

Question 1

Consider the constant-input system

$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Separate the equilibrium response from the transient response.

Tasks

1. Find the equilibrium and transform the equation into a homogeneous system by translating the state.
2. Use the homogeneous eigenmodes to find the complete family, then solve the stated IVP.
3. Determine whether either component overshoots its equilibrium value for $t \geq 0$. Give the limiting state and justify the signs of the derivatives.
4. Explain why the nonhomogeneous solution family is not a vector space. Which linear combinations of two solutions necessarily solve the same forced equation?

Question 1 – Solution

Strategy. A constant particular solution shifts the equilibrium; the translated state evolves through the familiar homogeneous modes.

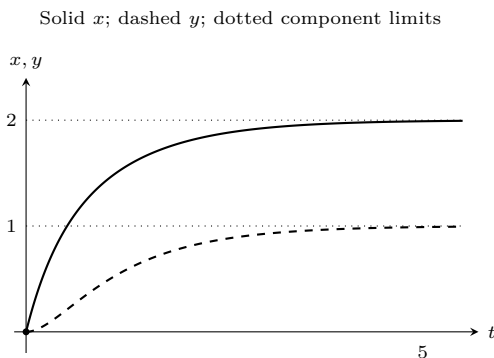
Step 1: Translate to the forced equilibrium. Solving $-2x + y + 3 = 0$, $x - 2y = 0$ gives $X_* = (2, 1)^T$. With $Z = X - X_*$, the equation becomes $Z' = AZ$. The origin is not an equilibrium of the forced system.

Step 2: Resolve the transient coefficients. The eigenpairs of A are $-1, (1, 1)^T$ and $-3, (1, -1)^T$. Thus every solution is $X_* + c_1 e^{-t}(1, 1)^T + c_2 e^{-3t}(1, -1)^T$. The initial state requires $c_1 = -3/2$, $c_2 = -1/2$, giving

$$x = 2 - \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y = 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

Step 3: Check monotonicity and the limit. Here $x' = \frac{3}{2}(e^{-t} + e^{-3t}) > 0$ for $t \geq 0$, while $y' = \frac{3}{2}(e^{-t} - e^{-3t}) > 0$ for $t > 0$ and $y'(0) = 0$. Both start at zero and tend to 2 and 1, respectively. Their monotone approach excludes overshoot; the initial velocity is $(3, 0)^T$, as the original equation requires. The two horizontal dotted lines in the figure are component limits, not additional time-dependent solutions.

Step 4: Identify the affine superposition rule. If U, V solve the same forced equation, then $(\alpha U + \beta V)' - A(\alpha U + \beta V) = (\alpha + \beta)(3, 0)^T$. It is a solution exactly when $\alpha + \beta = 1$. In particular the zero function is not a solution, and the sum of two solutions generally fails. Differences of solutions are homogeneous; adding any homogeneous solution to one particular solution gives the complete affine family derived above.



Question 2

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both homogeneous eigenvalues are negative, but the forcing has an unbounded component. Use a polynomial particular solution.

Tasks

1. Find a particular solution of the form $X_p = ut + v$, with constant vectors u, v . Explain why a constant trial cannot suffice.
2. Add the complete homogeneous family and impose the initial data.
3. Verify the result in the original equations and determine the limits of $x(t) - t$ and $y(t)$.
4. Decide which components are bounded and whether two solutions with different initial data approach one another. Explain why homogeneous stability does not force convergence to a constant here.

Question 2 – Solution

Strategy. Match polynomial coefficients before imposing initial data; stability controls differences of responses, not the size of an unbounded input.

Step 1: Match the polynomial coefficients. Substituting $X_p = ut + v$ gives $u = A(ut + v) + (t, 1)^T$. The coefficient of t requires $Au = -(1, 0)^T$, hence $u = (1, 0)^T$. The constant equation $u = Av + (0, 1)^T$ gives $v = (0, 1)^T$. Thus $X_p = (t, 1)^T$. A constant trial cannot cancel the forcing's nonzero coefficient of t .

Step 2: Complete the family and solve the IVP. The homogeneous solution is $e^{-t}(c_1 + c_2t, c_2)^T$. Initial zero gives $c_1 = 0$, $c_2 = -1$, so

$$x = t(1 - e^{-t}), \quad y = 1 - e^{-t}.$$

Step 3: Check the original equation and asymptotics. We have $x' = 1 - e^{-t} + te^{-t}$ and $y' = e^{-t}$. The right sides $-x + y + t$ and $-y + 1$ give those same expressions. The initial values vanish. Moreover $x - t = -te^{-t} \rightarrow 0$ and $y \rightarrow 1$. The first component tracks a growing ramp; its tracking error decays, but its value does not tend to a finite constant.

Step 4: Separate stability from bounded response. On $t \geq 0$, y is bounded between zero and one, while $x \rightarrow \infty$. For any two solutions, their difference has the form $e^{-t}(d_1 + d_2t, d_2)^T \rightarrow 0$. Thus initial-data effects disappear even though the common forced response grows. Negative eigenvalues of A guarantee decay of the homogeneous transient; they do not turn an unbounded polynomial forcing into a bounded steady state. A complete answer therefore describes both the particular response and the transient.

Question 3

For real constants a, b , consider

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} X + e^t \begin{pmatrix} a \\ b \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A student says that because the forcing exponent equals an eigenvalue, a factor t must occur in a particular solution for every nonzero (a, b) .

Tasks

1. Determine exactly when a trial $X_p = e^t v$ with constant vector v works, and describe all such v .
2. Construct a particular solution valid for every (a, b) and then solve the zero-state IVP.
3. Explain how the component of the forcing in the resonant eigendirection decides whether a polynomial factor is unavoidable.
4. For $a = 0, b \neq 0$, determine the forward growth of the solution. Is growth by itself evidence of resonance? Address the student's claim.

Question 3 – Solution

Strategy. Resonance is a compatibility problem for the forcing direction, not merely a comparison of two scalar exponents.

Step 1: Test the simple exponential trial. Substitution gives $(I - A)v = (a, b)^T$, or $0 = a$, $3v_2 = b$. Thus the trial works exactly when $a = 0$, with $v = (c, b/3)^T$, $c \in \mathbb{R}$. The free first component is a homogeneous contribution, not a new forced effect.

Step 2: Construct the general forced response. A particular solution for all parameters is $X_p = (ate^t, (b/3)e^t)^T$. Adding $(c_1e^t, c_2e^{-2t})^T$ and imposing zero initial data gives

$$x = ate^t, \quad y = \frac{b}{3}(e^t - e^{-2t}).$$

Differentiation gives $x' - x = ae^t$ and $y' + 2y = be^t$, verifying both forcing components and the initial values.

Step 3: Identify the resonant component. The resonant eigendirection is $(1, 0)^T$. Its forced scalar equation is $(e^{-t}x)' = a$, so a nonzero a necessarily creates ate^t . When $a = 0$, there is no forcing in that direction and no polynomial factor is needed. No choice of homogeneous constants can cancel a nonzero ate^t for all time. The condition is directional, not just spectral.

Step 4: Separate forced growth from resonance. For $a = 0, b \neq 0$, the zero-state solution has $x = 0$ and $y/e^t \rightarrow b/3$. It grows because its forcing grows, although the forced second component has homogeneous rate -2 and is not resonant. The first eigenvalue 1 is present in the matrix but receives no input. Thus growth alone does not prove resonance, and the student's claim fails for every nonzero forcing vector of the form $(0, b)^T$.

Question 4

Consider the time-dependent system

$$X' = A(t)X + f(t), \quad A(t) = \begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Use variation of parameters and check the correct two-time transition matrix.

Tasks

1. Find a fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ and its inverse.
2. Derive the variation-of-parameters equation by writing $X = \Phi C$, then compute the IVP solution.
3. Find $T(t, s) = \Phi(t)\Phi(s)^{-1}$ and express the same solution as an integral of $T(t, s)f(s)$. Evaluate it.
4. A proposed shortcut replaces $T(t, s)$ by $\Phi(t - s)$. Compute the resulting candidate and its residual in the original system. Explain why the shortcut fails.

Question 4 – Solution

Strategy. For time-dependent coefficients, propagation depends on both the departure time and the arrival time.

Step 1: Normalize the homogeneous matrix. Solving $y' = 0$, $x' = 2ty$ gives

$$\Phi(t) = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}, \quad \Phi(t)^{-1} = \begin{pmatrix} 1 & -t^2 \\ 0 & 1 \end{pmatrix}.$$

Its determinant is one, and direct differentiation verifies $\Phi' = A(t)\Phi$ and $\Phi(0) = I$.

Step 2: Derive and integrate the varying coefficients. Substituting $X = \Phi C$ cancels $\Phi' C = A\Phi C$, leaving $C' = \Phi^{-1}f = (t - t^2, 1)^T$. Zero initial data gives $C = (t^2/2 - t^3/3, t)^T$. Therefore

$$X = (t^2/2 + 2t^3/3, t)^T.$$

Indeed $x' = t + 2t^2 = 2ty + t$ and $y' = 1$.

Step 3: Use the genuine two-time transition. Matrix multiplication yields

$$T(t, s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.$$

Thus $X(t) = \int_0^t T(t, s)f(s) ds = \int_0^t (s + t^2 - s^2, 1)^T ds$, reproducing the same formula. This expression also follows directly from $\Phi(t) \int_0^t \Phi(s)^{-1}f(s) ds$.

Step 4: Expose the incorrect shortcut. Using $\Phi(t-s)$ instead gives $\tilde{X} = (t^2/2 + t^3/3, t)^T$. Its residual is

$$\tilde{X}' - A(t)\tilde{X} - f(t) = (-t^2, 0)^T,$$

which is not identically zero. The two matrices differ because $t^2 - s^2 \neq (t-s)^2$ in general. A fundamental matrix normalized at zero cannot automatically be shifted in time when the coefficient matrix itself changes with time.

Question 5

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The forcing matches a repeated homogeneous exponent and enters through the second component of a length-two chain.

Tasks

1. Use $Z = e^t X$ to derive a simpler forced system and solve the IVP.
2. Test the trial $X_p = te^{-t}v$ with constant vector v . Explain why it cannot supply a particular solution here.
3. Find the forward maximum of each component and its time. Do the maxima occur together?
4. Determine the long-time behavior and compare the polynomial degree of this forced response with the homogeneous family. Does resonance necessarily imply an unbounded response?

Question 5 – Solution

Strategy. Removing the common exponential reveals successive integrations along the nilpotent chain.

Step 1: Integrate along the chain. With $N = A + I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the transformed system is $Z' = NZ + (0, 1)^T$, $Z(0) = 0$. Thus $z_2' = 1$, $z_1' = z_2$, giving $z_2 = t$, $z_1 = t^2/2$ and

$$x = \frac{1}{2}t^2e^{-t}, \quad y = te^{-t}.$$

Substitution verifies the original equation and zero initial state.

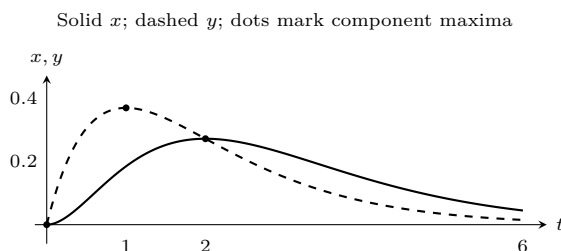
Step 2: Reject an insufficient polynomial trial. The trial would give $Z = tv$, so its equation requires $v = tNv + (0, 1)^T$ for every time. Matching coefficients forces $Nv = 0$ and $v = (0, 1)^T$, but $N(0, 1)^T = (1, 0)^T \neq 0$. These conditions contradict each other. A factor t multiplying a constant vector is insufficient; the chain requires a quadratic term in the first component.

Step 3: Locate the separate response peaks. For $t \geq 0$, $x' = e^{-t}t(2-t)/2$ and $y' = e^{-t}(1-t)$. Their sign changes show

$$x_{\max} = 2e^{-2} \text{ at } t = 2, \quad y_{\max} = e^{-1} \text{ at } t = 1.$$

The maxima occur at different states and different times. The figure marks these component peaks on the two time graphs.

Step 4: Interpret resonance without growth to infinity. Both components tend to zero, since every fixed polynomial times e^{-t} vanishes. The homogeneous family is $e^{-t}(c_1 + c_2t, c_2)^T$, whose largest degree is one; the forcing raises the first component's degree to two. This is a resonant polynomial response, even though the negative exponential ultimately wins. Resonance describes an enhanced polynomial factor, not a universal guarantee of an unbounded solution. Adding any homogeneous solution still leaves forward decay.



Question 6

For $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, consider

$$X' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Investigate the response to a forcing that rotates at the homogeneous oscillation frequency.

Tasks

1. Use $X = R(t)Z$ to derive and solve a constant-forcing equation for Z . Give the full family.
2. Find the unique 2π -periodic solution and prove its uniqueness.
3. For $X(0) = 0$, find the state, its distance from the periodic solution at the same time, and its distance from the unit circle.
4. Does the zero-state response become exactly periodic at a finite time or tend to a single point? Explain how its state curve should be interpreted in this time-dependent forced problem.

Question 6 – Solution

Strategy. A rotating coordinate frame removes the oscillation and leaves scalar damping toward a constant vector.

Step 1: Pass to the rotating frame. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $R' = JR$ and $A = -I + J$. Substitution cancels the rotational terms, giving $Z' = -Z + (1, 0)^T$. Hence

$$X(t) = R(t) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t} \left(c - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right],$$

where $c = X(0)$ is an arbitrary real vector. This gives every solution.

Step 2: Identify and prove the periodic response. Taking $c = (1, 0)^T$ gives $P(t) = (\cos t, \sin t)^T$. If another solution were 2π -periodic, its difference from P would be $e^{-t}R(t)d$ and would satisfy $d = e^{-2\pi}d$ at times zero and 2π . Thus $d = 0$, proving uniqueness. Damping prevents a secular term despite the matched rotation frequency.

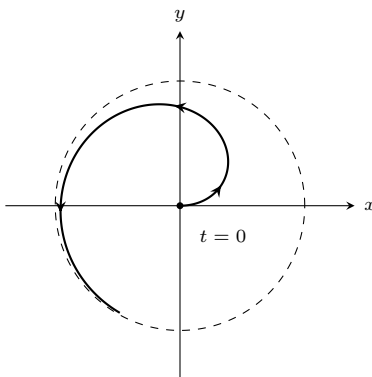
Step 3: Quantify zero-state convergence. For $c = 0$,

$$X(t) = (1 - e^{-t})(\cos t, \sin t)^T \quad (t \geq 0).$$

Since rotations preserve length, $\|X(t) - P(t)\| = e^{-t}$. Its radius is $1 - e^{-t}$, so its distance to the unit circle is also e^{-t} . Its initial velocity is $(1, 0)^T$, agreeing with the input at zero.

Step 4: Distinguish asymptotic response from exact return. The radius is strictly increasing at every finite forward time, so the solution neither repeats a state nor becomes exactly periodic later. Along $t = 2\pi n$ and $t = 2\pi n + \pi$, its states tend to $(1, 0)^T$ and $(-1, 0)^T$, respectively; there is no single point limit. The drawn curve is a nonautonomous state path approaching a periodic response, not an autonomous phase portrait or a claim of a finite-time merger with the circle. Only a finite initial segment is drawn.

Solid: initial state path; dashed: periodic-response circle



Question 7

On $t \geq 0$, consider the bounded input in

$$x' = x + e^{-t}, \quad y' = -y + 1, \quad (x(0), y(0)) = (p, q).$$

Investigate which initial states suppress the unstable homogeneous mode.

Tasks

1. Find the complete solution and classify every initial state yielding a bounded forward response.
2. Find the limiting state for those bounded solutions. Is that limiting vector, held constant, itself a solution of the original forced equation?
3. Express the bounded first component using an integral from the current time to infinity. Explain how this condition selects its initial value.
4. Perturb the selected initial first component by δ , with $0 < |\delta| < 1$. Find when its difference from the bounded response first reaches magnitude one. Does bounded forcing ensure bounded responses here?

Question 7 – Solution

Strategy. Boundedness can require an exact cancellation of an unstable mode rather than follow from bounded forcing alone.

Step 1: Solve and isolate the growing coefficient. Integrating factors give

$$x = (p + \frac{1}{2})e^t - \frac{1}{2}e^{-t}, \quad y = 1 + (q - 1)e^{-t}.$$

The second component is bounded for every q . The first is bounded exactly when $p = -1/2$, so the full bounded family has that fixed first initial value and arbitrary q .

Step 2: Check the limit against the original equation. For bounded data, $X(t) \rightarrow (0, 1)^T$. However the constant function $X = (0, 1)^T$ has zero derivative, whereas the original right side is $(e^{-t}, 0)^T \neq 0$ at finite times. It is the limiting equilibrium of the limiting equation, not an exact constant solution of the stated time-dependent equation.

Step 3: Recover the cancellation from a future integral. Solving the first equation with no growing term gives

$$x(t) = - \int_t^\infty e^{t-s} e^{-s} ds = -\frac{1}{2}e^{-t}.$$

At zero it selects $p = -1/2$. This is a boundedness condition imposed over the whole future, used to choose an initial value. It does not replace the ordinary forward initial-value evolution once that value is fixed.

Step 4: Measure sensitivity to the unstable mode. Changing p by δ adds δe^t to the first component. Its absolute difference starts below one and grows strictly, first reaching one at $t = \ln(1/|\delta|)$. The input remains bounded throughout, but every such perturbed response is unbounded. The unstable homogeneous coefficient, not the input size alone, determines the failure. Exact cancellation selects a special family; it does not make the underlying system stable.

Question 8

For $t \geq 0$, let

$$x' = -x + y, \quad y' = -y + u(t), \quad x(0) = y(0) = 0,$$

where $u(t) = 1$ for $0 \leq t < 1$ and $u(t) = 0$ for $t > 1$. Take the state to be continuous at the switch; use one-sided derivatives there. The value assigned to $u(1)$ does not change the continuous solution.

Tasks

1. Solve on $0 \leq t \leq 1$, then continue for $t \geq 1$ using $\tau = t - 1$ and the state at the switch.
2. Find the one-sided state derivatives at $t = 1$. Which component derivative is continuous, and which jumps?
3. Find the global forward maximum of x and its time. Explain why the maximum occurs after the input has switched off.
4. Find the long-time limit and the integrals $\int_0^\infty x(t) dt$ and $\int_0^\infty y(t) dt$ by integrating the differential equations.

Question 8 – Solution

Strategy. Match the state at the switch, then track the stored second-component response as it continues to feed the first.

Step 1: Solve and match the two intervals. For $0 \leq t \leq 1$, integration gives $x = 1 - (1+t)e^{-t}$, $y = 1 - e^{-t}$. Set $a = 1 - 2/e$, $b = 1 - 1/e$, the switch values. For $\tau = t - 1 \geq 0$, the homogeneous continuation is

$$x = e^{-\tau}(a + b\tau), \quad y = be^{-\tau}.$$

At $\tau = 0$ these match the first formulas, so both components are continuous.

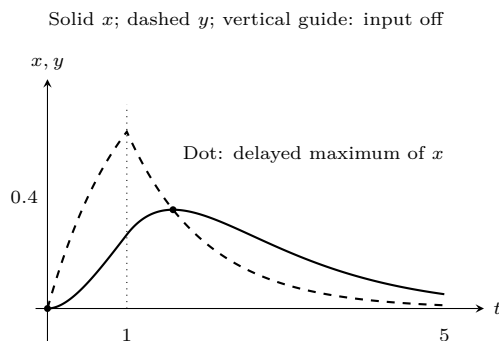
Step 2: Compare one-sided derivatives. Before the switch, $(x', y') = (te^{-t}, e^{-t})$. Thus $X'(1^-) = (1/e, 1/e)^T$. Immediately after it, $X'(1^+) = (-a+b, -b)^T = (1/e, -1+1/e)^T$. The derivative x' is continuous; y' has jump -1 . A finite step input changes a derivative, not the state itself.

Step 3: Find the delayed first-component peak. Before switching, $x' > 0$ for $t > 0$. Afterward, $x' = e^{-\tau}(b - a - b\tau)$, with $b - a = 1/e$. It changes sign once, at $\tau = 1/(e - 1)$, hence

$$t_{\max} = 1 + \frac{1}{e - 1}, \quad x_{\max} = (1 - 1/e)e^{-1/(e-1)}.$$

The still-positive y initially exceeds x after the input ends, so $x' = -x + y$ remains positive until the two components become equal.

Step 4: Integrate the total response. Both components tend to zero exponentially times a linear factor, so their integrals converge. Integrating $y' = -y + u$ over $[0, \infty)$ and using its zero endpoints gives $\int y = \int u = 1$. Integrating $x' = -x + y$ similarly gives $\int_0^\infty x(t) dt = \int_0^\infty y(t) dt = 1$. Equal total areas do not mean equal graphs or simultaneous peaks. The figure marks the switch and the delayed maximum of x .



Question 9

Consider the input-driven system

$$x' = y, \quad y' = u(t), \quad x(0) = y(0) = 0, \quad 0 \leq t \leq 1.$$

Choose an input so that $(x(1), y(1)) = (1, 0)$. First restrict attention to inputs of the form $u(t) = a + bt$.

Tasks

1. Express the endpoint conditions as two integral conditions on a general continuous input u .
2. Find the unique affine input $a + bt$ satisfying both conditions. Explain why a constant input cannot work.
3. Compute the entire state path for that input. Determine whether displacement is monotone and find the maximum velocity.
4. Is the input unique among all continuous functions? Test additions of $c(6t^2 - 6t + 1)$, with arbitrary real c , and distinguish endpoint agreement from identical intermediate motion.

Question 9 – Solution

Strategy. The endpoint conditions constrain two weighted integrals of the input, not all of its time-dependent values.

Step 1: Derive the two input moments. Integrating gives $y(t) = \int_0^t u(s) ds$. A second integration yields $x(t) = \int_0^t (t-s)u(s) ds$, verified by differentiation and the zero initial data. The endpoint requirements are

$$\int_0^1 u(s) ds = 0, \quad \int_0^1 (1-s)u(s) ds = 1.$$

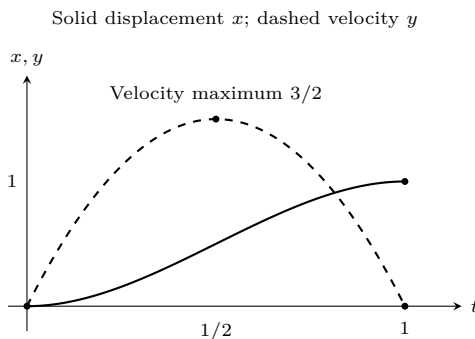
Step 2: Solve within the affine family. For $u = a + bt$, these conditions become $a + b/2 = 0$ and $a/2 + b/6 = 1$. Their unique solution is $a = 6, b = -12$, so $u = 6 - 12t$. A constant input would need $a = 0$ from the final velocity condition, then could not create displacement one.

Step 3: Recover and interpret the state path. Integration gives

$$y = 6t(1-t), \quad x = 3t^2 - 2t^3.$$

The state starts at zero and ends at $(1, 0)$. Since $y > 0$ for $0 < t < 1$, displacement strictly increases through the interior. Velocity has its unique maximum $3/2$ at $t = 1/2$, where the input changes sign. The negative input in the second half slows the positive velocity to zero; it does not reverse the displacement.

Step 4: Expose the remaining freedom. Let $g(t) = 6t^2 - 6t + 1$. Direct integration gives $\int_0^1 g = 0$ and $\int_0^1 sg(s) ds = 0$, hence also $\int_0^1 (1-s)g(s) ds = 0$. Therefore every $u = 6 - 12t + cg(t)$ reaches the same endpoint. For $c \neq 0$, g is not identically zero, so the velocity derivative and intermediate motion differ. Uniqueness holds within the affine trial family, not among all continuous inputs. No monotonicity claim is made for the added-input family.



Question 10

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Consider

$$X' = JX + e^{-t^2} R(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t \geq 0.$$

You may use $\int_0^\infty e^{-s^2} ds = \sqrt{\pi}/2$. Exact definite-integral answers are expected; no elementary antiderivative is assumed.

Tasks

1. Use variation of parameters to find the solution for arbitrary $X(0) = c$. Specialize to $c = 0$.
2. For $c = 0$, find the limiting norm. Does the state converge to a point or become periodic? Explain why forcing that tends to zero need not produce a state that tends to zero.
3. Find exactly the initial state that makes $X(t) \rightarrow 0$. Express that solution with an integral from t to infinity.
4. Prove a quantitative decay bound for this selected solution using $\int_t^\infty e^{-s^2} ds \leq e^{-t^2}/(2t)$ for $t > 0$, deriving the inequality as part of the solution.

Question 10 – Solution

Strategy. A rotating frame converts the forcing into accumulated scalar amplitude; cancelling its total integral removes the persistent free rotation.

Step 1: Integrate in rotating coordinates. Since $R' = JR$ and $R^{-1} = R(-t)$, writing $X = R(t)Z$ gives $Z' = e^{-t^2}(1, 0)^T$. Define $G(t) = \int_0^t e^{-s^2} ds$. Then

$$X(t) = R(t) \begin{bmatrix} c + G(t) \\ 0 \end{bmatrix}.$$

For $c = 0$ the state is $G(t)(\cos t, \sin t)^T$. Differentiation verifies the equation, including the non-elementary amplitude derivative.

Step 2: Interpret the zero-state asymptotics. Let $L = \sqrt{\pi}/2$. For $c = 0$, the norm is $G(t) \uparrow L$, strictly increasing at every finite time, so the solution is not periodic. At times $2\pi n$ and $2\pi n + \pi$ it tends to $(L, 0)^T$ and $(-L, 0)^T$, so no point limit exists. The input decays, but the undamped homogeneous rotation retains the accumulated amplitude; there is no dissipative mechanism to erase it.

Step 3: Select the unique cancelling initial state. Because rotations preserve length, $\|X(t)\| \rightarrow \|c + L(1, 0)^T\|$. Thus $X(t) \rightarrow 0$ exactly when $c = (-L, 0)^T$. For that state,

$$X(t) = -R(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \int_t^\infty e^{-s^2} ds.$$

Any other initial state leaves a nonzero limiting amplitude.

Step 4: Bound the remaining tail. For $s \geq t > 0$, $1 \leq s/t$. Therefore

$$0 \leq \int_t^\infty e^{-s^2} ds \leq \frac{1}{t} \int_t^\infty se^{-s^2} ds = \frac{e^{-t^2}}{2t}.$$

The selected solution consequently obeys $\|X(t)\| \leq e^{-t^2}/(2t) \quad (t > 0)$. This is a tail estimate, not a formula at $t = 0$; the initial norm there is L . The definite integral and the bound avoid inventing an elementary antiderivative or replacing the exact response by numerical samples.