

Question 1

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f(t) = e^{-t}$ and $g(t) = e^{-2t}$ for $t \geq 0$. A student claims that multiplying their transforms gives the transform of their pointwise product.

Tasks

1. Compute $c = f * g$ directly and prove $f * g = g * f$ by a change of variable.
2. Find the pointwise product $p = fg$. Compare $c(0)$, $p(0)$ and their transforms to assess the claim.
3. Derive the convolution theorem for these functions by changing variables in a double integral. State a real range of s that justifies the interchange of integrals.
4. Find the unique global maximum of c , its total area, and its exact real transform domain. Explain why the convolution initially vanishes even though both factors are positive at zero.

Question 1 – Solution

Strategy. Convolution integrates over a moving interval; pointwise multiplication does not.

Step 1: Integrate and reverse the factors. For $t \geq 0$,

$$c(t) = e^{-t} \int_0^t e^{-u} du = \boxed{e^{-t} - e^{-2t}}.$$

The substitution $v = t - u$ changes $\int_0^t f(t-u)g(u) du$ to $\int_0^t g(t-v)f(v) dv$, proving commutativity directly.

Step 2: Test the student's claim. Pointwise multiplication gives $p(t) = e^{-3t}$, so $p(0) = 1$ whereas $c(0) = 0$. Their transforms are

$$P(s) = \frac{1}{s+3}, \quad C(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}.$$

The product $F(s)G(s)$ equals $C(s)$, not $P(s)$. For example, at $s = 0$ they have values $1/2$ and $1/3$ respectively.

Step 3: Derive the product formula. In the triangular region $0 \leq u \leq t < \infty$, put $v = t - u$. Its image is the quadrant $u, v \geq 0$, and its Jacobian has absolute value 1. For real $s > -1$, the resulting integrand has a finite absolute integral:

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^\infty \int_0^t e^{-st} e^{-(t-u)} e^{-2u} du dt \\ &= \int_0^\infty \int_0^\infty e^{-(s+1)v} e^{-(s+2)u} du dv = F(s)G(s). \end{aligned}$$

Thus the interchange is justified by absolute integrability; it is not a formal rule for pointwise products.

Step 4: Locate the peak and inspect the tail. Since $c' = e^{-t}(2e^{-t} - 1)$, the derivative changes from positive to negative only at $t = \ln 2$. Consequently

$$\boxed{\max_{t \geq 0} c(t) = \frac{1}{4}, \quad \int_0^\infty c(t) dt = 1 - \frac{1}{2} = \frac{1}{2}.$$

The positive tail is asymptotic to e^{-t} , giving exact real domain $\boxed{s > -1}$; at $s = -1$ the transformed integrand tends to 1. At zero the integration interval has length zero. Finite endpoint values cannot produce a nonzero integral over that interval.

Question 2

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Two delayed rectangular pulses are

$$f = H_1 - H_3, \quad g = H_2 - H_5,$$

and $w = f * g$.

Tasks

1. Express $w(t)$ as the length of an intersection of intervals, then derive a complete piecewise formula.
2. Find its transform and invert it as a sum of delayed ramps. Check agreement with the geometric formula.
3. Determine the support, all maximizing times, continuity, and the jumps of the first derivative.
4. Compute the area in two independent ways and give the exact real transform domain, including the value at $s = 0$.

Question 2 – Solution

Strategy. For unit-height rectangles, the integral measures overlap length.

Step 1: Track the overlap. Using commutativity, integrate $f(u)g(t-u)$. Both factors are 1 on $[1, 3] \cap [t-5, t-2]$, so

$$w = \max\{0, \min(3, t-2) - \max(1, t-5)\} = \begin{cases} 0, & t < 3, \\ t-3, & 3 \leq t < 5, \\ 2, & 5 \leq t < 6, \\ 8-t, & 6 \leq t < 8, \\ 0, & t \geq 8. \end{cases}$$

Endpoints of the intervals do not change their length.

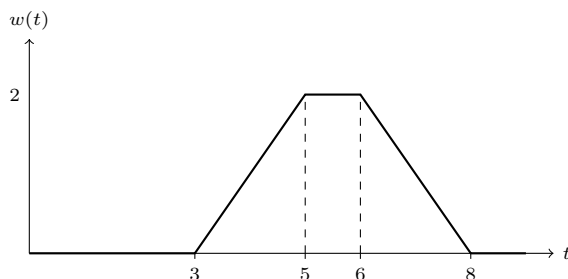
Step 2: Transform and recover the ramps. For $s \neq 0$,

$$W = \frac{e^{-3s}(1 - e^{-2s})(1 - e^{-3s})}{s^2}.$$

The inverse is $w = (t-3)H_3 - (t-5)H_5 - (t-6)H_6 + (t-8)H_8$. Activating these four ramps successively gives slopes 1, 0, -1, 0, exactly as in the piecewise formula.

Step 3: Describe the geometry and regularity. The support (closure of the nonzero set) is $[3, 8]$; $w > 0$ precisely on $(3, 8)$. Its maximum is 2, attained at every $t \in [5, 6]$. It is continuous everywhere. The first derivative has jumps $+1, -1, -1, +1$ at 3, 5, 6, 8, respectively, and is undefined there in the ordinary two-sided sense.

Step 4: Check area and convergence. Two triangles and the plateau have total area $2 + 2 + 2 = 6$. Independently, the convolution area is $(\int f)(\int g) = 2 \cdot 3 = 6$, by integrating over the rectangle in the two integration variables. Compact support makes the transform finite for every real s . The displayed expression has removable value $W(0) = 6$.



Question 3

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

An undamped oscillator receives a rapidly vanishing force:

$$y'' + y = e^{-t^2}, \quad y(0) = 2, \quad y'(0) = -1.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2 < 1$.

Tasks

1. Find $Y(s)$ in terms of $G(s) = \mathcal{L}\{e^{-t^2}\}$ and write a convolution formula for y .
2. Verify the equation and both initial conditions by differentiating your convolution.
3. Find a limiting sinusoidal expression $y_\infty(t)$ with constant coefficients and prove $|y - y_\infty| \leq e^{-t^2}/(2t)$ for $t > 0$.
4. Prove that the response does not tend to zero, despite the vanishing force. Identify which feature of the homogeneous equation makes this possible.

Question 3 – Solution

Strategy. Keep the Gaussian force inside an integral and separate the permanent oscillation from the remaining tail integral.

Step 1: Transform and invert. The initial terms give $(s^2 + 1)Y - 2s + 1 = G$, hence

$$Y = \frac{2s - 1 + G}{s^2 + 1}, \quad \boxed{y = 2 \cos t - \sin t + z(t), \quad z(t) = \int_0^t \sin(t - u)e^{-u^2} du.}$$

These transform operations are valid for real $s > 0$.

Step 2: Differentiate with the endpoint terms. The sine kernel vanishes at its endpoint, so $z' = \int_0^t \cos(t - u)e^{-u^2} du$. Differentiating again gives $z'' = e^{-t^2} - z$ because $\cos 0 = 1$. Also $z(0) = z'(0) = 0$. Thus the formula satisfies the full IVP.

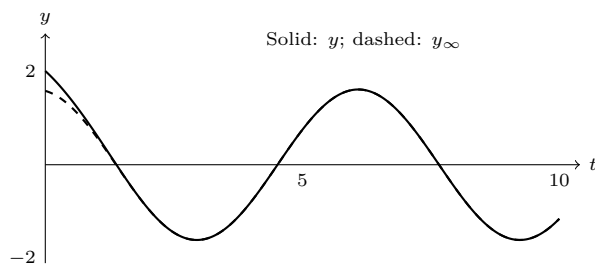
Step 3: Isolate the asymptotic oscillation. Set $A = \int_0^\infty e^{-u^2} \cos u du$ and $B = \int_0^\infty e^{-u^2} \sin u du$, both absolutely convergent. Expanding $\sin(t - u)$ gives

$$y_\infty = (2 - B) \cos t + (A - 1) \sin t, \quad y - y_\infty = - \int_t^\infty \sin(t - u)e^{-u^2} du.$$

For $t > 0$, use $1 \leq u/t$ on the tail:

$$|y - y_\infty| \leq \int_t^\infty e^{-u^2} du \leq \frac{1}{t} \int_t^\infty ue^{-u^2} du = \boxed{\frac{e^{-t^2}}{2t}}.$$

Step 4: Establish persistent motion. Since $|A| \leq \sqrt{\pi}/2 < 1$, the coefficient $A - 1$ is nonzero. The limiting sinusoid therefore has positive amplitude and has recurring positive and negative extrema. Along their times the error tends to zero, so y cannot tend to zero (or to any single constant). The homogeneous modes $\cos t, \sin t$ have no damping. The figure compares the exact integral solution with y_∞ ; the two quickly become indistinguishable.



Question 4

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let f be real and piecewise continuous on each finite interval, with $|f(t)| \leq 1$. Consider

$$y'' + 2y' + 2y = f(t), \quad y(0) = y'(0) = 0.$$

We seek the smallest constant M such that $|y(t)| \leq M$ for every admissible input and every $t \geq 0$.

Tasks

1. Find the impulse-response kernel h and express y by convolution. Compute $\int_0^\infty h$.
2. Compute $\int_0^\infty |h|$ exactly by summing half-periods, and prove it is a uniform response bound.
3. Construct an input attaining $\int_0^T |h|$ at any prescribed $T > 0$. Prove that your uniform bound is sharp, and decide whether it can be attained at a finite time.
4. Give an explicit two-level input on $[0, 2\pi]$ whose response at 2π exceeds $1/2$. Explain the error in using the signed kernel integral as the universal bound.

Question 4 – Solution

Strategy. A sign-changing kernel can reinforce a suitably chosen sign-changing input. Its signed area loses that information.

Step 1: Recover the kernel. The zero-data transform gives $Y = F/[(s+1)^2 + 1]$, so

$$h(t) = e^{-t} \sin t, \quad y = h * f, \quad \int_0^\infty h(t) dt = \frac{1}{2}.$$

For example, the last integral follows from the antiderivative $-e^{-t}(\sin t + \cos t)/2$.

Step 2: Sum the absolute areas. Put $q = e^{-\pi}$. The first positive lobe has area $A_0 = (1+q)/2$. Each later absolute lobe is q times its predecessor, so

$$M = \int_0^\infty |h| = \frac{1+q}{2(1-q)}.$$

For every admissible input, $|y(t)| \leq \int_0^t |h(t-u)| du \leq M$.

Step 3: Prove sharpness and distinguish attainment. For any $T > 0$, set

$$f_T(u) = \operatorname{sgn}(\sin(T-u)) \quad (0 \leq u \leq T), \quad f_T(u) = 0 \quad (u > T),$$

with arbitrary values in $[-1, 1]$ at zeros. This input is admissible and $y(T) = \int_0^T |h(T-u)| du = \int_0^T |h(v)| dv$. These values approach M as $T \rightarrow \infty$, so any smaller proposed constant fails for some finite T . However, the absolute kernel has a strictly positive tail after every finite T , so $|y(T)| < M$ for every input at every finite time. The universal supremum is sharp but is not attained.

Step 4: Refute the signed-area bound explicitly. Take $f = -1$ on $(0, \pi)$, $f = 1$ on $(\pi, 2\pi)$ and zero afterward. This is the preceding construction for $T = 2\pi$. Therefore

$$y(2\pi) = A_0(1+q) = \frac{(1+q)^2}{2} > \frac{1}{2}.$$

The signed integral describes cancellation within the kernel, as for a constant input's limiting response. An arbitrary bounded input can reverse its own sign so that both lobes contribute positively; the absolute area is the appropriate worst-case quantity.

Question 5

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

For a real parameter λ , solve the Volterra integral equation

$$u(t) = 1 + \lambda \int_0^t e^{-(t-\tau)} u(\tau) d\tau, \quad t \geq 0.$$

Seek a continuous solution on the entire half-line.

Tasks

1. Use transforms to derive the solution, treating every exceptional parameter value explicitly.
2. Derive an equivalent first-order IVP and prove the converse implication, establishing uniqueness among continuous solutions.
3. Classify boundedness, positivity and long-time behavior for all real λ , including $\lambda = 0$.
4. Give the exact real transform domain in every case. Explain why exponential decay of the memory kernel does not by itself guarantee a bounded solution.

Question 5 – Solution

Strategy. Feedback changes the effective growth rate. Verify the transformed candidate through an equivalent IVP so no growth assumption is needed for uniqueness.

Step 1: Solve the transformed equation. Initially for sufficiently large s ,

$$U = \frac{1}{s} + \frac{\lambda U}{s+1}, \quad U = \frac{s+1}{s(s+1-\lambda)}.$$

Partial fractions give

$$u(t) = \frac{1 - \lambda e^{(\lambda-1)t}}{1 - \lambda} \quad (\lambda \neq 1), \quad u(t) = 1 + t \quad (\lambda = 1).$$

In particular $\lambda = 0$ gives $u = 1$; the factor $s + 1$ then cancels in U .

Step 2: Prove equivalence without assuming a transform. For continuous u , define $v = e^{-t} * u$. Differentiation gives $v' = u - v$, $v(0) = 0$. The integral equation makes u continuously differentiable and yields

$$u' = (\lambda - 1)u + 1, \quad u(0) = 1.$$

Conversely, let u solve this IVP and form v as above. The residual $w = u - 1 - \lambda v$ satisfies $w' = -u + 1 + \lambda v = -w$ and $w(0) = 0$. Hence $w = 0$ and the integral equation holds, including at $\lambda = 0$. The linear IVP has a unique global solution, which proves the asserted existence and uniqueness.

Step 3: Classify all parameters. For $\lambda < 1$, the solution is bounded and tends to $1/(1 - \lambda) > 0$. For $\lambda = 1$ it grows linearly; for $\lambda > 1$ it grows exponentially with positive leading coefficient $\lambda/(\lambda - 1)$. For $\lambda \neq 1$, $u' = \lambda e^{(\lambda-1)t}$: it decreases from 1 to a positive limit when $\lambda < 0$, is constant when $\lambda = 0$, and increases when $\lambda > 0$. Thus it is positive for every real λ and every $t \geq 0$.

Step 4: Inspect the actual transform domain. For $\lambda < 1$ the positive nonzero limiting constant forces $s > 0$; for $\lambda = 1$ the linear term also requires $s > 0$. For $\lambda > 1$ the nonzero exponential requires $s > \lambda - 1$. Consequently

$$s > 0 \quad (\lambda \leq 1), \quad s > \lambda - 1 \quad (\lambda > 1).$$

Positive feedback can offset or exceed the kernel's decay. The rate in the equivalent equation is $\lambda - 1$, not simply the kernel's rate -1 .

Question 6

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Three identical smoothing stages each have causal kernel $k_a(t) = ae^{-at}$, where $a > 0$. Their cascade has kernel $r_a = k_a * k_a * k_a$.

Tasks

1. Compute the two-stage and three-stage kernels by direct integration.
2. Find the transform of r_a and justify why the grouping of the convolutions does not matter.
3. Find the unique peak of r_a , its height and its initial value and slope. Compare these with a single stage.
4. Compute the mass and mean time of both kernels. Explain what is preserved by cascading and what changes. You may derive needed exponential moments by integration by parts.

Question 6 – Solution

Strategy. Equal exponential factors leave polynomial integrals inside the convolution. Unit mass is preserved while the response is spread over time.

Step 1: Integrate the cascade. Directly,

$$(k_a * k_a)(t) = a^2 e^{-at} \int_0^t 1 \, du = a^2 t e^{-at}, \quad r_a(t) = a^3 e^{-at} \int_0^t u \, du = \boxed{\frac{1}{2} a^3 t^2 e^{-at}}.$$

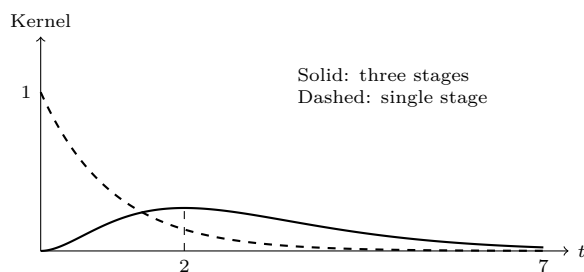
Step 2: Check transforms and grouping. For real $s > -a$, $R_a(s) = [a/(s+a)]^3$. Both groupings integrate the same product $k_a(t-u-v)k_a(u)k_a(v)$ over $u, v \geq 0$, $u+v \leq t$. The continuous integrand is integrable on this bounded triangle, so reversing the integration order proves associativity directly.

Step 3: Compare the peak and onset. For $t > 0$, $r'_a = \frac{1}{2} a^3 e^{-at} t(2-at)$. Thus the unique maximum is at $t = 2/a$, with height $2ae^{-2}$. Also $r_a(0) = r'_a(0) = 0$. A single stage has maximum a at zero and right derivative $-a^2$ there. The cascade begins more smoothly, reaches its peak later and has a lower peak.

Step 4: Compute mass and mean time. Integration by parts yields $\int_0^\infty t^n e^{-at} dt = n!/a^{n+1}$ for nonnegative integers n . Hence

$$\int k_a = 1, \quad \int r_a = 1, \quad \int t k_a(t) dt = \frac{1}{a}, \quad \int t r_a(t) dt = \frac{3}{a}.$$

Since both masses are one, these first moments are their mean times. The total area is preserved; the mean time triples. The cascade remains positive at every $t > 0$, so its later peak is not a literal waiting interval with zero output. The figure uses $a = 1$ and the same vertical scale for both.



Question 7

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let the unstable kernel be $h(t) = e^t$. First allow a piecewise continuous input f supported in $[0, T]$, where $T > 0$. Then specialize to

$$f_c(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -c, & 1 \leq t < 2, \\ 0, & t \geq 2, \end{cases} \quad c > 0,$$

with output $y = h * f_c$.

Tasks

1. For the general input, derive a necessary and sufficient integral condition for the output to vanish for every $t \geq T$.
2. For f_c , compare the choice that makes the ordinary input area zero with the choice that eliminates the output tail.
3. For the tail-eliminating choice, derive the complete piecewise output and find its maximum.
4. Find the transformed output for general c and compare its exact real transform domain before and after cancellation. Explain why the common transform domain of the factors need not be the output's full domain.

Question 7 – Solution

Strategy. An unstable kernel remembers a weighted input area. Equal positive and negative ordinary areas need not cancel that memory.

Step 1: Factor the growing exponential. For an input supported in $[0, T]$,

$$(h * f)(t) = e^t \int_0^{\min(t, T)} e^{-u} f(u) du.$$

Thus the output is zero for every $t \geq T$ if and only if $\int_0^T e^{-u} f(u) du = 0$, since e^t never vanishes.

Step 2: Distinguish the two cancellations. The ordinary area is $1 - c$, so it vanishes at $c = 1$. The weighted area is $(1 - e^{-1}) - c(e^{-1} - e^{-2})$; its unique zero is $c = e$. At $c = 1$ it equals $(1 - e^{-1})^2 > 0$, leaving a growing tail.

Step 3: Recover the selected response. Using the partial weighted integrals with $c = e$ gives

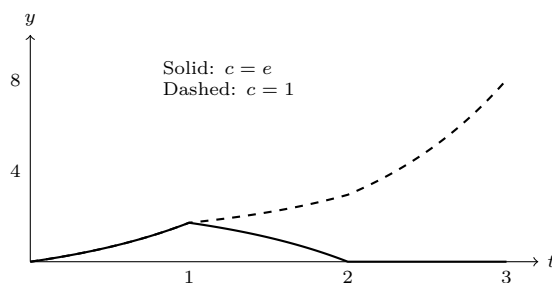
$$y_*(t) = \begin{cases} e^t - 1, & 0 \leq t < 1, \\ e - e^{t-1}, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

The pieces agree at both junctions. It increases up to 1 and decreases from 1 to 2, so its unique maximum is $e - 1$ at $t = 1$.

Step 4: Track the transform domain. The convolution theorem initially gives, for $s > 1$,

$$Y(s) = \frac{1 - (1 + c)e^{-s} + ce^{-2s}}{s(s - 1)}.$$

If $c \neq e$, the output has a nonzero e^t tail and exact real domain $s > 1$. If $c = e$, the output is compactly supported, so its transform exists for every real s ; both apparent singularities at 0 and 1 are removable. The product formula agrees with it on $s > 1$ and extends by removal of these singularities. Cancellation changes this output, not the unstable kernel. The dashed comparison uses $c = 1$, which cancels only the ordinary input area.



Question 8

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Suppose an ordinary continuous input f produces $z = e^{-t} * f$. No impulses and no nonzero initial state are allowed. Consider the desired output $z_0(t) = te^{-t}$ and the perturbed outputs

$$z_{\varepsilon,N}(t) = te^{-t} + \varepsilon e^{-t} \sin(Nt), \quad \varepsilon > 0, \quad N \in \{1, 2, \dots\}.$$

Tasks

1. Characterize all continuously differentiable outputs obtainable from such inputs, and prove uniqueness of the recovered input.
2. Recover the input for z_0 . Decide whether the target $z(t) = e^{-t}$ is obtainable, explaining why differentiating alone can give a misleading answer.
3. Recover the input for $z_{\varepsilon,N}$ and bound its output error in the uniform norm on $[0, \infty)$. Compute the input error at zero.
4. Choose a sequence $\varepsilon = \varepsilon_N \rightarrow 0$ for which the output errors tend uniformly to zero but the input errors grow without bound. Interpret what this means for reconstruction from approximate measurements.

Question 8 – Solution

Strategy. Inversion differentiates the output. Check the initial-value restriction before using that derivative formula.

Step 1: Derive and prove the inverse rule. Differentiation under the integral gives $z' = f - z$, $z(0) = 0$. Therefore a C^1 target is obtainable exactly when it starts at zero, and its only possible continuous input is $\boxed{f = z' + z}$. Conversely, for this input, the target and the convolution solve the same IVP $w' + w = f$, $w(0) = 0$. Uniqueness proves they agree.

Step 2: Check two targets. For $z_0 = te^{-t}$, $z'_0 + z_0 = e^{-t}$, so the input is $f_0 = e^{-t}$. The target e^{-t} is impossible because its value at zero is 1. Although its derivative plus itself is zero, a zero input with zero initial state produces the zero output. Differentiation discards the missing initial condition.

Step 3: Differentiate the perturbation. The two exponential-derivative terms cancel, leaving

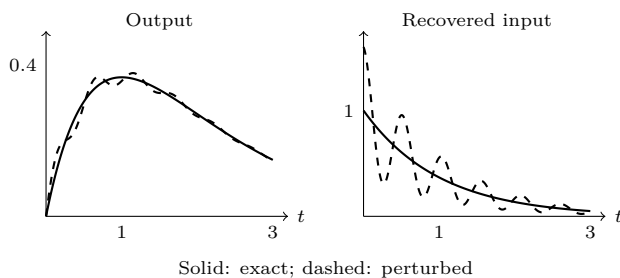
$$\boxed{f_{\varepsilon,N} = e^{-t} + \varepsilon N e^{-t} \cos(Nt)}.$$

All these inputs are continuous, and the target outputs vanish at zero. The uniform output error is at most ε because $e^{-t}|\sin Nt| \leq 1$. The input error at zero is εN ; in fact this is its uniform norm since $e^{-t}|\cos Nt| \leq 1$.

Step 4: Exhibit the instability. Take $\varepsilon_N = N^{-1/2}$. Then

$$\|z_{\varepsilon_N,N} - z_0\|_\infty \leq N^{-1/2} \rightarrow 0, \quad \|f_{\varepsilon_N,N} - f_0\|_\infty = \sqrt{N} \rightarrow \infty.$$

Thus the inverse is not continuous in the uniform output norm, even on smooth attainable outputs. Small high-frequency measurement errors can produce large reconstruction errors. The figure uses $\varepsilon = 0.05$, $N = 12$; each panel compares its own exact and perturbed functions.



Question 9

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Two normalized stages $k_\alpha(t) = \alpha e^{-\alpha t}$ and $k_\beta(t) = \beta e^{-\beta t}$, with $\alpha, \beta > 0$, are followed by a delay $d \geq 0$:

$$r(t) = H_d(t)(k_\alpha * k_\beta)(t - d).$$

The observed response is positive precisely for $t > 1$, has mass 1, mean $\mu = 5/2$ and variance $v = 5/4$. For a nonnegative unit-mass kernel p , $\mu_p = \int_0^\infty tp(t) dt$ and $v_p = \int_0^\infty (t - \mu_p)^2 p(t) dt$.

Tasks

1. Prove that convolution adds the means and variances of two nonnegative unit-mass kernels with finite second moments.
2. Compute the mean and variance of k_α , and explain the effect of a delay on mass, mean and variance.
3. Recover d and the two rates from the observations. State precisely what is and is not uniquely identifiable.
4. For general prescribed delay d , mean μ and variance v , give necessary and sufficient conditions for two finite positive rates to exist. Interpret the equality case and the excluded upper boundary.

Question 9 – Solution

Strategy. Work with reciprocal rates. Convolution moment identities reduce identification to a quadratic equation.

Step 1: Prove the moment identities. In the convolution integral use $t = u + w$, $u, w \geq 0$. Nonnegativity permits interchange, and finite second moments make the following values finite:

$$\begin{aligned}\int (p * q) &= \left(\int p\right)\left(\int q\right) = 1, \\ \mu_{p*q} &= \iint (u + w)p(u)q(w) du dw = \mu_p + \mu_q, \\ \int t^2(p * q)(t) dt &= \int u^2 p(u) du + 2\mu_p \mu_q + \int w^2 q(w) dw.\end{aligned}$$

Subtracting $(\mu_p + \mu_q)^2$ from the last line proves $\boxed{v_{p*q} = v_p + v_q}$.

Step 2: Calculate the stage statistics. Integration by parts gives mass 1, first moment $1/\alpha$ and second moment $2/\alpha^2$ for k_α . Its variance is therefore $1/\alpha^2$. A delay replaces time by $t = d + u$: mass stays 1, the mean increases by d , and the centered second moment is unchanged. Hence

$$\mu = d + \frac{1}{\alpha} + \frac{1}{\beta}, \quad v = \frac{1}{\alpha^2} + \frac{1}{\beta^2}.$$

Step 3: Recover the parameters. Both stage kernels are positive for positive times, so their convolution is positive at every positive time and zero at zero. The observed onset thus forces $d = 1$. Put $x = 1/\alpha$, $z = 1/\beta$. Then

$$x + z = \frac{3}{2}, \quad x^2 + z^2 = \frac{5}{4}, \quad xz = \frac{1}{2}, \quad (X - x)(X - z) = X^2 - \frac{3}{2}X + \frac{1}{2}.$$

Its roots are 1 and $1/2$, giving $\boxed{\{\alpha, \beta\} = \{1, 2\}}$. The delay and unordered rate pair are unique. Stage order cannot be identified even from the entire output because convolution is commutative.

Step 4: Give the full feasibility condition. Write $m = \mu - d$. Positive x, z require and are possible exactly when

$$\boxed{m > 0, \quad \frac{m^2}{2} \leq v < m^2.}$$

Indeed $xz = (m^2 - v)/2 > 0$ and $(x - z)^2 = 2v - m^2 \geq 0$; conversely these conditions make $(m \pm \sqrt{2v - m^2})/2$ real and strictly positive. At the lower equality the rates coincide. At the upper boundary one reciprocal rate would be zero, requiring an infinite rate, excluded here.

Question 10

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f \in C^1[0, \infty)$ satisfy $|f'(t)| \leq L$, with $L \geq 0$. For $\varepsilon > 0$ define

$$k_\varepsilon(t) = \frac{1}{\varepsilon} e^{-t/\varepsilon}, \quad y_\varepsilon = k_\varepsilon * f.$$

This causal averaging kernel becomes concentrated near zero as ε decreases.

Tasks

1. Derive an IVP for y_ε and use integration by parts to express $y_\varepsilon - f$ in terms of $f(0)$ and f' .
2. Prove an explicit error bound containing both the initial transient and a term at most $L\varepsilon$.
3. Distinguish uniform convergence on $[0, \infty)$ from uniform convergence on $[t_0, \infty)$ for $t_0 > 0$. Show the $L\varepsilon$ bound is sharp when $f(0) = 0$.
4. For $f(t) = 1 + t$, decide separately whether $\varepsilon = 1/20$ and $\varepsilon = 1/40$ ensure $|y_\varepsilon(t) - f(t)| \leq 1/20$ for every $t \geq 1$. Justify the strict inequalities involved.

Question 10 – Solution

Strategy. Separate the error due to starting from rest from the error due to variation of the input.

Step 1: Obtain the equation and exact error identity. Differentiation gives $\varepsilon y'_\varepsilon + y_\varepsilon = f$, $y_\varepsilon(0) = 0$. Integrating the convolution by parts in u gives

$$y_\varepsilon(t) - f(t) = -e^{-t/\varepsilon} f(0) - \int_0^t e^{-(t-u)/\varepsilon} f'(u) du.$$

Step 2: Bound the two contributions. Taking absolute values yields

$$|y_\varepsilon - f| \leq |f(0)|e^{-t/\varepsilon} + L\varepsilon(1 - e^{-t/\varepsilon}).$$

This accounts for the kernel mass missing before time zero as well as the variation over its recent history.

Step 3: Distinguish the convergence statements. If $f(0) = 0$, the error is uniformly at most $L\varepsilon$ on the half-line. For the ramp $f = Lt$ it equals $L\varepsilon(1 - e^{-t/\varepsilon})$ in magnitude, whose supremum is $L\varepsilon$ (approached as $t \rightarrow \infty$ when $L > 0$). Thus the constant is sharp. If $f(0) \neq 0$, the error at zero is always $|f(0)|$, preventing uniform convergence on $[0, \infty)$. On $[t_0, \infty)$ it is bounded by $|f(0)|e^{-t_0/\varepsilon} + L\varepsilon \rightarrow 0$ for every fixed $t_0 > 0$.

Step 4: Test the two proposed averaging scales. For $f = 1 + t$, $y_\varepsilon = t + 1 - \varepsilon - (1 - \varepsilon)e^{-t/\varepsilon}$. For $0 < \varepsilon < 1$ the error magnitude is $E_\varepsilon = \varepsilon + (1 - \varepsilon)e^{-t/\varepsilon}$. At $\varepsilon = 1/20$ this is strictly greater than $1/20$ at every finite time, so that choice fails. At $\varepsilon = 1/40$ and $t \geq 1$,

$$E_\varepsilon \leq \frac{1}{40} + \frac{39}{40}e^{-40} < \frac{1}{20},$$

since $e^{40} > 1 + 40 = 41 > 39$. Thus the smaller choice succeeds. The figure shows the initial error layer and the $1/20$ tolerance.

