

Question 1

For $t \geq 0$, consider the coupled initial-value problem

$$x' = -2x + y, \quad y' = 2x - 3y, \quad x(0) = 1, \quad y(0) = 0.$$

Write $F(s) = \mathcal{L}\{x\}(s)$ and $G(s) = \mathcal{L}\{y\}(s)$, using one-sided Laplace transforms. Recover both components by solving the transformed algebraic system.

Tasks

1. Derive the two transformed equations, retaining the initial terms. Solve for F and G and state a common half-plane of convergence.
2. Invert both transforms and verify the initial values and original differential equations.
3. Find the transform and time response of $w = x - y$. Explain why one pole of the full system is absent from this observation.
4. Find the exact time and value of the largest $y(t)$ on $t \geq 0$. Explain how y can initially increase while both system modes decay.

Question 1 – Solution

Strategy. Keep the initial data on the right side of the transformed system, then distinguish component responses from observed combinations.

Step 1: Solve the algebraic system. The derivative rule gives $(s+2)F - G = 1$ and $-2F + (s+3)G = 0$. Their determinant is $(s+1)(s+4)$, so

$$F = \frac{s+3}{(s+1)(s+4)}, \quad G = \frac{2}{(s+1)(s+4)}.$$

Both transforms converge for $\operatorname{Re} s > -1$. Dropping the initial term 1 would incorrectly give the zero solution.

Step 2: Invert and check. Partial fractions give

$$x = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}, \quad y = \frac{2}{3}(e^{-t} - e^{-4t}).$$

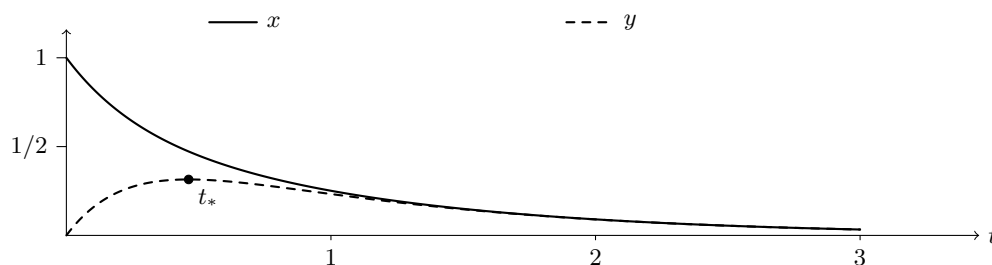
At zero these are $(1, 0)$. Differentiation gives $x' = -\frac{2}{3}e^{-t} - \frac{4}{3}e^{-4t}$ and $y' = -\frac{2}{3}e^{-t} + \frac{8}{3}e^{-4t}$, which equal $-2x + y$ and $2x - 3y$.

Step 3: Explain the cancelled pole. Subtracting the transforms gives $F - G = 1/(s+4)$, hence $w = e^{-4t}$. The slow mode is a multiple of $(1, 1)^T$; subtracting its components annihilates it. The state still contains this mode. Cancellation in an observation does not remove a mode from the system.

Step 4: Locate the component peak. Since $y' = \frac{2}{3}e^{-t}(4e^{-3t} - 1)$, it changes sign once, from positive to negative, at $t_* = \log(4)/3$. Thus

$$\max_{t \geq 0} y(t) = \frac{1}{2} 4^{-1/3}, \quad t_* = \log(4)/3.$$

Here $y'(0) = 2$ because the first component feeds the second. Decay of each mode does not imply monotonic decay of every component: their signed combination initially grows. The curve tends back to zero.



Question 2

Let $Z = (x, y, z)^T$ satisfy $Z' = AZ$ for $t \geq 0$, where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \quad Z(0) = (a, b, c)^T.$$

The characteristic polynomial has a triple root. Investigate how the initial vector determines the poles actually present in each transformed component.

Tasks

1. Compute $(sI - A)^{-1}$ directly, and verify its product with $sI - A$ is I . State where the inverse matrix exists.
2. Use the resolvent to find all three Laplace transforms and invert them for arbitrary real a, b, c .
3. For the initial state $(0, 0, 1)^T$, verify the original equations and explain the different polynomial degrees in the three components.
4. Classify the pole order of $\mathcal{L}\{x\}$ at $s = -2$ for every (a, b, c) , including the case of no pole. Does a triple characteristic root force a triple pole in every component?

Question 2 – Solution

Strategy. Invert the triangular matrix before substituting data; its powers of $(s+2)^{-1}$ display how a chain transmits the initial state.

Step 1: Compute the resolvent. Put $q = s + 2$. For $q \neq 0$,

$$(sI - A)^{-1} = \begin{pmatrix} q^{-1} & q^{-2} & q^{-3} \\ 0 & q^{-1} & q^{-2} \\ 0 & 0 & q^{-1} \end{pmatrix}.$$

Multiplication by the upper triangular matrix with diagonal q and superdiagonal -1 gives diagonal entries 1; each off-diagonal entry cancels. The inverse exists exactly for $s \neq -2$. This algebraic domain is larger than the convergence half-plane $\operatorname{Re} s > -2$ for the nonzero responses.

Step 2: Multiply by the initial vector. The transforms are $a/q + b/q^2 + c/q^3$, $b/q + c/q^2$, and c/q . Since $\mathcal{L}\{t^n e^{-2t}\} = n!/(s+2)^{n+1}$,

$$x = e^{-2t}(a + bt + ct^2/2), \quad y = e^{-2t}(b + ct), \quad z = ce^{-2t}.$$

The factorial is essential: the inverse of q^{-3} is $t^2 e^{-2t}/2$.

Step 3: Check transmission along the chain. For the specified initial state,

$$Z = (t^2 e^{-2t}/2, te^{-2t}, e^{-2t})^T.$$

Directly, $z' = -2z$, $y' = -2y + z$ and $x' = -2x + y$, with the required initial state. After multiplication by e^{2t} , the equations become $(e^{2t}z)' = 0$, $(e^{2t}y)' = e^{2t}z$ and $(e^{2t}x)' = e^{2t}y$. Two successive integrations produce the quadratic factor.

Step 4: Classify actual pole orders. The numerator of the first transform over q^3 is $aq^2 + bq + c$. Its order at $q = 0$ gives the exhaustive classification:

initial-data condition	pole order of $\mathcal{L}\{x\}$
$c \neq 0$	3
$c = 0, b \neq 0$	2
$c = b = 0, a \neq 0$	1
$a = b = c = 0$	no pole

Thus the characteristic multiplicity supplies a possible order, not an order every response must attain. Even for $c \neq 0$, the third component has only a simple pole, whereas the first has a triple pole.

Question 3

A delayed input drives the zero-state system

$$x' = -x + H(t - 2), \quad y' = x - 2y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

Here $H(t - a)$ is zero for $t < a$ and one for $t > a$; its value at the switch does not affect the continuous state. Interpret derivatives there one-sidedly.

Tasks

1. Find the transforms of both components and identify the factor that encodes the input delay.
2. Invert them to obtain a piecewise solution. Explain why multiplying an undelayed response by e^{-2t} does not implement a delay.
3. Determine continuity of x, y, x', y' at $t = 2$, and find the one-sided values of y'' there.
4. Prove both components increase after the switch and approach their limits without overshoot. Sketch their time responses, marking the switch and limiting levels.

Question 3 – Solution

Strategy. First invert the undelayed rational factors, then translate the entire response using the second shifting theorem.

Step 1: Transform both equations. With $F = \mathcal{L}\{x\}$ and $G = \mathcal{L}\{y\}$, $(s+1)F = e^{-2s}/s$ and $(s+2)G = F$. Hence

$$F = \frac{e^{-2s}}{s(s+1)}, \quad G = \frac{e^{-2s}}{s(s+1)(s+2)}, \quad \operatorname{Re} s > 0.$$

It is e^{-2s} , rather than an exponential in time, that encodes the delay.

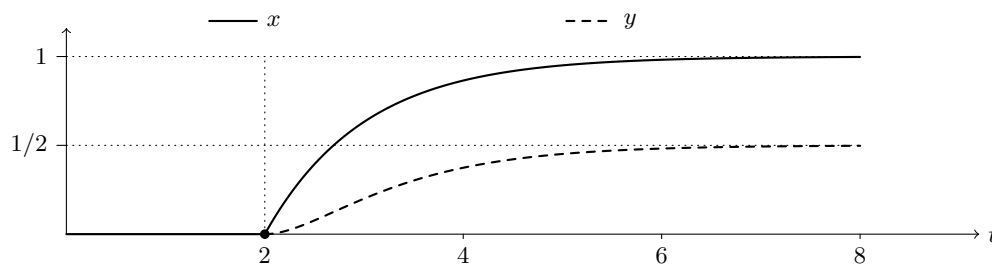
Step 2: Shift the inverse transforms. The undelayed inverses are $1 - e^{-t}$ and $\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} = \frac{1}{2}(1 - e^{-t})^2$. Writing $\tau = t - 2$, the solution is zero for $t < 2$, and for $t \geq 2$,

$$x = 1 - e^{-\tau}, \quad y = \frac{1}{2}(1 - e^{-\tau})^2.$$

These formulas give zero at the joining point. Multiplication by e^{-2t} would change decay rates while generally leaving a nonzero response before $t = 2$; it is a different transform operation.

Step 3: Check the switch regularity. Both states are continuous. Before the switch their derivatives vanish; after it $x' = e^{-\tau}$ and $y' = e^{-\tau} - e^{-2\tau}$. Thus $x'(2^-) = 0$, $x'(2^+) = 1$, whereas $y'(2^-) = y'(2^+) = 0$. Afterward $y'' = -e^{-\tau} + 2e^{-2\tau}$, so $y''(2^-) = 0$, $y''(2^+) = 1$. In particular y is continuously differentiable at the switch but is not twice differentiable there.

Step 4: Establish shape and limits. For $\tau > 0$, $x' > 0$ and $y' = e^{-\tau}(1 - e^{-\tau}) > 0$. Also $0 < x < 1$ and $0 < y < 1/2$, with limits 1 and $1/2$. These inequalities prove the absence of overshoot for all later times. The dotted horizontal guides in the sketch are limits; the solid and dashed curves are the components. Both remain zero throughout the waiting interval.



Question 4

For $a > 0$, a single impulse acts on an undamped system:

$$x' = y + \alpha\delta(t - a), \quad y' = -x + \beta\delta(t - a), \quad (x(0), y(0)) = (1, 0).$$

Here $\delta(t - a)$ has unit mass, α, β are real, and states are piecewise smooth with jumps determined by integrating the equations. Use right-hand values at the impulse when writing the response.

Tasks

1. Transform the system and find $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ for general a, α, β .
2. Invert the transforms and verify both jump conditions directly. Explain why this impulse may create jumps in the states themselves.
3. Find the unique pair (α, β) that leaves the state identically zero for all $t > a$.
4. If only the second equation can receive an impulse ($\alpha = 0$), find all admissible stopping times and amplitudes. Identify the earliest time and sketch its two component responses, showing one-sided values.

Question 4 – Solution

Strategy. The shifted impulse produces a shifted homogeneous response; stopping requires cancellation of the complete pre-impulse state.

Step 1: Solve in the transform domain. The equations are $sF - G = 1 + \alpha e^{-as}$ and $F + sG = \beta e^{-as}$. Thus, for $\operatorname{Re} s > 0$,

$$F = \frac{s}{s^2 + 1} + e^{-as} \frac{\alpha s + \beta}{s^2 + 1}, \quad G = \frac{-1}{s^2 + 1} + e^{-as} \frac{-\alpha + \beta s}{s^2 + 1}.$$

This half-plane suffices for all parameter choices, even when cancellation makes the actual response have compact support.

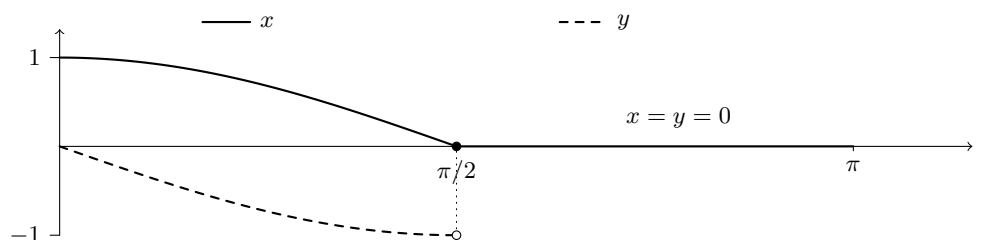
Step 2: Invert and check jumps. For $t < a$, $(x, y) = (\cos t, -\sin t)$. For $t \geq a$, with $\tau = t - a$,

$$x = \cos t + \alpha \cos \tau + \beta \sin \tau, \quad y = -\sin t - \alpha \sin \tau + \beta \cos \tau.$$

Hence $[x]_a = \alpha$, $[y]_a = \beta$, where $[f]_a = f(a^+) - f(a^-)$. Integrating the original equations across a gives these same jumps: the ordinary bounded terms contribute zero in the shrinking interval, while the impulses contribute their masses. Away from a the ordinary equations hold.

Step 3: Cancel the full state. The state just before a is $(\cos a, -\sin a)$. The only kick making the right-hand state zero is $(\alpha, \beta) = (-\cos a, \sin a)$. The ensuing homogeneous IVP then stays zero by uniqueness. Conversely, an identically zero future requires that same right-hand state, so the kick is unique.

Step 4: Restrict the available impulse. With $\alpha = 0$, cancellation requires $\cos a = 0$. Thus $a = \pi/2 + k\pi$, $\beta = (-1)^k$, $k = 0, 1, 2, \dots$. The earliest is $a = \pi/2$, $\beta = 1$. There x is continuous at zero, while y jumps from -1 to 0 . Both vanish afterward. The vertical dotted segment marks a jump, not values traversed continuously by the solution.



Question 5

Consider, for $t \geq 0$,

$$Z' = AZ + B, \quad A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad Z(0) = \begin{pmatrix} p \\ q \end{pmatrix}.$$

A student cancels a factor $s - 2$ in the zero-initial-state transform and concludes that every solution of this forced system is bounded.

Tasks

1. Compute $(sI - A)^{-1}B$ and find the response when $p = q = 0$ using Laplace transforms.
2. Explain the cancellation using the direction of B , and identify the system mode that the constant input does not excite from zero.
3. Retain the initial-state term in the transformed system and determine exactly which (p, q) produce bounded solutions on $[0, \infty)$.
4. Perturb the zero initial state to $(\varepsilon, 2\varepsilon)^T$, with $\varepsilon \neq 0$. Find the exact difference from the zero-state response and assess the student's conclusion.

Question 5 – Solution

Strategy. Separate the input contribution from the initial-state contribution before cancelling rational factors.

Step 1: Calculate the zero-state response. The determinant is $s^2 - s - 2 = (s - 2)(s + 1)$, and

$$(sI - A)^{-1}B = \frac{1}{(s - 2)(s + 1)} \begin{pmatrix} s - 1 & 1 \\ 2 & s \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{s + 1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The constant input contributes an additional $1/s$. Inverting yields $\boxed{Z_0(t) = (1 - e^{-t})(1, -1)^T}$, which is bounded. Its transform converges for $\operatorname{Re} s > 0$; the cancelled expression extends through $s = 2$, where the full matrix resolvent itself is undefined.

Step 2: Identify why the pole disappears. We have $AB = -B$, so the input lies entirely in the stable eigendirection. The other eigenpair is $2, (1, 2)^T$. From zero, this input never excites that direction. The pole cancellation describes the chosen input and data; it does not change either eigenvalue of A .

Step 3: Include all initial conditions. Write $(p, q)^T = c_+(1, 2)^T + c_-(1, -1)^T$, where $c_+ = (p + q)/3$ and $c_- = (2p - q)/3$. The transform contains

$$\frac{c_+}{s - 2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{c_-}{s + 1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{s(s + 1)} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore

$$Z = c_+ e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + [1 + (c_- - 1)e^{-t}] \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The growing vector cannot be cancelled by a bounded term, so boundedness holds exactly when $\boxed{p + q = 0}$. Then the limit is $(1, -1)^T$. A common convergence half-plane for the general family is $\operatorname{Re} s > 2$.

Step 4: Test sensitivity to the omitted term. For the specified perturbation, $c_+ = \varepsilon$ and $c_- = 0$. The exact difference is $\boxed{Z - Z_0 = \varepsilon e^{2t}(1, 2)^T}$, whose Euclidean norm is $\sqrt{5}|\varepsilon|e^{2t} \rightarrow \infty$. Thus arbitrarily small initial perturbations in that direction destroy boundedness. The student's inference confuses a bounded zero-state response with boundedness for all initial states.

Question 6

Two inputs with the same frequency drive the same system from rest:

$$x' = -y + \cos t, \quad y' = x + \sigma \sin t, \quad x(0) = y(0) = 0, \quad \sigma \in \{1, -1\}.$$

Use Laplace transforms to investigate why a frequency match alone does not settle whether this vector input generates growing oscillations.

Tasks

1. Derive both transformed components for general σ , showing the denominator before any cancellation.
2. For $\sigma = 1$, invert the transforms and verify the original equations. Determine the Euclidean norm for $t \geq 0$.
3. For $\sigma = -1$, simplify before inversion, solve the IVP and prove boundedness.
4. Compare the pole orders and the input directions in the two cases. Is every sinusoidal vector forcing at the natural frequency resonant in the sense of an unbounded zero-state response?

Question 6 – Solution

Strategy. Solve the coupled algebra first; vector numerators can either retain or cancel the repeated oscillator poles.

Step 1: Derive the rational responses. The transformed equations are $sF + G = s/(s^2 + 1)$ and $-F + sG = \sigma/(s^2 + 1)$. Solving gives, on the sufficient half-plane $\operatorname{Re} s > 0$,

$$F = \frac{s^2 - \sigma}{(s^2 + 1)^2}, \quad G = \frac{(1 + \sigma)s}{(s^2 + 1)^2}.$$

The numerator, not the unsimplified denominator alone, decides actual pole order.

Step 2: Invert the co-rotating input. For $\sigma = 1$, differentiating the standard transforms with respect to s gives $\mathcal{L}\{t \cos t\} = (s^2 - 1)/(s^2 + 1)^2$ and $\mathcal{L}\{t \sin t\} = 2s/(s^2 + 1)^2$. Hence

$$x = t \cos t, \quad y = t \sin t, \quad \|Z(t)\| = t.$$

Indeed $x' = \cos t - t \sin t = -y + \cos t$ and $y' = \sin t + t \cos t = x + \sin t$, with zero initial state. The response is unbounded even though each forcing component is bounded.

Step 3: Invert the opposite rotation. For $\sigma = -1$, cancellation gives $F = 1/(s^2 + 1)$ and $G = 0$. Consequently $x = \sin t, y = 0$, with norm $|\sin t| \leq 1$. The first equation reads $\cos t = 0 + \cos t$; the second reads $0 = \sin t - \sin t$. Both initial conditions hold. No polynomial factor survives in this response.

Step 4: Interpret the distinction. For $\sigma = 1$, both transforms have double poles at $s = \pm i$; for $\sigma = -1$, the first has simple poles and the second is zero. The free system rotates counterclockwise. The input $(\cos t, \sin t)$ rotates with it, whereas $(\cos t, -\sin t)$ rotates oppositely. More explicitly, write $Z = R(t)V$, where R is the counterclockwise rotation matrix. Then $V' = (1, 0)^T$ in the first case and $V' = (\cos 2t, -\sin 2t)^T$ in the second. Their integrals respectively grow linearly and remain bounded. A frequency match alone therefore does not force an unbounded response; the vector direction and resulting pole cancellations matter.

Question 7

A scalar input drives a two-stage system:

$$x' = -x + u(t), \quad y' = x - 3y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

Assume u is continuous and of exponential order. In the inverse task, assume a prescribed target y and its first two derivatives are continuous and of exponential order. All claims about input bounds apply for every $t \geq 0$.

Tasks

1. Derive both transfer expressions in the Laplace domain and invert their kernels to express x and y as convolutions with u .
2. If $0 \leq u \leq 1$, prove a sharp constant upper bound on $y(t)$. Explain in what sense the bound is sharp.
3. Recover x and u from a prescribed y . State the initial compatibility conditions and justify uniqueness of the recovered input.
4. Apply the inverse formula to $y(t) = t^2 e^{-t}$. Find x, u , decide whether $0 \leq u \leq 1$ is possible, and give a second obstruction using the target's maximum.

Question 7 – Solution

Strategy. Positive inverse kernels give all-time bounds; elimination then turns the same system into an exact input-recovery formula.

Step 1: Invert the transfer kernels. If $U = \mathcal{L}\{u\}$, the zero initial values give $F = U/(s+1)$ and $G = U/[(s+1)(s+3)]$. Partial fractions yield

$$x(t) = \int_0^t e^{-(t-r)} u(r) dr, \quad y(t) = \frac{1}{2} \int_0^t (e^{-(t-r)} - e^{-3(t-r)}) u(r) dr.$$

These identities hold on any common right half-plane of convergence and, after inversion, for all $t \geq 0$.

Step 2: Use positivity to obtain a sharp bound. The kernel $k(v) = (e^{-v} - e^{-3v})/2$ is nonnegative for $v \geq 0$. For $0 \leq u \leq 1$,

$$0 \leq y(t) \leq \int_0^t k(v) dv = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} < \frac{1}{3} \quad (t < \infty).$$

The strict upper inequality follows from the positive remaining tail of k ; at $t = 0$ it also holds. With $u \equiv 1$, $y(t) \rightarrow 1/3$. Thus $1/3$ is the least uniform constant upper bound, although it is not attained at finite time from zero.

Step 3: Recover the input and its compatibility conditions. The second equation forces $x = y' + 3y$. Substitution into the first gives

$$u = y'' + 4y' + 3y, \quad y(0) = 0, \quad y'(0) = 0.$$

The conditions are necessary for both initial states to vanish. Conversely, under the stated regularity, these formulas satisfy both equations and the zero initial values whenever the conditions hold. They prove uniqueness: any input producing that y must give exactly this x and u .

Step 4: Test the proposed target. For $y = t^2 e^{-t}$, the compatibility conditions hold, and direct calculation gives $x = (2t + 2t^2)e^{-t}$, $u = (2 + 4t)e^{-t}$. The unique input is nonnegative but already has $u(0) = 2 > 1$; its maximum is $4e^{-1/2} > 1$ at $t = 1/2$. Independently, the target peaks at $t = 2$ with value $4e^{-2} > 1/3$, contradicting the kernel bound. It is therefore achievable by a continuous input, but not by an input confined to $[0, 1]$.

Question 8

For $\varepsilon \geq 0$, let

$$x' = -\varepsilon x - y + 1, \quad y' = x - \varepsilon y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

A student evaluates $\lim_{s \rightarrow 0^+} s\mathcal{L}\{x\}$ and $\lim_{s \rightarrow 0^+} s\mathcal{L}\{y\}$ and declares these the limiting state for every ε . For rational transforms, the final-value theorem requires all poles of the simplified $s\mathcal{L}\{f\}$ to have negative real part.

Tasks

1. Find both transformed components for general ε and calculate the student's two proposed limits.
2. Solve the case $\varepsilon = 0$. Check the theorem's hypothesis and determine whether either component has a limit as $t \rightarrow \infty$.
3. For $\varepsilon > 0$, invert the transforms and establish the true limiting state. Explain why the theorem now applies.
4. Show that $y_\varepsilon \rightarrow y_0$ at each fixed time as $\varepsilon \rightarrow 0^+$ but not uniformly on $[0, \infty)$. Use times $2\pi n$ to obtain a lower bound on the uniform error. Sketch y_0 and $y_{1/2}$.

Question 8 – Solution

Strategy. Check pole locations before applying the final-value theorem, and distinguish fixed-time convergence from an infinite-time assertion.

Step 1: Solve the transformed system. With $D = (s + \varepsilon)^2 + 1$, the equations give

$$F = \frac{s + \varepsilon}{sD}, \quad G = \frac{1}{sD}.$$

A sufficient convergence half-plane is $\operatorname{Re} s > 0$. The proposed final values are $\varepsilon/(1 + \varepsilon^2)$ and $1/(1 + \varepsilon^2)$.

Step 2: Examine the undamped case. At $\varepsilon = 0$, inversion yields $\boxed{x = \sin t, y = 1 - \cos t}$. Neither has a limit: the values along successive maxima and minima differ. Both sF and sG have poles at $\pm i$, so the stated theorem's hypothesis fails. The finite algebraic values 0 and 1 do not prove convergence in time.

Step 3: Resolve the damped response. For $\varepsilon > 0$, decompose, for example,

$$G = \frac{1}{1 + \varepsilon^2} \left(\frac{1}{s} - \frac{s + \varepsilon}{D} - \frac{\varepsilon}{D} \right).$$

Inverting this and the analogous decomposition of F gives

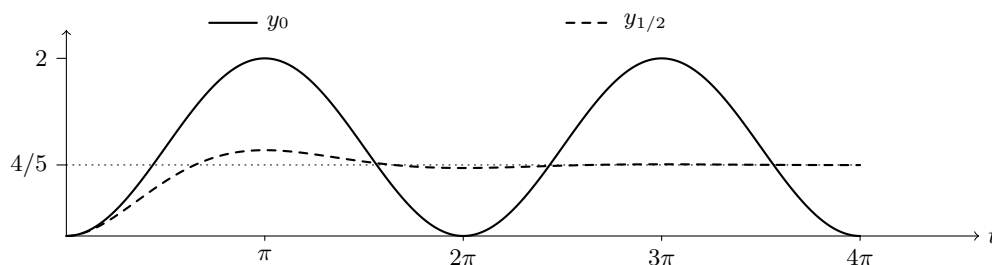
$$x_\varepsilon = \frac{\varepsilon - e^{-\varepsilon t}(\varepsilon \cos t - \sin t)}{1 + \varepsilon^2}, \quad y_\varepsilon = \frac{1 - e^{-\varepsilon t}(\cos t + \varepsilon \sin t)}{1 + \varepsilon^2}.$$

Both start at zero. The exponentially decaying terms leave the limit $\boxed{(\varepsilon, 1)^T / (1 + \varepsilon^2)}$. Now the poles of sF and sG are $-\varepsilon \pm i$, strictly in the left half-plane, as required. The limit also solves the equilibrium equations.

Step 4: Test uniformity on the whole half-line. At fixed t , the formula gives $y_\varepsilon(t) \rightarrow 1 - \cos t = y_0(t)$. But $y_0(2\pi n) = 0$, whereas $y_\varepsilon(2\pi n) = (1 - e^{-2\pi n \varepsilon})/(1 + \varepsilon^2)$. Consequently

$$\sup_{t \geq 0} |y_\varepsilon(t) - y_0(t)| \geq \frac{1}{1 + \varepsilon^2}.$$

This lower bound tends to 1, not 0. Fixed-time convergence therefore cannot be used to pass a limiting-state conclusion to the undamped system. The dotted guide is the damped limit $4/5$ for $\varepsilon = 1/2$.



Question 9

A rectangular input drives an initially resting oscillator:

$$x' = y, \quad y' = -4x + u_T(t), \quad x(0) = y(0) = 0, \quad u_T(t) = H(t) - H(t - T), \quad T > 0.$$

Use the causal step convention for $t \geq 0$. States are continuous at T , and equations at the switch are understood one-sidedly.

Tasks

1. Find both Laplace transforms and express the response as an undelayed step response minus its delayed copy.
2. Calculate $(x(T), y(T))$ and determine exactly which positive pulse durations leave the state at rest forever after the pulse ends.
3. For arbitrary T , calculate the constant value of $y^2 + 4x^2$ after switch-off. Relate complete cancellation to zeros of the input transform at the oscillator poles.
4. Compare $T = \pi$ and $T = \pi/2$: give their post-pulse responses and sketch $x(t)$ on $0 \leq t \leq 2\pi$. Explain why a nonzero, nonnegative pulse can leave no residual oscillation.

Question 9 – Solution

Strategy. Shift a single step response, then check both state components at switch-off; displacement alone is not a stopping condition.

Step 1: Transform and invert by shifting. Since $sF = G$ and $sG = -4F + (1 - e^{-Ts})/s$,

$$F = \frac{1 - e^{-Ts}}{s(s^2 + 4)}, \quad G = \frac{1 - e^{-Ts}}{s^2 + 4}, \quad \text{Re } s > 0.$$

Let $h(t) = (1 - \cos 2t)/4$ and $j(t) = \sin 2t/2$. Then

$$x = h(t) - H(t - T)h(t - T), \quad y = j(t) - H(t - T)j(t - T).$$

Both delayed terms begin at zero, proving state continuity.

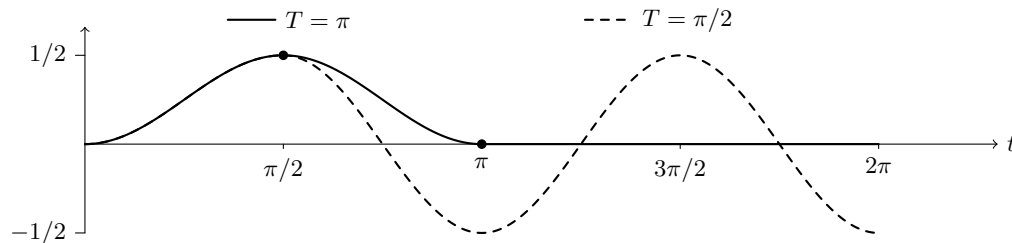
Step 2: Enforce complete rest at the endpoint. At T , the state is $((1 - \cos 2T)/4, \sin 2T/2)^T$. Both components vanish exactly when $\cos 2T = 1$, giving $T = k\pi$, $k = 1, 2, \dots$; the earliest is π . Zero endpoint data then yield the identically zero unforced continuation. Conversely, rest for all later times requires these endpoint values by continuity.

Step 3: Compute the residual and its spectral cancellation. After the pulse, differentiation gives $(y^2 + 4x^2)' = 2y(-4x) + 8xy = 0$. Evaluation at T yields

$$y^2 + 4x^2 = \frac{1}{4} \sin^2 2T + \frac{1}{4} (1 - \cos 2T)^2 = \sin^2 T.$$

The finite-pulse transform $U_T = (1 - e^{-Ts})/s$ has zeros at $s = \pm 2i$ exactly when $e^{2iT} = 1$. These are precisely the stopping durations; they cancel the oscillator poles. At $s = 0$ its singularity is removable, with value T , so a nonzero total input area does not prevent this cancellation.

Step 4: Compare two durations. For $T = \pi$, $x = y = 0$ after π . For $T = \pi/2$, writing $\tau = t - \pi/2 \geq 0$, the endpoint state is $(1/2, 0)$ and $x = \frac{1}{2} \cos 2\tau$, $y = -\sin 2\tau$ afterward. During either pulse, $x = h(t)$ and $y = j(t)$; the plotted curves coincide up to $\pi/2$. A full-period pulse lets the forced trajectory return to rest before switch-off; accumulated state contributions have different oscillatory phases and can cancel even when the input never becomes negative. Equal input sign is not equal state phase.



Question 10

A known system has an unknown continuous input u and unknown initial state:

$$x' = -x + y, \quad y' = -2x - 4y + u(t), \quad t \geq 0.$$

The input and states are of exponential order. A proposed pair of measured one-sided transforms is

$$F(s) = \frac{s+3}{(s+1)(s+2)}, \quad G(s) = \frac{1}{s+2}.$$

Assess compatibility before trying to infer the input.

Tasks

1. Recover both initial values from the transforms using the initial-value rule, then invert F and G .
2. Check the first transformed equation, including its initial term, and verify the corresponding time-domain equation.
3. Recover $\mathcal{L}\{u\}$ and $u(t)$ from the second equation. Prove whether knowledge of $x(t)$ alone would determine both $y(t)$ and $u(t)$ under the stated assumptions.
4. Suppose F is unchanged but the proposed second transform is instead $\tilde{G}(s) = 1/(s+3)$. Can any continuous input in the second equation repair the discrepancy? Quantify the violated transformed identity.

Question 10 – Solution

Strategy. Initial terms and the equation without the unknown input provide independent consistency checks before input reconstruction.

Step 1: Recover the initial state and signals. The initial-value rule gives $x(0) = \lim_{s \rightarrow \infty} sF(s) = 1$ and $y(0) = \lim_{s \rightarrow \infty} sG(s) = 1$. Partial fractions give

$$F = \frac{2}{s+1} - \frac{1}{s+2}, \quad \boxed{x = 2e^{-t} - e^{-2t}, \quad y = e^{-2t}.$$

Both recovered signals indeed start at one; their transforms share the convergence half-plane $\operatorname{Re} s > -1$.

Step 2: Check the input-free constraint. The first transformed equation is $(s+1)F - G = x(0) = 1$. Using the measured expressions, $(s+1)F - G = (s+3)/(s+2) - 1/(s+2) = 1$. In time, $x' = -2e^{-t} + 2e^{-2t} = -x + y$. Thus the two measurements pass a constraint that adjusting the second-equation input cannot alter.

Step 3: Reconstruct and establish uniqueness. The second equation gives $U = sG - y(0) + 2F + 4G$. Therefore

$$U = \frac{s+4}{s+2} - 1 + \frac{2(s+3)}{(s+1)(s+2)} = \frac{4}{s+1}, \quad \boxed{u = 4e^{-t}.$$

Directly, $y' = -2e^{-2t} = -2x - 4y + 4e^{-t}$. Even x alone determines $y = x' + x$ and then $\boxed{u = x'' + 5x' + 6x}$. The continuous input makes these derivatives well defined through the system. Thus two inputs giving the same exact x must give the same y and the same u ; the initial state is recovered as well. This is an exact-data uniqueness claim, not a claim about noise sensitivity.

Step 4: Reject an incompatible second signal. The replacement still has initial value one, but

$$(s+1)F - \tilde{G} - 1 = \frac{1}{s+2} - \frac{1}{s+3} = \boxed{\frac{1}{(s+2)(s+3)} \neq 0}.$$

Equivalently, $x' + x - \tilde{y} = e^{-2t} - e^{-3t}$ is nonzero for $t > 0$. No choice of input confined to the second equation can repair this failure of the first equation. Matching initial values is necessary but does not guarantee compatibility of the complete measured signals.