1 and preserve the counterclockwise orientation. The parallelograms nevertheless differ. A determinant encodes signed area scaling, not the full image of each vector. The plot uses equal scales on both axes.



Question 3

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$A_\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \tau & 1 \\ 1 & 1 & \tau \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \tau \in \mathbb{R}.$$

Tasks

1. Find $\det A_\tau$ by determinant-preserving row operations and classify invertibility.
2. For the invertible cases, derive A_τ^{-1} by solving $A_\tau x = c$ for a general column c . Verify the formula.
3. Solve $A_\tau x = b$ for every τ . At the singular parameter, classify consistency for an arbitrary right-hand side c .
4. Compare the behavior near $\tau = 1$ for right-hand sides $(1, 2, 3)^T$ and $(1, 1, 1)^T$. Explain why invertibility for each nearby parameter does not imply a uniform bound on recovered vectors.

Question 3 – Solution

Strategy. Subtracting the first row exposes the parameter that controls both inversion and the exceptional case.

Step 1: Compute the determinant. Replace rows two and three by themselves minus row one. These operations do not change the determinant and produce

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \tau - 1 & 0 \\ 0 & 0 & \tau - 1 \end{pmatrix}.$$

Therefore $\det A_\tau = (\tau - 1)^2$, and inversion is possible exactly when $\tau \neq 1$.

Step 2: Solve for a general right-hand side. Put $d = \tau - 1 \neq 0$. The subtracted equations give $x_2 = (c_2 - c_1)/d$, $x_3 = (c_3 - c_1)/d$, and $x_1 = c_1 - x_2 - x_3$. Thus

$$A_\tau^{-1} = \begin{pmatrix} 1 + 2/d & -1/d & -1/d \\ -1/d & 1/d & 0 \\ -1/d & 0 & 1/d \end{pmatrix}.$$

Substitution in the three original equations returns c_1, c_2, c_3 for every c , so multiplication $A_\tau A_\tau^{-1} = I_3$ is verified. The displayed matrices are symmetric, and transposing that equality also verifies multiplication in the opposite order.

Step 3: Handle the prescribed and singular systems. For $\tau \neq 1$,

$$x = \left(1 - \frac{3}{d}, \frac{1}{d}, \frac{2}{d} \right)^T.$$

At $\tau = 1$, all three left sides are $x_1 + x_2 + x_3$, while the entries of b differ, so there is no solution. For a general c , consistency at 1 is equivalent to $c_1 = c_2 = c_3$. If they coincide, the whole plane $x_1 + x_2 + x_3 = c_1$ is the solution set; otherwise it is empty.

Step 4: Distinguish invertibility from bounded recovery. For $b = (1, 2, 3)^T$, the coordinate $x_2 = 1/(\tau - 1)$ becomes unbounded near 1. For $c = (1, 1, 1)^T$, the unique nearby solution is always $(1, 0, 0)^T$. Thus some data remain harmless while other fixed data yield arbitrarily large solutions. The factors $1/d$ in the inverse prevent a uniform bound near the singular parameter, even though every matrix with $d \neq 0$ is invertible.

Question 4

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Use the ordered basis $u = (1, 1)^T$, $v = (1, -1)^T$ of the plane. Let P have columns u, v , so a coordinate column c represents the physical vector Pc . A transformation in standard coordinates is

$$T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Tasks

1. Find P^{-1} and the coordinates of $w = (3, 1)^T$ in the ordered basis.
2. Derive the matrix representing T in this basis, explaining the order of all three operations.
3. Apply the coordinate matrix to your coordinate column and check the result against direct standard-coordinate multiplication.
4. For arbitrary coordinate columns, derive the relation between coordinate length and physical length. Compare the determinants of the two representations and interpret the physical area factor.

Question 4 – Solution

Strategy. A vector and its coordinate column describe the same object in different languages. Convert in the correct order.

Step 1: Build and invert the basis matrix. Here

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad [w]_{u,v} = P^{-1}w = \boxed{(2, 1)^T}.$$

Thus $w = 2u + v$, as the figure illustrates.

Step 2: Express the transformation in the new coordinates. Start with a coordinate column c . Convert it to Pc , apply T , and convert the result back with P^{-1} . Consequently

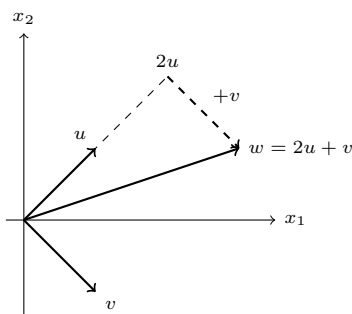
$$C = P^{-1}TP = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}}.$$

Step 3: Verify both routes. The coordinate route gives $C(2, 1)^T = (4, 3)^T$, representing $P(4, 3)^T = (7, 1)^T$. Directly, $T(3, 1)^T = (7, 1)^T$ as well. Equivalently, $PC = TP$ verifies the agreement for every input, not only w .

Step 4: Compare lengths and area factors. Since $P^T P = 2I_2$,

$$\boxed{\|Pc\|^2 = c^T P^T P c = 2\|c\|^2.}$$

The basis vectors are perpendicular but have length $\sqrt{2}$, so physical length is $\sqrt{2}$ times coordinate length. Also $\det C = \det(P^{-1}) \det T \det P = \det T = 2$. The physical transformation doubles areas and preserves orientation, regardless of which basis is used to describe it. The negative determinant of P describes the basis orientation; it does not change this area factor.



Question 5

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let $a = (1, 2)^T$ and $b = (2, 1)^T$. We seek the point on the line through a closest to b . All points on that line have the form ta , $t \in \mathbb{R}$.

Tasks

1. Minimize $\|b - ta\|^2$ exactly, giving the unique closest point and the minimum distance.
2. Construct a matrix Q that sends any vector to its closest point on this same line. Compute Q^T and Q^2 .
3. Describe all vectors sent to zero and all vectors fixed by Q . Verify that the error $b - Qb$ is perpendicular to the line.
4. Prove $\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2$ for every x , and deduce a sharp length inequality with its equality condition.

Question 5 – Solution

Strategy. Complete the square to find the best coefficient, then express that coefficient as a linear function of the input.

Step 1: Minimize the distance. Using $a^T a = 5$, $a^T b = 4$ and $b^T b = 5$,

$$\|b - ta\|^2 = 5t^2 - 8t + 5 = 5(t - 4/5)^2 + 9/5.$$

The unique minimizer is $t = 4/5$, so

$$Qb = (4/5, 8/5)^T, \quad \text{minimum distance} = 3/\sqrt{5}.$$

Step 2: Build the projection matrix. For any input x , the minimizing coefficient is $(a^T x)/(a^T a)$. Therefore

$$Q = \frac{aa^T}{5} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad Q^T = Q, \quad Q^2 = Q.$$

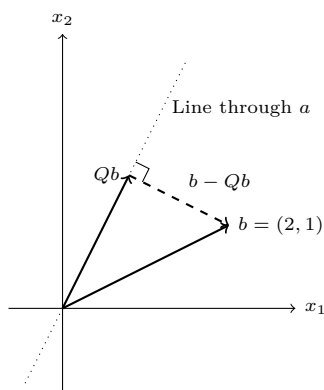
The last equality follows from $aa^T aa^T = a(a^T a)a^T = 5aa^T$.

Step 3: Describe the two perpendicular directions. The condition $Qx = 0$ is equivalent to $x_1 + 2x_2 = 0$, so the vectors sent to zero form the line through $(2, -1)^T$. The fixed vectors are exactly the multiples of a : the image lies on that line, and $Q(ta) = ta$. For the stated input, $b - Qb = (6/5, -3/5)^T$, whose dot product with a is zero.

Step 4: Prove the length identity and equality case. In general $a^T(x - Qx) = 0$, so Qx is perpendicular to $x - Qx$. Expanding the squared length of their sum gives

$$\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2, \quad \|Qx\| \leq \|x\|.$$

Equality holds exactly when $x - Qx = 0$, that is, when x lies on the line through a , including the zero vector. For b , the squared lengths are $5 = 16/5 + 9/5$. The plot uses equal axis scales and marks the right angle.



Question 6

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In three dimensions, let

$$u = (1, 1, 0)^T, \quad v = (1, 0, 1)^T, \quad w_\tau = (\tau, 1, 1)^T.$$

The plane of interest passes through $p = (1, 2, 3)^T$ and has direction vectors u, v . Use the scalar triple product $(u \times v) \cdot w$ as signed volume.

Tasks

1. Find a normal vector and a Cartesian equation for the plane. Verify that the specified point and directions satisfy the required conditions.
2. Find the area of the parallelogram spanned by u, v , and the signed and unsigned volumes spanned by u, v, w_τ .
3. Classify every τ for which the three direction vectors are linearly independent. At each dependent value, give an explicit nontrivial relation.
4. Explain how reversing the order of u, v affects the normal, plane equation, signed volume and unsigned volume. Distinguish a direction lying in the plane from a point lying on this translated plane.

Question 6 – Solution

Strategy. A cross product encodes both a perpendicular direction and an orientation. Translation affects a plane's points, not its direction space.

Step 1: Find and verify the normal. Direct calculation gives

$$n = u \times v = (1, -1, -1)^T.$$

Both dot products $n \cdot u$ and $n \cdot v$ vanish. The plane equation is $n \cdot (x - p) = 0$, or

$$x_1 - x_2 - x_3 = -4.$$

The point p satisfies it, and adding any combination of u, v to p leaves its left side unchanged.

Step 2: Compute area and volume. The parallelogram area is $\|n\| = \sqrt{3}$. The scalar triple product is

$$n \cdot w_\tau = \tau - 2, \quad \text{unsigned volume} = |\tau - 2|.$$

For example, $\tau = 0$ gives signed volume -2 and volume 2 .

Step 3: Prove the independence classification. If $\alpha u + \beta v + \gamma w_\tau = 0$, dotting with n gives $\gamma(\tau - 2) = 0$. For $\tau \neq 2$, this forces $\gamma = 0$. The second and third coordinates of $\alpha u + \beta v = 0$ then give $\alpha = \beta = 0$. Thus the vectors are independent exactly for $\tau \neq 2$. At $\tau = 2$, $w_2 = u + v$ is a nontrivial dependence, and the volume collapses to zero.

Step 4: Separate orientation from translation. Reversing the order gives $v \times u = -n$. The plane equation becomes $-x_1 + x_2 + x_3 = 4$, describing the same plane. Signed volume changes sign, while unsigned volume does not. As a direction, w_τ lies in the plane's direction space exactly when $n \cdot w_\tau = 0$, or $\tau = 2$. As a point with those coordinates, it lies on the translated plane exactly when $\tau - 2 = -4$, or $\tau = -2$. These are different questions because the plane does not pass through the origin.

Question 7

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

For real t , let

$$M(t) = \begin{pmatrix} 1 & t \\ 0 & 2 \end{pmatrix}, \quad v(t) = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}, \quad y(t) = M(t)v(t).$$

Differentiate matrices entry by entry.

Tasks

1. Derive the matrix-vector product rule from component sums. Compute y and y' in this example by two routes.
2. Find $N(t) = M(t)^{-1}$. Derive the inverse differentiation rule from $MN = I_2$ and verify it here.
3. Compute $(M^2)'$, $M'M + MM'$ and $2MM'$. Decide whether the scalar shortcut for differentiating a square is valid.
4. Compute $\det M(t)$ and examine $\|M(t)(0, 1)^T\|$. Explain why a constant nonzero determinant does not force a constant matrix or constant vector lengths.

Question 7 – Solution

Strategy. Scalar differentiation still applies to each entry, but matrix factors must retain their multiplication order.

Step 1: Derive and apply the product rule. For component i , $y_i = \sum_j M_{ij}v_j$. Differentiating the finite sum gives $y'_i = \sum_j (M'_{ij}v_j + M_{ij}v'_j)$, hence $y' = M'v + Mv'$. Direct multiplication and differentiation give

$$y = \begin{pmatrix} t^2 + te^t \\ 2e^t \end{pmatrix}, \quad y' = \begin{pmatrix} 2t + (1+t)e^t \\ 2e^t \end{pmatrix}.$$

The product-rule route gives $M'v = (e^t, 0)^T$ and $Mv' = (2t + te^t, 2e^t)^T$, confirming the same derivative.

Step 2: Differentiate the inverse with the correct order. The inverse is $N = \begin{pmatrix} 1 & -t/2 \\ 0 & 1/2 \end{pmatrix}$. Differentiating $MN = I_2$ gives $M'N + MN' = 0$; multiplying on the left by N yields

$$N' = -NM'N = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}.$$

This agrees with direct entrywise differentiation of N .

Step 3: Test the square rule explicitly. Since $M^2 = \begin{pmatrix} 1 & 3t \\ 0 & 4 \end{pmatrix}$,

$$(M^2)' = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}}_{M'M} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{MM'}.$$

But $2MM' = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, which is different. The shortcut works when M and M' commute; they do not here.

Step 4: Interpret the constant determinant. The determinant is always 2, whereas $M(t)(0, 1)^T = (t, 2)^T$ has length $\sqrt{t^2 + 4}$. Thus signed area scaling remains fixed while shear and some vector lengths change. Determinant information alone cannot determine every entry or every geometric effect of a matrix.

Question 8

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

For $n = (1, 2)^T$, define

$$H = I_2 - \frac{2nn^T}{n^T n}.$$

Consider its action on the triangle with ordered vertices $O = (0, 0)$, $a = (3, 1)$ and $b = (1, 2)$.

Tasks

1. Compute H . Prove from a perpendicular decomposition that it reflects the plane across a named line.
2. Verify $H^T H = I_2$ and $H^2 = I_2$. Deduce preservation of lengths and dot products and describe the inverse operation.
3. Find the reflected triangle, its area and orientation relative to the original. Check the midpoint and displacement of a and its image.
4. Find every vector fixed by H and every vector sent to its negative. Derive these directions directly from the reflection formula.

Question 8 – Solution

Strategy. Separate a vector into components parallel and perpendicular to the normal. The formula changes the sign of only the normal component.

Step 1: Identify the reflection axis. Here $n^T n = 5$, so

$$H = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}.$$

Write $x = x_{\perp} + x_{\parallel}$, where $x_{\parallel} = n(n^T x)/5$ and $n^T x_{\perp} = 0$. Then $Hx = x_{\perp} - x_{\parallel}$: it reflects across the line $x_1 + 2x_2 = 0$, whose direction is $(2, -1)^T$.

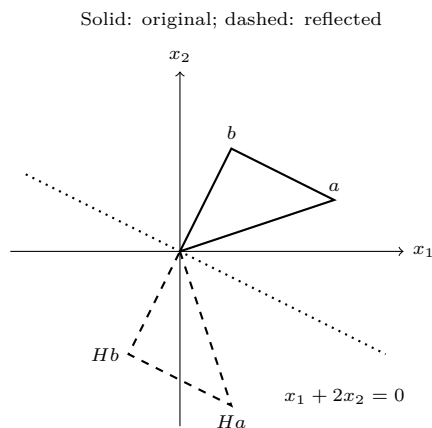
Step 2: Verify the metric and inverse identities. The matrix is symmetric, and its numerator squared is $\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$. Thus $H^2 = H^T H = I_2$. For any x, z , $(Hx)^T(Hz) = x^T H^T H z = x^T z$, preserving dot products and hence lengths. Also $H^{-1} = H$: reflecting twice restores the original vector.

Step 3: Transform and inspect the triangle. The images are

$$HO = O, \quad Ha = (1, -3)^T, \quad Hb = (-1, -2)^T.$$

The original ordered edge determinant is $3 \cdot 2 - 1 \cdot 1 = 5$, so its area is $5/2$. The reflected edge determinant is $1(-2) - (-1)(-3) = -5$. Its area remains $5/2$ but its orientation reverses, consistent with $\det H = -1$. The midpoint of a and Ha is $(2, -1)$ on the axis; the displacement $Ha - a = (-2, -4) = -2n$ is perpendicular to that axis.

Step 4: Classify fixed and reversed vectors. From the formula, $Hx = x$ exactly when $n^T x = 0$, giving the axis. For $Hx = -x$, the decomposition gives $x_{\perp} = 0$, so x is a multiple of n . The zero vector belongs to both sets. These conclusions follow from the components directly, without a characteristic-polynomial calculation. The plot uses equal axis scales.



Question 9

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

An unknown real 2×2 matrix A satisfies

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Tasks

1. Recover A uniquely, determine whether it is invertible, and verify both supplied measurements.
2. Predict $A(3, 1)^T$ without multiplying by the recovered matrix. Decide whether an additional reported image $(4, 5)^T$ is compatible.
3. In a separate experiment, replace the second input by $(2, 2)^T$ and its image by $(4, 0)^T$, keeping the first measurement. Find every matrix compatible with this experiment.
4. Classify the invertible matrices in the second experiment. Explain why one more image of an input outside the line through $(1, 1)^T$ would determine a unique matrix.

Question 9 – Solution

Strategy. Images of a basis determine a linear map, while repeated information along one input direction leaves freedom.

Step 1: Recover the columns from independent inputs. The first and second measurements are respectively the sum and difference of the columns of A . Half their sum gives column one; half their difference gives column two. Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \quad \det A = -4 \neq 0.$$

The column sum is $(2, 0)^T$ and the difference is $(0, 4)^T$, checking both measurements. Both columns are forced, so recovery is unique.

Step 2: Predict by linearity and test a new report. Because $(3, 1)^T = 2(1, 1)^T + (1, -1)^T$, linearity predicts $A(3, 1)^T = 2(2, 0)^T + (0, 4)^T = (4, 4)^T$. Thus a report of $(4, 5)^T$ is inconsistent with the two original measurements for any linear map, not just with a particular computation of its entries.

Step 3: Describe the dependent-input experiment. Now the second input and output are exactly twice the first, so the second measurement adds no restriction. Write the first column as $(\alpha, \beta)^T$. The required column sum $(2, 0)^T$ forces the second column, giving

$$A_{\alpha, \beta} = \begin{pmatrix} \alpha & 2 - \alpha \\ \beta & -\beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

Every member gives both required outputs; every compatible matrix has this form.

Step 4: Classify inversion and the missing information. The determinant is $-\alpha\beta - (2 - \alpha)\beta = -2\beta$. Thus exactly the members with $\beta \neq 0$ are invertible. If another input w is outside the line through $(1, 1)^T$, those two inputs form a basis. Each standard coordinate vector has a unique expression in that basis, so the two measured images uniquely determine both columns. Any assigned image for this independent input defines one linear map; invertibility would additionally require independent output images.

Question 10

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = I_3 + N.$$

A discrete update is $v_{m+1} = Av_m$, starting from $v_0 = (0, 0, 1)^T$. The index m is a nonnegative integer, and each update uses the entire old vector.

Tasks

1. Compute N^2, N^3 , $\det A$ and A^{-1} . Verify the inverse without a general inverse algorithm.
2. Prove a finite formula for A^m for all nonnegative integers m , using induction or the binomial identity with justification.
3. Find v_m and describe its coordinate growth. Explain why $\det A = 1$ does not imply bounded iterates.
4. A program first updates $y \leftarrow y + z$ and then $x \leftarrow x + y$, using the already updated y . Find the matrix it actually applies and its trajectory from the same initial vector. Compare the two after two steps.

Question 10 – Solution

Strategy. The powers of the strictly upper triangular part terminate. A matrix update is simultaneous unless a different order is explicitly encoded.

Step 1: Exploit the terminating powers. Multiplication gives $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $N^3 = 0$. The upper triangular matrix A has determinant 1. Moreover

$$A^{-1} = I_3 - N + N^2$$

because $(I_3 + N)(I_3 - N + N^2) = I_3 + N^3 = I_3$; the reverse product is the same.

Step 2: Derive the finite power formula. The identity matrix commutes with N , so the binomial expansion terminates:

$$A^m = I_3 + mN + \frac{m(m-1)}{2}N^2.$$

Alternatively, the formula is true at $m = 0$. Multiplication by $I_3 + N$ increases the coefficient of N to $m + 1$ and that of N^2 to $m(m-1)/2 + m = m(m+1)/2$, establishing the induction step.

Step 3: Find the exact iterates. Applying the power formula to v_0 gives

$$v_m = (m(m-1)/2, m, 1)^T.$$

The first coordinate grows quadratically, the second linearly and the third stays fixed. Determinant 1 preserves oriented volumes of three-dimensional sets, not individual lengths or boundedness under repetition. The plot marks integer-time values only; coincident coordinate values share a location.

Step 4: Diagnose the sequential-update error. The program computes $y_{\text{new}} = y + z$ and then $x_{\text{new}} = x + y + z$. It actually applies

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I_3 + N + N^2.$$

Starting from v_0 , its third coordinate remains 1, its second becomes m , and its first accumulates $1 + 2 + \dots + m = m(m+1)/2$. After two steps the correct vector is $(1, 2, 1)^T$, while the program gives $(3, 2, 1)^T$. Both matrices are invertible with determinant 1, so checking only the determinant would not expose this order-of-update mistake.

