

Question 1

Two perfectly mixed tanks contain 100 L and 200 L of solution. Two pumps transfer 10 L/min in each direction; there is no external inflow, outflow, or reaction. Initially the first tank contains 10 kg of salt and the second contains none. Let $x(t), y(t)$ be their salt amounts in kg, with t in minutes.

Tasks

1. Derive the amount equations from salt fluxes, explaining why the volumes stay constant and why the two transfer coefficients differ.
2. Find a conserved quantity and the equilibrium amounts. Does equilibrium mean equal amounts or equal concentrations?
3. Solve the initial-value problem and verify nonnegativity and conservation for all $t \geq 0$.
4. Find the earliest time at which the concentration difference is at most 0.005 kg/L, and show it stays below this tolerance thereafter. Sketch the two concentrations and their common limiting level.

Question 1 – Solution

Strategy. Use volume flow times concentration for each salt flux, then use conservation to reduce the coupled balances.

Step 1: Construct the two balances. Each tank gains and loses 10 L/min, so its volume is fixed. Its outgoing salt rate is 10 times its own concentration. Thus

$$x' = -\frac{1}{10}x + \frac{1}{20}y, \quad y' = \frac{1}{10}x - \frac{1}{20}y.$$

The coefficients have units min^{-1} : equal volume flows remove different fractions of the unequal tank volumes per minute.

Step 2: Use the conserved total. Adding gives $(x + y)' = 0$, so $x + y = 10$ kg. At equilibrium $x/100 = y/200$. Consequently $(x_*, y_*) = (10/3, 20/3)$ kg and the common concentration is $1/30$ kg/L. Equal amounts would leave unequal concentrations and hence a nonzero net salt transfer.

Step 3: Solve and check the physical range. Substituting $y = 10 - x$ gives $x' = 1/2 - 3x/20$. The initial data yield

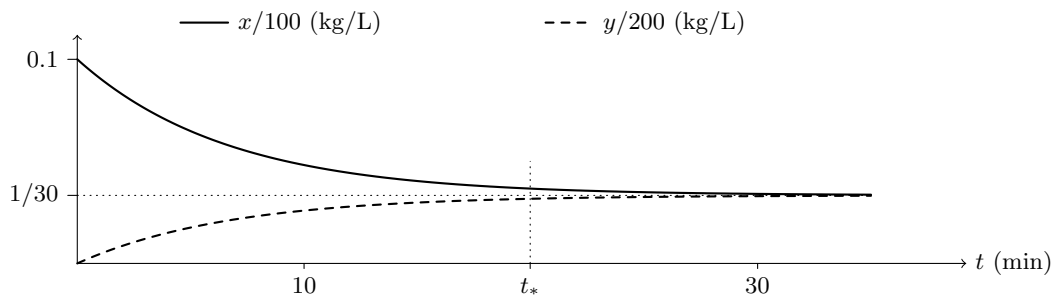
$$x = \frac{10}{3} + \frac{20}{3}e^{-3t/20}, \quad y = \frac{20}{3}(1 - e^{-3t/20}).$$

Their sum is 10, both are nonnegative, and their initial values are $(10, 0)$. Differentiation gives $x' = -e^{-3t/20}$ and $y' = e^{-3t/20}$ kg/min; substitution into the original flux balances gives the same rates.

Step 4: Compute a lasting tolerance time. The concentration difference is

$$\frac{x}{100} - \frac{y}{200} = \frac{1}{10}e^{-3t/20} > 0.$$

It decreases strictly, so its first value 0.005 occurs at $t = \frac{20}{3} \log 20 \text{ min} \approx 19.97 \text{ min}$. The inequality holds for this time and every later time. The first concentration approaches the dotted limit from above; the second approaches it from below.



Question 2

An idealized drug model has a central compartment and a peripheral compartment. Let x, y be drug amounts in mg, with time in hours. Each hour, transfer from central to peripheral is 1 times the current central amount; reverse transfer is 1 times the peripheral amount. Elimination occurs only from the central compartment, at rate 1 times its amount per hour. Immediately after a bolus, $(x(0), y(0)) = (D, 0)$, where $D > 0$ is the numerical dose in mg; no further drug enters.

Tasks

1. Derive the two amount equations and the rate of change of total drug. State which modeling assumptions justify linear transfer laws.
2. Explain why nonnegative initial amounts remain nonnegative. Add an eliminated-drug state $e(t)$ and find the conserved total, with $e(0) = 0$.
3. Solve the two-compartment IVP exactly in terms of D . Verify the initial data and determine the limiting retained and eliminated amounts.
4. Find $\int_0^\infty x(t) dt$ and $\int_0^\infty y(t) dt$ directly from the balance equations. Explain their units and why adding them does not count an eliminated mass twice.

Question 2 – Solution

Strategy. Distinguish retained mass from cumulative elimination; integrated compartment balances recover exposure without integrating long formulas.

Step 1: Write transfer and loss rates. Well-mixed compartments and constant, unsaturated first-order rate constants give $x' = -2x + y$, $y' = x - y$. The numerical coefficients have units h^{-1} . Adding gives $(x + y)' = -x$: internal transfers cancel, but central elimination removes drug. Amounts, rather than concentrations, are the states; no unspecified compartment volume is needed for these laws.

Step 2: Establish the physical state constraints. On $x = 0, y \geq 0$, $x' = y \geq 0$; on $y = 0, x \geq 0$, $y' = x \geq 0$. These inward boundary directions and uniqueness preserve the nonnegative quadrant. With $e' = x$, $e(0) = 0$, $x + y + e = D$. Thus retained amounts lie between 0 and D , and e is nondecreasing. The third state records removal, not return flow.

Step 3: Resolve the two decay modes. The roots are $\lambda_{\pm} = (-3 \pm \sqrt{5})/2$, both negative. Solving for the constants gives, with $r = \sqrt{5}$,

$$x = De^{-3t/2}(\cosh(rt/2) - r^{-1} \sinh(rt/2)), \quad y = \frac{2D}{r}e^{-3t/2} \sinh(rt/2).$$

These give $x(0) = D, y(0) = 0, x'(0) = -2D, y'(0) = D$. Differentiating the formulas gives $x' = -2x + y$ and $y' = x - y$ for all t . Since $r < 3$, both modal exponentials decay: $x, y \rightarrow 0$ and $e \rightarrow D$.

Step 4: Integrate the balances. Decay makes both integrals finite. Integrating $(x + y)' = -x$ gives $\int_0^\infty x \, dt = D$ numerically, and integrating $y' = x - y$ then gives $\int_0^\infty y \, dt = D$. Restoring the rate units,

$$\int_0^\infty x(t) \, dt = \int_0^\infty y(t) \, dt = D \text{ mg} \cdot \text{h}.$$

These are amount–time exposures in $\text{mg} \cdot \text{h}$, not eliminated masses. Their sum is $2D \text{ mg} \cdot \text{h}$ of total residence exposure, while the eliminated mass is only $D \text{ mg}$. A molecule can contribute residence time in both compartments.

Question 3

Two uniform bodies each have heat capacity $1 \text{ kJ}/^\circ\text{C}$. Their mutual thermal conductance is $1 \text{ kJ}/(\text{min}\cdot^\circ\text{C})$, and each has that same conductance to a room maintained at 20°C . Heat flow equals conductance times temperature difference; there are no internal heat sources. Initially $T_1 = 80^\circ\text{C}$ and $T_2 = 20^\circ\text{C}$. Time is in minutes.

Tasks

1. Derive the two energy balances. Introduce excess temperatures $\theta_i = T_i - 20$ and write the homogeneous system.
2. Find the total excess thermal energy and its decay law. Explain which heat transfers cancel in this sum.
3. Solve for both temperatures. Find the exact maximum temperature of the initially cooler body and when it occurs.
4. Explain why one body can warm while total excess energy decreases. Prove neither body leaves the interval $[20, 80]^\circ\text{C}$, and sketch the temperatures.

Question 3 – Solution

Strategy. Write heat-capacity balances before shifting the reference temperature; sum and difference variables isolate two decay rates.

Step 1: Balance heat for each body. For body 1, capacity times T_1' equals $(T_2 - T_1) + (20 - T_1)$ in kJ/min; the analogous balance holds for body 2. Hence

$$\theta_1' = -2\theta_1 + \theta_2, \quad \theta_2' = \theta_1 - 2\theta_2, \quad (\theta_1(0), \theta_2(0)) = (60, 0).$$

The numerical decay coefficients have units min^{-1} .

Step 2: Track total excess energy. With each capacity equal to 1 kJ/°C, excess energy is $E = \theta_1 + \theta_2$ numerically in kJ. Interbody exchange cancels, leaving $E' = -E$ and $E(t) = 60e^{-t}$ kJ. Only heat delivered to the room decreases this total.

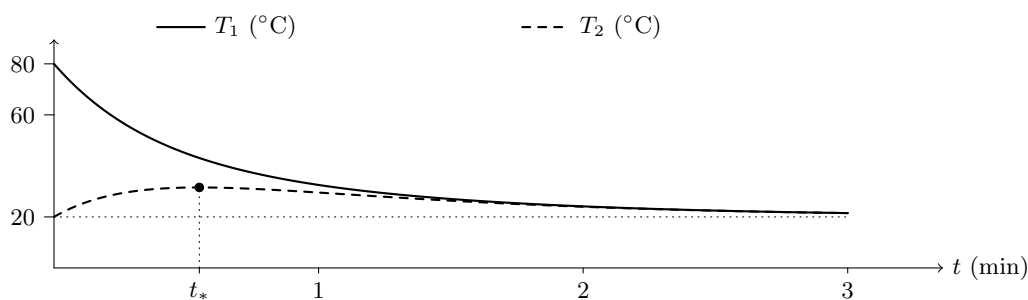
Step 3: Solve the modes and locate the warm-up peak. The difference obeys $(\theta_1 - \theta_2)' = -3(\theta_1 - \theta_2)$. Together with the sum this gives

$$T_1 = 20 + 30(e^{-t} + e^{-3t}), \quad T_2 = 20 + 30(e^{-t} - e^{-3t}).$$

Here $T_2' = 30(-e^{-t} + 3e^{-3t})$, which changes sign from positive to negative at $t_* = (\log 3)/2$. Therefore

$$T_{2,\max} = (20 + 20/\sqrt{3})^\circ\text{C}, \quad t_* = (\log 3)/2 \text{ min.}$$

Step 4: Reconcile local warming with total cooling. Initially the hotter body supplies heat to body 2 faster than body 2 loses heat to the room. Nevertheless $E' < 0$ throughout. The formulas give $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 60e^{-t} \leq 60$, so each temperature stays in $[20, 80]^\circ\text{C}$. Both tend to room temperature; the second has a genuine temporary maximum, not a higher final equilibrium.



Question 4

Two 1 kg masses slide without friction on a horizontal line. Each is attached to its outer fixed wall by a spring of stiffness 1 N/m, and a third spring of stiffness 1 N/m joins the masses. Let q_1, q_2 be displacements in metres from equilibrium, positive to the right. Initially $q_1 = a$, $q_2 = 0$ and both velocities vanish, where $a > 0$. Time is in seconds; Hooke's law applies.

Tasks

1. Derive the forces, paying attention to the change in length of the middle spring. Write a first-order system with four states and its initial vector.
2. Use the sum and difference of the displacements to solve for q_1, q_2 and their velocities.
3. Construct the total mechanical energy from the three springs and two masses. Show it is constant and find its initial value.
4. Determine whether the full initial state ever returns at a positive time. Justify your answer exactly, and sketch q_1, q_2 for $a = 0.1$ m on $0 \leq t \leq 10$ s.

Question 4 – Solution

Strategy. The coupling spring exerts opposite forces on the masses; symmetric and antisymmetric motions then become independent oscillators.

Step 1: Derive the complete state model. The middle spring's extension is $q_2 - q_1$, giving force $q_2 - q_1$ on mass 1 and its negative on mass 2. Thus $q_1'' = -2q_1 + q_2$ and $q_2'' = q_1 - 2q_2$, with numerical acceleration coefficients in s^{-2} . For $Z = (q_1, v_1, q_2, v_2)^T$,

$$Z' = (v_1, -2q_1 + q_2, v_2, q_1 - 2q_2)^T, \quad Z(0) = (a, 0, 0, 0)^T.$$

Step 2: Solve the two normal modes. Set $s = q_1 + q_2$, $d = q_1 - q_2$. Then $s'' = -s$, $d'' = -3d$, with $s(0) = d(0) = a$ and zero initial derivatives. Therefore

$$q_1 = \frac{a}{2}(\cos t + \cos \sqrt{3}t), \quad q_2 = \frac{a}{2}(\cos t - \cos \sqrt{3}t).$$

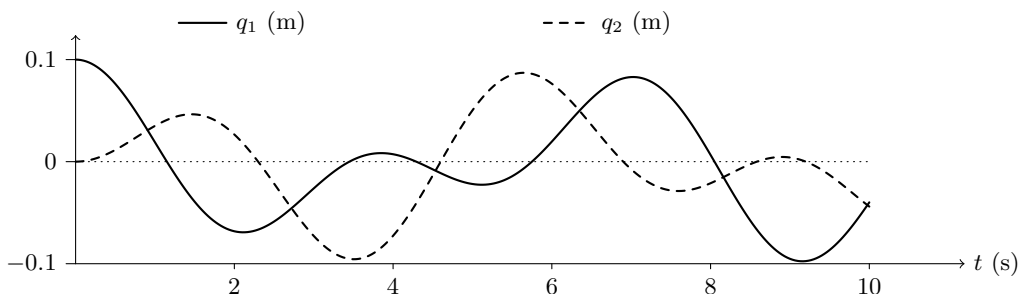
The velocities are $v_1 = -a(\sin t + \sqrt{3} \sin \sqrt{3}t)/2$ and $v_2 = -a(\sin t - \sqrt{3} \sin \sqrt{3}t)/2$. They satisfy all four initial data.

Step 3: Check the energy balance. In joules, with the stated numerical masses and stiffnesses,

$$E = \frac{1}{2}(v_1^2 + v_2^2) + \frac{1}{2}(q_1^2 + q_2^2 + (q_2 - q_1)^2).$$

Its derivative is $v_1(q_1'' + 2q_1 - q_2) + v_2(q_2'' + 2q_2 - q_1) = 0$. At the initial state $E = a^2 \text{ J}$ when a is the numerical initial displacement in metres. Energy includes the middle spring as well as the wall springs.

Step 4: Test an exact return. A return of the full state requires $s = d = a$ and $s' = d' = 0$. Thus $t = 2\pi m$ and $\sqrt{3}t = 2\pi n$ for integers m, n . For $t > 0$ this would imply $\sqrt{3} = n/m$, impossible. No exact positive-time return occurs. A near return visible in a plot cannot prove periodicity.



Question 5

Two magnetically coupled loops have inductances $L_1 = L_2 = 2$ H, mutual inductance $M = 1$ H, and resistances $R_1 = R_2 = 1$ Ω . Reference current directions and coil orientations are chosen so the mutual derivative terms have positive signs in both voltage-drop balances. At $t = 0$ a constant 3 V source is connected to loop 1; loop 2 has no source. Initially both currents are zero. Time is in seconds; the ideal linear circuit model applies.

Tasks

1. Use Kirchhoff's voltage law to derive the coupled derivative equations, then solve them for i'_1, i'_2 . Explain why the inductance matrix is invertible.
2. Solve the initial-value problem using current sum and difference, and find the limiting currents.
3. Determine the sign of $i_2(t)$ for $t > 0$ and its most negative value and time. Explain what a negative current means here, and sketch both currents.
4. With magnetic energy $W = (2i_1^2 + 2i_1i_2 + 2i_2^2)/2$, derive the power balance. Explain what changes when both voltage sources are zero.

Question 5 – Solution

Strategy. Invert the inductance matrix, not just its diagonal; the sum and difference currents see different inductances.

Step 1: Form the coupled voltage balances. Voltage drops give $2i_1' + i_2' = 3 - i_1$ and $i_1' + 2i_2' = -i_2$. The inductance determinant is $4 - 1 = 3 \text{ H}^2 > 0$, so

$$i_1' = 2 - \frac{2}{3}i_1 + \frac{1}{3}i_2, \quad i_2' = -1 + \frac{1}{3}i_1 - \frac{2}{3}i_2.$$

Each right side has units A/s. Ignoring the mutual terms would predict $i_2 \equiv 0$ and would fail the second voltage balance during the transient.

Step 2: Solve the sum and difference. Let $s = i_1 + i_2$, $d = i_1 - i_2$. Adding and subtracting the voltage equations gives $3s' = 3 - s$ and $d' = 3 - d$, with $s(0) = d(0) = 0$. Thus $s = 3(1 - e^{-t/3})$, $d = 3(1 - e^{-t})$, and

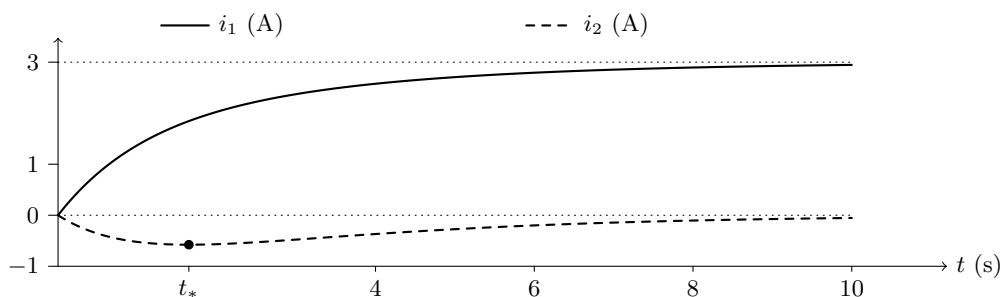
$$i_1 = 3 - \frac{3}{2}(e^{-t/3} + e^{-t}), \quad i_2 = \frac{3}{2}(e^{-t} - e^{-t/3}).$$

They tend to (3, 0) A, the resistive steady state. Their initial slopes are (2, -1) A/s, agreeing with the inverted model.

Step 3: Locate the induced-current extremum. For $t > 0$, $e^{-t} < e^{-t/3}$, so $i_2 < 0$. Its derivative vanishes when $e^{-t/3}/3 = e^{-t}$, at $t_* = (3/2) \log 3$ s. The derivative changes from negative to positive, giving

$i_{2,\min} = -1/\sqrt{3} \text{ A}$. The sign means actual current flows opposite its chosen reference direction; it is not a negative stored energy or a violation of the circuit model.

Step 4: Verify the power identity. Writing $L = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $W' = i^T L i' = i^T (V - i)$ gives $W' = 3i_1 - i_1^2 - i_2^2$ in watts. With $V = 0$ instead, $W' = -i_1^2 - i_2^2 \leq 0$. Since L has positive eigenvalues 1, 3, $W > 0$ for nonzero currents. The driven circuit can gain magnetic energy from its source; the source-free circuit can only dissipate it.



Question 6

In an idealized closed-population epidemic model, s, i, r are susceptible, infectious and removed fractions. There are no births, deaths or reinfections. Homogeneous mixing gives new infections at rate $2si$ per day and removal at rate i per day. Initially $(s, i, r) = (0.99, 0.01, 0)$, and t is in days. Assume the stated rates remain constant throughout the modeled outbreak.

Tasks

1. Derive the three balance equations and explain conservation and nonnegativity. Reduce to two independent states.
2. Eliminate time to derive a relation between i and s . Find the susceptible fraction and infectious fraction at the unique infectious peak; justify that this peak is reached.
3. Show $i(t) \rightarrow 0$ and $s(t) \rightarrow s_\infty > 0$. Derive an equation uniquely selecting s_∞ in $(0, 1/2)$ and estimate it to three decimals.
4. Sketch the physical path in the (s, i) plane with equal scales, its initial point, peak and limiting endpoint. Explain why the peak condition does not mean infections have ended.

Question 6 – Solution

Strategy. Conservation confines the dynamics; eliminating time gives an exact curve and identifies both the peak and final-size constraint.

Step 1: Balance the three populations. The equations are $\boxed{s' = -2si, i' = 2si - i, r' = i}$, with $s + i + r = 1$. Boundary rates cannot create negative fractions. More explicitly, $s = 0.99e^{-2\int_0^t i}$ and $i = 0.01e^{\int_0^t (2s-1)}$ are positive at finite times; $r' = i \geq 0$. Use $r = 1 - s - i$ to reduce the system.

Step 2: Eliminate time and locate the peak. Since $s, i > 0$, $di/ds = -1 + 1/(2s)$. Integrating through the initial point gives

$$\boxed{i = 1 - s + \frac{1}{2} \log(s/0.99)}.$$

Here $i' = i(2s - 1)$ changes sign at $s = 1/2$. This value must be reached: if $s \geq 1/2$ forever, then $i \geq 0.01$ and $s' \leq -0.02s$, a contradiction. Strict decrease of s permits only one crossing. Hence

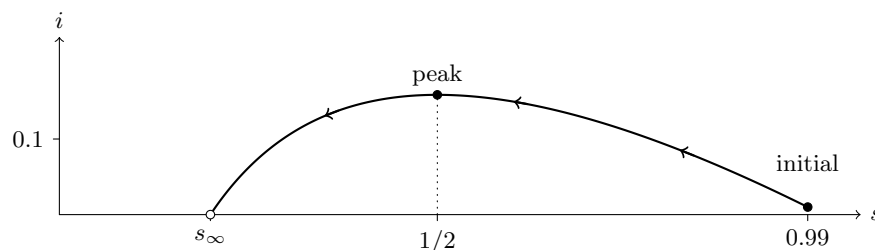
$$\boxed{i_{\max} = \frac{1}{2} + \frac{1}{2} \log(0.5/0.99) \approx 0.15845}.$$

Step 3: Identify the physical final size. Because $\int_0^\infty i = r_\infty \leq 1$, $s \geq 0.99e^{-2} > 0$. Thus decreasing s has a positive limit. Also i is integrable and has a bounded derivative on the invariant simplex, so $i \rightarrow 0$: nonvanishing peaks would have widths bounded below and contradict integrability. The peak crossing implies $s_\infty < 1/2$. Substitution gives

$$\boxed{1 - s_\infty + \frac{1}{2} \log(s_\infty/0.99) = 0, \quad s_\infty \approx 0.200}.$$

On $(0, 1/2)$ the left side is strictly increasing, tends to $-\infty$ at zero and is positive at $1/2$, so this root is unique.

Step 4: Interpret the physical trajectory. Time moves toward smaller s , up to the peak and then down toward zero infectious fraction. The limiting endpoint is open because $i(t) > 0$ at finite times. At the peak, incidence and removal balance; both are positive. The peak marks zero net infectious growth, not the end of infection.



Question 7

A continuous two-stage population model uses juvenile and adult abundances J, A , measured in thousands, with time in years. Juveniles mature at per-capita rate 1 yr^{-1} and die at rate 1 yr^{-1} . Adults die at rate 1 yr^{-1} and produce juveniles at per-capita rate $b \text{ yr}^{-1}$, where $b \geq 0$. Assume no crowding and strictly positive initial J_0, A_0 .

Tasks

1. Derive the stage equations and explain why maturation must appear with opposite signs in the two balances.
2. Find the eigenvalues and classify long-time extinction, a critical threshold, and unbounded growth as b varies. Justify that the growing mode is actually present for the stated initial data.
3. At the critical value, find a conserved weighted total, the exact solution and its limiting stage abundances.
4. For every $b \geq 0$, find the limiting ratio J/A and interpret it. Explain why this linear model cannot determine a realistic carrying capacity.

Question 7 – Solution

Strategy. Separate transfers between stages from births and deaths, then use the dominant mode and the exceptional zero eigenvalue.

Step 1: Construct the stage balances. Maturation removes juveniles and adds adults, whereas juvenile death removes individuals without adding to the adult stage. Therefore $J' = bA - 2J$, $A' = J - A$. All coefficients have units yr^{-1} . Nonnegative off-diagonal rates preserve the nonnegative quadrant; the strictly positive initial populations remain positive.

Step 2: Determine the growth threshold. The characteristic equation is $\lambda^2 + 3\lambda + 2 - b = 0$, so $\lambda_{\pm} = (-3 \pm \sqrt{1 + 4b})/2$. Put $d = \sqrt{1 + 4b}$. The adult solution has dominant coefficient

$$A(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}, \quad c_+ = \frac{J_0 + (1 + d)A_0/2}{d} > 0.$$

Thus both populations tend to zero for $0 \leq b < 2$, approach a nonzero critical equilibrium for $b = 2$, and grow without bound for $b > 2$. For $b > 2$, the dominant juvenile coefficient $(\lambda_+ + 1)c_+$ is also positive.

Step 3: Solve the critical case exactly. When $b = 2$, $K = J + 2A = J_0 + 2A_0$ is conserved. Since $A' = K - 3A$,

$$A = K/3 + (A_0 - K/3)e^{-3t}, \quad J = K/3 - 2(A_0 - K/3)e^{-3t}.$$

Both tend to $K/3$ thousand. The critical equilibria form the line $J = A$; there is no single equilibrium amount independent of initial data.

Step 4: Interpret the asymptotic stage ratio. Since $J = A' + A$ and $c_+ > 0$, division by the dominant adult mode gives

$$\lim_{t \rightarrow \infty} \frac{J}{A} = \lambda_+ + 1 = \frac{\sqrt{1 + 4b} - 1}{2}.$$

For $b = 0$, this limit is zero: juveniles decay at rate 2, adults at rate 1. The ratio describes relative stage composition even when the entire population tends to zero; it does not imply constant abundance. For $b > 2$ there is no crowding term to halt growth. A carrying capacity requires additional density-dependent assumptions absent from this model.

Question 8

An ideal temperature controller is modeled in dimensionless variables. The room temperature is 0, the target is 1, and the scaled heater output is $P(t)$. The plant obeys $T' = -T + P$. A controller with memory sets

$$P(t) = k \int_0^t [1 - T(\tau)] d\tau, \quad T(0) = 0, \quad k > 0.$$

There is no heater saturation, delay or heat-source noise in this ideal model.

Tasks

1. Convert the plant and controller into a two-state autonomous IVP. Find its equilibrium and explain why P is a state rather than a fixed parameter.
2. Shift to the equilibrium and show it attracts every initial state for every $k > 0$. Classify the eigenvalues at the critical gain.
3. For $k = 1/4$, solve the specified IVP. Prove both T and P stay in $[0, 1)$ at finite times and increase toward 1. Sketch them.
4. Replace the memory controller by the proportional rule $P = k(1 - T)$, with the same $k > 0$. Find its limiting temperature and explain the difference in steady error and the practical scope of the ideal memory model.

Question 8 – Solution

Strategy. The accumulated error supplies an additional state; its equilibrium condition eliminates the temperature offset of proportional feedback.

Step 1: Introduce the controller state. Differentiation gives

$$T' = -T + P, \quad P' = k(1 - T), \quad (T(0), P(0)) = (0, 0).$$

At equilibrium, $T = 1$ and $P = 1$. Heater output depends on accumulated past error, so two states with the same temperature can have different future motion if their current P values differ.

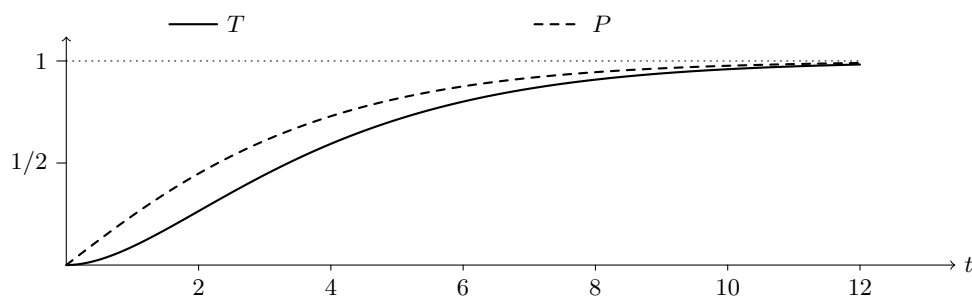
Step 2: Check stability and the critical gain. For $x = T - 1$, $u = P - 1$, the matrix is $\begin{pmatrix} -1 & 1 \\ -k & 0 \end{pmatrix}$. Its roots are $(-1 \pm \sqrt{1 - 4k})/2$. For $0 < k < 1/4$ both are real and negative; at $k = 1/4$ there is a repeated root $-1/2$; for $k > 1/4$ they are complex with real part $-1/2$. Every homogeneous mode decays, including the polynomial times exponential at the repeated root. Thus the equilibrium attracts every initial state within this ideal linear model.

Step 3: Solve the critical-gain response. Eliminating P gives $T'' + T' + T/4 = 1/4$, with $T(0) = T'(0) = 0$. The repeated-root solution and $P = T' + T$ yield

$$T = 1 - (1 + t/2)e^{-t/2}, \quad P = 1 - (1 + t/4)e^{-t/2}.$$

The initial values vanish and $T' = te^{-t/2}/4 \geq 0$, $P' = (1/4 + t/8)e^{-t/2} > 0$. The latter equals $(1 - T)/4$, checking the controller equation. The positive subtracted terms and monotonicity imply $0 \leq T, P < 1$ at finite t , and both tend to 1. Here the heater output stays nonnegative without needing a saturation rule.

Step 4: Compare steady errors. Proportional feedback instead gives $T' = k - (1 + k)T$, so $T \rightarrow k/(1 + k)$ and the steady error is $1/(1 + k) > 0$. The memory controller can retain $P = 1$ even when the current error is zero; a proportional controller would then set $P = 0$. Real actuator limits and delays can change stability or feasibility, so the ideal conclusions apply under the stated assumptions, not to every physical controller.



Question 9

In a closed, well-mixed vessel of constant volume, the reversible reaction $A \rightleftharpoons 2B$ has forward reaction rate $k_f a$ and reverse reaction rate $k_r b^2$ in $\text{mol}/(\text{L}\cdot\text{min})$, where a, b are concentrations in mol/L . Here $k_f = 1 \text{ min}^{-1}$ and $k_r = 1 \text{ L}/(\text{mol}\cdot\text{min})$. Initially $a = 1$, $b = 0 \text{ mol}/\text{L}$. There are no other reactions. Use numerical concentration values in these units.

Tasks

1. Derive the concentration equations from the stoichiometric coefficients. Identify a conserved weighted concentration and explain why $a + b$ is not conserved.
2. Reduce to a scalar equation for b and identify its physically allowed interval. Show this interval is invariant.
3. Find the unique physical equilibrium and prove the stated initial response approaches it monotonically in b .
4. Solve explicitly for $b(t)$ by separation, then recover $a(t)$. Explain why the second algebraic root of the equilibrium equation is inadmissible and why the physical equilibrium is not reached at finite time.

Question 9 – Solution

Strategy. Each forward reaction consumes one A and creates two B; the resulting weighted conservation law reduces the nonlinear system.

Step 1: Respect the reaction stoichiometry. A net forward reaction rate $a - b^2$ gives

$$\boxed{a' = -a + b^2, \quad b' = 2a - 2b^2}$$

in the chosen numerical units. Hence $(2a + b)' = 0$ and $\boxed{2a + b = 2}$ mol/L. The sum $a + b$ has derivative $a - b^2$ and is not conserved: the reaction changes the number of molecules, while preserving the weighted building-block total.

Step 2: Reduce within the physical domain. With $a = 1 - b/2$, the scalar equation is $\boxed{b' = 2 - b - 2b^2}$, with $0 \leq b \leq 2$ required by $a, b \geq 0$. At $b = 0$ the derivative is $2 > 0$, and at $b = 2$ it is $-8 < 0$. Uniqueness and inward boundary directions make the interval invariant. The concentrations cannot be extended to negative values merely because an algebraic equation has another root.

Step 3: Identify the attracting physical equilibrium. Let $d = \sqrt{17}$ and $r_{\pm} = (-1 \pm d)/4$. Then $b' = -2(b - r_+)(b - r_-)$, with $r_- < 0 < r_+ < 2$. The only physical equilibrium is

$$\boxed{b_* = r_+, \quad a_* = 1 - r_+/2.}$$

For $0 \leq b < r_+$, $b' > 0$. Uniqueness prevents crossing the equilibrium, so the solution from zero increases and stays below r_+ . Its limit must be a zero of the scalar right side; otherwise its positive derivative would remain bounded away from zero. Thus it tends to r_+ .

Step 4: Separate and impose the initial concentration. Since $2(r_+ - r_-) = d$, integration gives

$$\log \left| \frac{b - r_+}{b - r_-} \right| = -dt + \log \left| \frac{r_+}{r_-} \right|.$$

The ratio is negative on the physical branch starting at zero. Solving gives

$$\boxed{b(t) = \frac{r_+(1 - e^{-dt})}{1 - (r_+/r_-)e^{-dt}}, \quad a(t) = 1 - \frac{b(t)}{2}.}$$

The denominator exceeds 1 at finite times because $r_+/r_- < 0$. The formulas give $(a, b) = (1, 0)$ initially and the stated equilibrium as $t \rightarrow \infty$. For every finite t , $b < r_+$, so arrival is asymptotic.

Question 10

A conserved tracer moves among three well-mixed compartments. Transfers occur only along $1 \rightleftharpoons 2 \rightleftharpoons 3$, with constant first-order rates, no external input and no loss. Let x_j be fractions of the total tracer, and use time in hours. Three experiments start with all tracer in compartment j ; their measured initial derivative vectors are

$$v_1 = (-2, 2, 0)^T, \quad v_2 = (1, -4, 3)^T, \quad v_3 = (0, 2, -2)^T.$$

Treat these measurements as exact within the stated model class.

Tasks

1. Recover all four transfer rates and the matrix in $X' = AX$. Explain why the experiment vectors give columns rather than rows.
2. Verify conservation and nonnegativity, and explain why the unit simplex is the physical state space.
3. Find the unique equilibrium with total fraction 1. Check balance of each pair of opposing transfer fluxes.
4. Solve the first experiment exactly using the three modes. Determine whether equilibrium is reached at finite time, and state what the identification result does and does not establish about alternative model classes.

Question 10 – Solution

Strategy. Initial states equal to coordinate vectors reveal matrix columns; conservation and nonnegative transfer rates then provide physical checks.

Step 1: Identify the directed rates. Since $X'(0) = Ae_j$, the j th experimental vector is column j . Thus

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 3 & -2 \end{pmatrix}.$$

The rates are $k_{1 \rightarrow 2} = 2$, $k_{2 \rightarrow 1} = 1$, $k_{2 \rightarrow 3} = 3$, $k_{3 \rightarrow 2} = 2 \text{ h}^{-1}$. Each diagonal entry is minus the total outgoing rate from its compartment. Transposing the measurements would incorrectly replace these column balances with row balances.

Step 2: Check the physical state space. Every column sums to zero, so $(x_1 + x_2 + x_3)' = 0$. At a boundary $x_j = 0$ with the other fractions nonnegative, the corresponding derivative is a sum of nonnegative incoming fluxes. The simplex $\boxed{x_j \geq 0, x_1 + x_2 + x_3 = 1}$ is therefore invariant. Conservation bounds every fraction by 1.

Step 3: Balance the equilibrium fluxes. The end-compartment equations require $x_2 = 2x_1$ and $2x_3 = 3x_2$. Normalization then gives $\boxed{X_* = (1, 2, 3)^T/6}$. Indeed the opposing fluxes on the first edge are $2(1/6) = 1(2/6)$, and on the second are $3(2/6) = 2(3/6)$. These two relations and the total also prove uniqueness of the normalized equilibrium.

Step 4: Resolve the first experiment and its interpretation. The eigenpairs are $0, (1, 2, 3)^T$, $-2, (1, 0, -1)^T$ and $-6, (1, -4, 3)^T$. Resolving $(1, 0, 0)^T$ gives

$$X = X_* + \frac{3}{4}e^{-2t}(1, 0, -1)^T + \frac{1}{12}e^{-6t}(1, -4, 3)^T.$$

Equivalently,

$$\boxed{x_1 = \frac{1}{6} + \frac{3}{4}e^{-2t} + \frac{1}{12}e^{-6t}, \quad x_2 = \frac{1}{3}(1 - e^{-6t}), \quad x_3 = \frac{1}{2} - \frac{3}{4}e^{-2t} + \frac{1}{4}e^{-6t}.$$

The coefficients reproduce the initial state, and each mode satisfies the matrix equation. Since $x_2 < 1/3$ at finite times, equilibrium is approached but never reached then. The experiments uniquely identify this constant linear transfer matrix; they do not rule out nonlinear or time-dependent models sharing the same three measured initial slopes.