

Question 1

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Consider the parameter-dependent system

$$x + y + z = 2, \quad 2x + 3y + z = 5, \quad 3x + 4y + 2z = k,$$

where k is real.

Tasks

1. Classify every k according to whether the system has no solution, one solution or infinitely many solutions.
2. For each consistent case, give the entire solution set with one free parameter and verify it in all three original equations.
3. Add the measurement $x - y = 0$. Determine all solutions of the enlarged system for every k .
4. Explain why three equations in three unknowns need not determine a unique answer. Compare the information supplied by the original third equation with that supplied by the new measurement.

Question 1 – Solution

Strategy. Inspect relationships between whole equations before committing to a long elimination.

Step 1: Test consistency. The third left-hand side is the sum of the first two. Any common solution must therefore satisfy $k = 2 + 5 = 7$. If $k \neq 7$, subtracting the first two equations from the third gives the contradiction $0 = k - 7$.

Step 2: Find and verify the full family. When $k = 7$, the third equation contributes no new restriction. Subtract twice the first equation from the second to get $y - z = 1$. Setting $z = t$ gives

$$(x, y, z) = (1 - 2t, 1 + t, t), \quad t \in \mathbb{R}.$$

The first left side is 2, the second is 5, and the third is 7 for every t . Conversely, every solution must have $y = 1 + z$ and $x = 1 - 2z$, so no solutions have been omitted. Thus $k = 7$ gives infinitely many solutions; no value of k gives exactly one solution in the original system.

Step 3: Use the additional measurement. On this family, $x - y = -3t$. The new measurement forces $t = 0$, giving

$$(x, y, z) = (1, 1, 0) \quad \text{if } k = 7.$$

If $k \neq 7$, the contradiction from the original equations remains, so adding a measurement cannot create a solution.

Step 4: Distinguish equation count from information. For $k = 7$, the third equation merely repeats a consequence already known. For $k \neq 7$, it contradicts that consequence. It never removes just the one remaining freedom. By contrast, $x - y$ varies as $-3t$ along the family, so its measured value selects exactly one member. Counting equations and unknowns alone cannot distinguish these situations; their relationships matter.

Question 2

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

For real λ , solve

$$(\lambda - 1)x + y = 1, \quad x + (\lambda - 1)y = 1.$$

Tasks

1. Use the sum and difference of the equations to classify the solution set for every λ .
2. Explain why dividing by λ or $\lambda - 2$ before checking their zeros can lose entire cases. Verify the exceptional cases directly.
3. Find the limits of the unique solution as $\lambda \rightarrow 2$ and as $\lambda \rightarrow 0$ from either side. Compare these limits with the solution sets at the exceptional values.
4. Classify all λ for which a solution with $x, y \geq 0$ exists, giving the full nonnegative solution set whenever it is not a single point.

Question 2 – Solution

Strategy. Adding and subtracting separates the sum of the unknowns from their difference, exposing both exceptional parameter values.

Step 1: Reduce reversibly. The sum and difference equations are

$$\lambda(x + y) = 2, \quad (\lambda - 2)(x - y) = 0.$$

They are equivalent to the originals: taking half their sum and half their difference recovers the two original equations. If $\lambda \neq 0, 2$, then

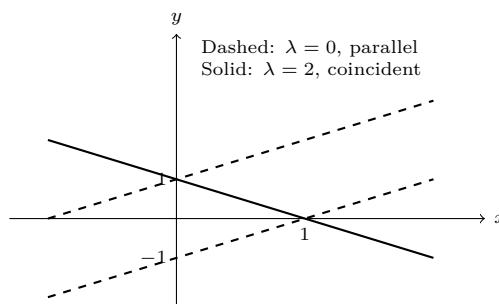
$$x = y = 1/\lambda.$$

At $\lambda = 0$, the sum is $0 = 2$, giving no solution. At $\lambda = 2$, only $x + y = 1$ remains, giving infinitely many solutions.

Step 2: Retain the exceptional equations. Directly, $\lambda = 0$ gives $-x + y = 1$ and $x - y = 1$, incompatible parallel lines. At $\lambda = 2$, both equations become $x + y = 1$. Dividing by a factor that can be zero discards these cases rather than solving them. The figure displays the two exceptional geometries.

Step 3: Compare limits with exact parameter values. As $\lambda \rightarrow 2$ through the unique-solution cases, $(x, y) \rightarrow (1/2, 1/2)$. This is one point of the entire line present at $\lambda = 2$, not the whole limiting parameter's solution set. As $\lambda \rightarrow 0^+$ both coordinates increase without bound; as $\lambda \rightarrow 0^-$ both decrease without bound. There is no finite limiting solution, consistent with the contradiction at zero.

Step 4: Enforce nonnegativity. For $\lambda \neq 0, 2$, the solution is nonnegative exactly when $\lambda > 0$. At $\lambda = 2$, the nonnegative solutions are $(x, y) = (u, 1 - u)$, $0 \leq u \leq 1$. Therefore a nonnegative solution exists exactly for $\lambda > 0$, with the entire stated segment at 2.



Question 3

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A system begins with an equation whose x coefficient is zero:

$$2y - z = 1, \quad x - y + 2z = 3, \quad 2x + y + z = 6 + \delta,$$

where δ is a real perturbation of the last measurement.

Tasks

1. Solve the system for $\delta = 0$ using valid elimination. Explain why the initial zero coefficient is not evidence of nonuniqueness.
2. Find the exact solution for arbitrary δ , keeping fractions exact.
3. Verify the perturbed solution in all original equations, and decide whether any δ makes the system inconsistent or nonunique.
4. Compute $x + 3y$ and recover δ from an exact measurement of z . Explain why measuring $x + 3y$ alone cannot detect the perturbation.

Question 3 – Solution

Strategy. Choose an equation with a nonzero coefficient for elimination. An inconvenient ordering does not change the information in a system.

Step 1: Eliminate without dividing by zero. The second equation gives $x = 3 + y - 2z$. Substituting into the third gives $3y - 3z = \delta$, or $y - z = \delta/3$. Together with $2y - z = 1$, this determines y, z , then x . For $\delta = 0$ the result is $(x, y, z) = (2, 1, 1)$. Swapping the first two equations would also provide an immediate nonzero x coefficient; neither approach requires a zero division.

Step 2: Solve for the perturbation exactly. Subtract $y - z = \delta/3$ from $2y - z = 1$ to obtain $y = 1 - \delta/3$. It follows that

$$x = 2 + \delta, \quad y = 1 - \frac{\delta}{3}, \quad z = 1 - \frac{2\delta}{3}.$$

Only the nonzero constants 1 and 3 were used as divisors, so this calculation applies to every real δ .

Step 3: Check the original measurements and uniqueness. Substitution gives

$$\begin{aligned} 2y - z &= 2 - 2\delta/3 - 1 + 2\delta/3 = 1, \\ x - y + 2z &= 2 + \delta - 1 + \delta/3 + 2 - 4\delta/3 = 3, \\ 2x + y + z &= 4 + 2\delta + 1 - \delta/3 + 1 - 2\delta/3 = 6 + \delta. \end{aligned}$$

Every solution was forced by reversible substitutions and nonzero divisions. Thus exactly one solution exists for every real δ .

Step 4: Identify a blind measurement. The combination $x + 3y = 2 + \delta + 3 - \delta = 5$ is independent of δ . In contrast, z determines $\delta = \frac{3}{2}(1 - z)$ uniquely. The first two equations fix certain combinations even when the last datum changes; a measurement that stays constant along that change cannot reveal it.

Question 4

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Three unknown quantities are measured through pairwise sums and a total:

$$x + y = a, \quad y + z = b, \quad x + z = c, \quad x + y + z = d.$$

The recorded values are $(a, b, c, d) = (2, 3, 4, 5)$.

Tasks

1. Derive a necessary and sufficient condition on arbitrary real a, b, c, d for consistency. Give the unique solution when it exists.
2. For the recorded data, solve the first three equations and identify the conflict with the fourth.
3. Assume the total $d = 5$ is exact and exactly one of the three pairwise readings is wrong. Find every possible corrected reading and corresponding triple (x, y, z) .
4. Decide whether these measurements identify which pairwise reading is wrong, even if x, y, z must be nonnegative. Explain the difference between detecting an error and locating it.

Question 4 – Solution

Strategy. Redundant measurements can reveal a contradiction without identifying its source.

Step 1: Derive the consistency condition and solution. Adding the three pairwise equations gives $2(x + y + z) = a + b + c$. Thus consistency requires $a + b + c = 2d$. The pairwise equations alone uniquely give

$$x = \frac{a + c - b}{2}, \quad y = \frac{a + b - c}{2}, \quad z = \frac{b + c - a}{2}.$$

These values satisfy all three pairwise equations, and their total is $(a + b + c)/2$. Therefore the displayed condition is also sufficient.

Step 2: Diagnose the recorded discrepancy. For $(a, b, c) = (2, 3, 4)$, the unique triple is $(3/2, 1/2, 5/2)$. Its total is $9/2$, not 5. Hence the full recorded system has no solution; rounding the computed triple would not repair an exact contradiction.

Step 3: Enumerate every single-reading correction. The exact total requires the sum of the pairwise readings to be 10 rather than 9. If exactly one pairwise reading changes, its correction must be +1. There are precisely three possibilities:

Changed reading	(a, b, c)	(x, y, z)
$a : 2 \rightarrow 3$	$(3, 3, 4)$	$(2, 1, 2)$
$b : 3 \rightarrow 4$	$(2, 4, 4)$	$(1, 1, 3)$
$c : 4 \rightarrow 5$	$(2, 3, 5)$	$(2, 0, 3)$

Each triple satisfies its corrected pairwise readings and has total 5.

Step 4: State the identification limit. All three corrected triples are nonnegative. Thus even with nonnegativity, the measurements cannot select the faulty reading. The redundancy detects that the four recorded equations cannot all be true. Locating the error requires another independent measurement or an additional justified restriction; it does not follow merely from detecting the inconsistency.

Question 5

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Two nearly parallel measurement equations are

$$x + y = 2, \quad x + (1 + \varepsilon)y = 2 + \delta,$$

where $\varepsilon > 0$ is known and δ is an unknown measurement error satisfying $|\delta| \leq \eta$, with $\eta > 0$.

Tasks

1. Find the exact solution for every admissible δ and explain what would happen if $\varepsilon = 0$ were allowed.
2. Find the smallest uniform bound on the largest absolute coordinate error relative to the nominal solution at $\delta = 0$. Prove sharpness.
3. For $\varepsilon = \eta = 10^{-3}$, find the full range of possible solutions. Compare the small equation residual of the nominal answer with its possible coordinate error.
4. Find a necessary and sufficient condition on ε ensuring both coordinate errors are at most 0.1 for every $|\delta| \leq \eta$. Interpret the condition geometrically.

Question 5 – Solution

Strategy. Subtracting almost identical equations exposes the small coefficient that amplifies uncertainty.

Step 1: Solve and retain the excluded case. Subtract the first equation from the second to get $\varepsilon y = \delta$. For $\varepsilon > 0$,

$$(x, y) = \left(2 - \frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon} \right).$$

If $\varepsilon = 0$ and $\delta \neq 0$, there is no solution; if both are zero, the two equations coincide and every point on $x + y = 2$ is a solution.

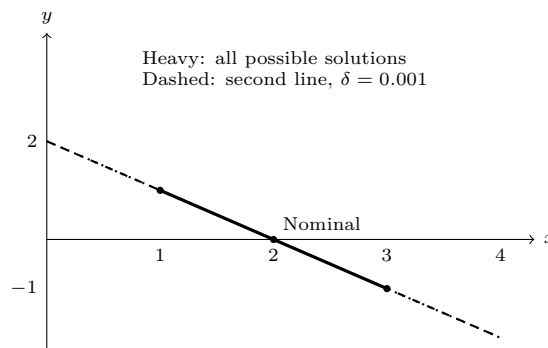
Step 2: Derive the exact uncertainty bound. The nominal solution is $(2, 0)$. Each coordinate changes in magnitude by $|\delta|/\varepsilon$. Therefore

$$\max(|x - 2|, |y|) \leq \eta/\varepsilon.$$

Equality occurs when $\delta = \eta$ or $\delta = -\eta$, so no smaller universal bound is possible.

Step 3: Quantify the nearly parallel example. For $\varepsilon = \eta = 10^{-3}$, let $u = \delta/\varepsilon \in [-1, 1]$. The full possible solution set is the segment $(x, y) = (2 - u, u)$, $-1 \leq u \leq 1$, from $(3, -1)$ to $(1, 1)$. The nominal answer satisfies the first equation exactly and has residual $-\delta$ in the second, of magnitude at most 0.001. Nevertheless its coordinate error can be 1. Small residuals do not always certify small solution errors. The graph uses true scales; the two measured lines are nearly indistinguishable, while the possible solution segment is visible.

Step 4: Give the exact design condition. The requested guarantee holds exactly when $\eta/\varepsilon \leq 0.1$, or $\varepsilon \geq 10\eta$. Necessity follows by taking an endpoint error; sufficiency follows from the bound. Increasing ε separates the line directions and reduces the effect of shifting one measurement's right-hand side.



Question 6

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A homogeneous system is

$$2x - y + z = 0, \quad x + y - 2z = 0.$$

Two proposed ways to choose a scale are $x + y + z = 9$ and $x^2 + y^2 + z^2 = 35$. These are alternative added conditions, not simultaneous unless explicitly stated.

Tasks

1. Find and verify the entire solution set of the homogeneous system, explaining the role of the zero solution.
2. Apply the linear normalization $x + y + z = 9$ and determine whether it selects a unique solution.
3. Apply the quadratic normalization $x^2 + y^2 + z^2 = 35$ instead. Explain its sign ambiguity and resolve it if nonnegative coordinates are additionally required.
4. For arbitrary real a, b, c, m , classify the effect of adding $ax + by + cz = m$: give exact conditions for no solution, one solution or infinitely many solutions.

Question 6 – Solution

Strategy. A homogeneous system can fix ratios while leaving the overall scale free. A normalization must actually measure that remaining freedom.

Step 1: Solve the homogeneous equations. From the second equation, $y = 2z - x$. Substitution into the first gives $3x - z = 0$, hence $z = 3x$ and $y = 5x$. Thus

$$(x, y, z) = t(1, 5, 3), \quad t \in \mathbb{R}.$$

Substitution gives $2t - 5t + 3t = 0$ and $t + 5t - 6t = 0$. Every solution has this form, including the zero solution at $t = 0$. Dividing by an unknown coordinate would risk losing that solution.

Step 2: Apply the linear normalization. The coordinate sum is $9t$. The condition $9t = 9$ fixes $t = 1$, giving $(1, 5, 3)$ uniquely.

Step 3: Compare the quadratic normalization. The squared length is $(1 + 25 + 9)t^2 = 35t^2$. Therefore the quadratic condition gives $t^2 = 1$ and exactly two solutions:

$$(1, 5, 3) \quad \text{and} \quad (-1, -5, -3).$$

Squaring loses the overall sign. If nonnegative coordinates are also required, only the positive triple remains. The zero triple does not satisfy either stated normalization.

Step 4: Classify an arbitrary linear measurement. On the homogeneous family, the extra left side is $(a + 5b + 3c)t$. Set $K = a + 5b + 3c$. Then

$$\begin{array}{ll} K \neq 0 : & t = m/K \text{ is the unique solution scale,} \\ K = 0, m \neq 0 : & \text{no solution,} \\ K = 0, m = 0 : & \text{every real } t \text{ remains possible.} \end{array}$$

A nonzero-looking added equation can still vanish on the entire family. What matters is its value on the remaining freedom, not the visual presence of three coefficients.

Question 7

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A sensor's response is assumed to be a quadratic polynomial $p(t) = a + bt + ct^2$. Exact readings at three times are

$$p(0) = 1, \quad p(1) = 2, \quad p(2) = 5.$$

Later, consider a perturbation of only the last reading to $5 + \epsilon$, where $|\epsilon| \leq \eta$ and $\eta > 0$.

Tasks

1. Find the unique polynomial from the exact readings using a system of equations. Predict $p(3)$.
2. Find the exact perturbed coefficients and the response error at an arbitrary time $T \geq 0$.
3. Identify all nonnegative times whose predicted value is insensitive to this perturbation. Find sharp uniform error bounds on $0 \leq T \leq 2$ and at $T = 10$.
4. Explain which conclusions depend on the quadratic model. Construct a nonquadratic smooth response with the same three original readings but a different value at 3.

Question 7 – Solution

Strategy. A model assumption converts function recovery into coefficient recovery, while extrapolation can amplify measurement errors.

Step 1: Solve the exact coefficient equations. The data give $a = 1$, $a + b + c = 2$ and $a + 2b + 4c = 5$. Thus $b + c = 1$ and $2b + 4c = 4$, yielding $c = 1$ and $b = 0$. Every coefficient is forced, so

$$p(t) = 1 + t^2, \quad p(3) = 10.$$

Step 2: Retain the measurement perturbation. The first two equations are unchanged, while $2b + 4c = 4 + \epsilon$. Solving gives

$$a = 1, \quad b = -\epsilon/2, \quad c = 1 + \epsilon/2, \quad p_\epsilon(T) - p(T) = \frac{\epsilon}{2}T(T-1).$$

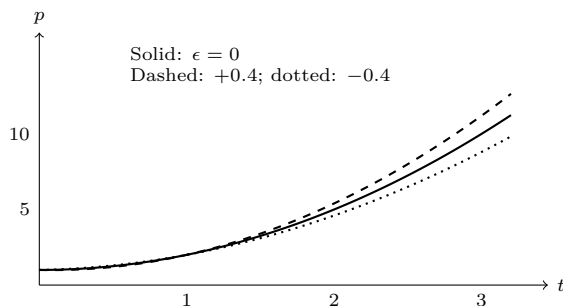
The error vanishes at both unchanged measurement times and equals ϵ at $T = 2$, as required.

Step 3: Compare interpolation and extrapolation. The error is independent of ϵ exactly when $T(T-1) = 0$, so the nonnegative insensitive times are 0 and 1. On $[0, 1]$, $|T(T-1)| \leq 1/4$; on $[1, 2]$, $T(T-1)$ increases from 0 to 2. Therefore

$$\sup_{0 \leq T \leq 2} |p_\epsilon - p| \leq \eta, \quad |p_\epsilon(10) - p(10)| \leq 45\eta.$$

Both bounds are sharp at $|\epsilon| = \eta$, the first at $T = 2$. The figure shows the true curves for $\epsilon = 0, \pm 0.4$.

Step 4: State the model's limitation. For any real K , the smooth function $q_K(t) = 1 + t^2 + Kt(t-1)(t-2)$ matches all three original readings. For $K \neq 0$ it is not quadratic, and $q_K(3) = 10 + 6K \neq 10$. Thus three readings uniquely determine a member of the quadratic family, not an arbitrary smooth response. Extrapolation bounds also rely on the quadratic family and on the stated location of the measurement error.



Question 8

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A steady flow network has directed internal flows p from A to B , q from B to C , and r from C to A . External flow enters A at rate 3; rates 1 and 2 leave B and C , respectively. There is no storage. All rates use the same units, and internal flows must be nonnegative.

Tasks

1. Write the three node-balance equations and identify any redundancy.
2. Find every nonnegative steady flow and explain why the external data do not determine the internal circulation uniquely.
3. Impose capacity bounds $p \leq 5$, $q \leq 4$ and $r \leq 3$. Find the entire feasible set and identify which capacities limit it.
4. Among feasible flows, minimize the cost rate $2p + q + 3r$. Prove uniqueness of the minimizer and interpret it in terms of circulation.

Question 8 – Solution

Strategy. Conservation fixes net transfers but can leave an arbitrary circulating flow around a closed loop.

Step 1: Balance every node. Inflow equals outflow, giving

$$A : r + 3 = p, \quad B : p = q + 1, \quad C : q = r + 2.$$

Equivalently $p - r = 3$, $q - p = -1$, $r - q = -2$. The three left sides sum to zero, as do their right sides. Any two balances imply the third because total external inflow equals total external outflow.

Step 2: Recover the free circulation. Set $r = t$. Then

$$(p, q, r) = (t + 3, t + 2, t), \quad t \geq 0.$$

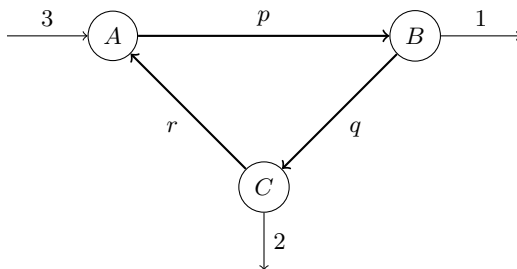
The condition $t \geq 0$ is necessary from $r \geq 0$ and sufficient for all three flows to be nonnegative. Adding the same amount to all three internal flows preserves every node balance: it is an additional circulation around the loop.

Step 3: Apply all capacities. The bounds give $t \leq 2$, $t \leq 2$ and $t \leq 3$, respectively. Thus the entire feasible family has $[0 \leq t \leq 2]$. The capacities on p and q both limit the upper endpoint; at $t = 2$ they are saturated, while $r = 2 < 3$ leaves unused capacity on the third link.

Step 4: Minimize the cost on the feasible family. Substituting the flow formulas yields

$$2p + q + 3r = 2(t + 3) + (t + 2) + 3t = 8 + 6t.$$

This increases strictly on $[0, 2]$. Therefore the unique minimizer is $(p, q, r) = (3, 2, 0)$, with cost rate $[8]$. It sends the required external transfers without an added loop circulation. The diagram shows directions, not arrows scaled in proportion to flow.



Question 9

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

You may design a second linear measurement to accompany $x + y = 1$:

$$ax + by = c.$$

It must select the target $(x, y) = (1/3, 2/3)$ uniquely. In a normalized design, require $a, b \geq 0$ and $a + b = 3$.

Tasks

1. Classify all real triples (a, b, c) that select the target uniquely. Explain the excluded case geometrically.
2. Under the normalization, describe every admissible design using one parameter and identify the excluded parameter value.
3. If the second right-hand side changes from c to $c + \delta$ while the first equation remains exact, derive the coordinate errors for an admissible design.
4. For $|\delta| \leq \eta$, where $\eta > 0$, find all normalized designs minimizing the worst possible absolute coordinate error. Give that minimum and verify the proposed designs directly.

Question 9 – Solution

Strategy. Passing through a target is necessary, but the measurement must also distinguish points along the first equation's line.

Step 1: Require both passage and uniqueness. The target lies on the second line exactly when $c = (a + 2b)/3$. Substituting $y = 1 - x$ gives $(a - b)x = c - b$. Hence unique selection requires

$$c = (a + 2b)/3, \quad a \neq b.$$

If $a = b$, the target condition forces $c = a$ and the second equation is a multiple of $x + y = 1$ (possibly $0 = 0$). The whole first line remains possible.

Step 2: List all normalized designs. Put $b = 3 - a$, with $0 \leq a \leq 3$. Then

$$b = 3 - a, \quad c = 2 - a/3, \quad a \in [0, 3] \setminus \{3/2\}.$$

The midpoint is excluded because both coefficients would equal $3/2$.

Step 3: Derive the exact measurement sensitivity. Subtract the target equations from the perturbed equations. With $\Delta x = x - 1/3$ and $\Delta y = y - 2/3$, one obtains $\Delta x + \Delta y = 0$ and $a\Delta x + b\Delta y = \delta$. Thus

$$\Delta x = \frac{\delta}{a - b}, \quad \Delta y = -\frac{\delta}{a - b}.$$

The worst coordinate error over $|\delta| \leq \eta$ is exactly $\eta/|a - b|$, attained at either endpoint of the allowed error interval.

Step 4: Optimize within the allowed normalization. Since $a + b = 3$ and both are nonnegative, $|a - b| \leq 3$, with equality exactly at $(a, b) = (3, 0)$ or $(0, 3)$. Therefore the minimum worst error is $\boxed{\eta/3}$, attained by exactly two designs:

$$3x = 1 \quad \text{or} \quad 3y = 2.$$

Perturbing these readings directly changes the measured coordinate by $\delta/3$ and the other by its negative. Both attain the claimed bound. Without the normalization, arbitrary rescaling would change the meaning of an absolute right-hand-side error, making this comparison ill-defined.

Question 10

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

Three nonnegative production amounts satisfy two exact resource totals:

$$x + y + z = 6, \quad x + 2y + 3z = 10, \quad x, y, z \geq 0.$$

The return is $P = 4x + 2y + z$. Fractional production amounts are permitted.

Tasks

1. Find all solutions of the two equalities, then determine the exact parameter interval allowed by non-negativity.
2. Find the minimum and maximum return over the feasible set, giving every allocation at which either is attained.
3. Add the capacity constraint $x \leq 3$. Find the new feasible set and the new maximizing allocation and return.
4. Explain why solving the equalities alone does not solve the allocation problem. Verify the optimizing allocations against both original totals and all applicable inequalities.

Question 10 – Solution

Strategy. Solve the equalities first, then optimize over the remaining feasible interval rather than over an unrestricted algebraic family.

Step 1: Parameterize and restrict. Subtract the first equality from the second to obtain $y + 2z = 4$. Set $z = t$, giving

$$(x, y, z) = (2 + t, 4 - 2t, t).$$

Nonnegativity imposes $t \geq 0$, $t \leq 2$ and $t \geq -2$. The complete feasible set is therefore

$$(x, y, z) = (2 + t, 4 - 2t, t), \quad 0 \leq t \leq 2.$$

Each member satisfies both original totals; every feasible member must arise from this parameterization.

Step 2: Optimize on the feasible interval. The return becomes $P = 4(2 + t) + 2(4 - 2t) + t = 16 + t$. It is strictly increasing, so the unique minimum is $\boxed{16}$ at $(2, 4, 0)$, and the unique maximum is $\boxed{18}$ at $(4, 0, 2)$. Neither an interior stationary-point calculation nor an integer search is needed; the stated amounts may be fractional.

Step 3: Include the new capacity. The constraint $x \leq 3$ becomes $2 + t \leq 3$, or $t \leq 1$. The new feasible interval is $[0, 1]$, with unique maximizing allocation

$$(x, y, z) = (3, 2, 1), \quad P = 17.$$

The minimum remains at $t = 0$. In the figure, the feasible sets are shown in the (z, y) plane; $x = 2 + z$ is determined by the equalities.

Step 4: Verify the optimizers and the role of inequalities. The endpoint triples $(2, 4, 0)$ and $(4, 0, 2)$ have totals $(6, 10)$ and are nonnegative. The new optimizer $(3, 2, 1)$ also has totals $(6, 10)$ and satisfies $x = 3$. The former maximizer has $x = 4$, so it is no longer allowed. Over the unrestricted equality family, $P = 16 + t$ has no finite maximum or minimum. The inequalities make the allowed interval bounded and turn this into a well-posed allocation problem.

