

Question 1

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Let $a, J > 0$ and $\varepsilon > 0$. A finite pulse and its impulse idealization drive

$$y'_\varepsilon + 2y_\varepsilon = \frac{J}{\varepsilon}[H_a(t) - H_{a+\varepsilon}(t)], \quad y' + 2y = J\delta(t-a), \quad y_\varepsilon(0) = y(0) = 0.$$

Tasks

1. Prove that the finite pulse tends to $J\delta(t-a)$ when integrated against a continuous test function. Derive the limiting input transform.
2. Find the impulsive response, including its jump, and solve the finite-pulse IVP explicitly.
3. For $t \geq a + \varepsilon$, prove $0 \leq y_\varepsilon(t) - y(t) \leq J\varepsilon$.
4. Describe pointwise convergence, including the exceptional time a , and decide whether convergence can be uniform on an interval containing a in its interior.

Question 1 – Solution

Strategy. Preserve pulse area rather than height. Compare the two exact responses before making any claim about convergence.

Step 1: Test the pulse and transform. Its action on a continuous ϕ is

$$\frac{J}{\varepsilon} \int_a^{a+\varepsilon} \phi(t) dt \longrightarrow J\phi(a).$$

The error is at most $J \sup_{[a, a+\varepsilon]} |\phi(t) - \phi(a)|$, which tends to zero. Taking $\phi = e^{-st}$ gives $F_\varepsilon = J e^{-as}(1 - e^{-\varepsilon s})/(\varepsilon s) \rightarrow J e^{-as}$, with the removable value $F_\varepsilon(0) = J$.

Step 2: Solve both equations. The impulse gives $Y = J e^{-as}/(s+2)$, so $y = J H_a(t) e^{-2(t-a)}$ and $[y]_a = J$. The finite-pulse solution is continuous and equals

$$y_\varepsilon(t) = \begin{cases} 0, & t < a, \\ \frac{J}{2\varepsilon} (1 - e^{-2(t-a)}), & a \leq t < a + \varepsilon, \\ \frac{J}{2\varepsilon} (1 - e^{-2\varepsilon}) e^{-2(t-a-\varepsilon)}, & t \geq a + \varepsilon. \end{cases}$$

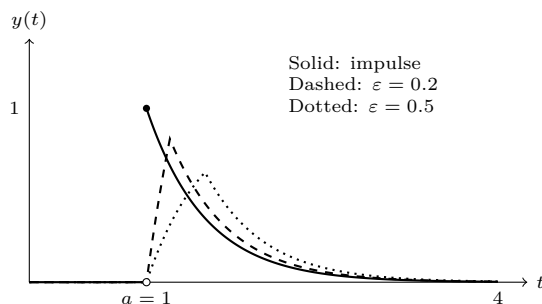
Each piece satisfies its ordinary equation, and the endpoints match.

Step 3: Bound the tail error. Put $\tau = t - a \geq \varepsilon$. The explicit tail can be rewritten as $y_\varepsilon = (J/\varepsilon) \int_0^\varepsilon e^{-2(\tau-u)} du$. Since $x \mapsto e^{-2x}$ is decreasing and has derivative magnitude at most 2 for $x \geq 0$,

$$0 \leq y_\varepsilon - y = \frac{J}{\varepsilon} \int_0^\varepsilon [e^{-2(\tau-u)} - e^{-2\tau}] du \leq \frac{J}{\varepsilon} \int_0^\varepsilon 2u du = J\varepsilon.$$

This representation follows directly from the finite-pulse formula.

Step 4: Distinguish the kinds of convergence. For each fixed $t \neq a$, $y_\varepsilon(t) \rightarrow y(t)$. At a , $y_\varepsilon(a) = 0$ but the right-hand representative has $y(a) = J$. Uniform convergence on an interval containing a is impossible: the approximants are continuous and the limit has a jump. Changing only $y(a)$ does not remove that jump or change its transform. The graph uses $a = J = 1$; open and filled dots mark the impulsive response.



Question 2

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Consider a weighted impulse and a reversed impulse argument:

$$y' + y = (t^2 - 1)\delta(2t - 4) + 3\delta(3 - t), \quad y(0) = 2.$$

You may use $\delta(c(t-a)) = \delta(t-a)/|c|$ for $c \neq 0$.

Tasks

1. Reduce the forcing to impulses at named times with constant weights. Explain the absolute value in the scaling rule.
2. Find $Y(s)$ and the complete time response, including the initial state.
3. Determine both jumps and verify the ordinary equation on each interval.
4. Compute the response area and its exact real transform domain. Explain why the response area is not just the sum of the impulse weights.

Question 2 – Solution

Strategy. Scale the delta first, then evaluate its smooth multiplier at the impulse time. The initial state contributes its own response area.

Step 1: Normalize and sift. Here $\delta(2t - 4) = \delta(t - 2)/2$ and $\delta(3 - t) = \delta(t - 3)$. The sampling property gives

$$f(t) = \frac{3}{2}\delta(t - 2) + 3\delta(t - 3).$$

The first weight is $(2^2 - 1)/2 = 3/2$. Reversing an argument does not turn a positive unit mass into a negative one: changing integration variables reverses the limits as well as introducing the negative derivative. Thus the scale factor is $1/|c|$.

Step 2: Transform and invert. The initial term is retained:

$$(s + 1)Y - 2 = \frac{3}{2}e^{-2s} + 3e^{-3s}, \quad Y = \frac{2 + \frac{3}{2}e^{-2s} + 3e^{-3s}}{s + 1}.$$

Consequently

$$y(t) = 2e^{-t} + \frac{3}{2}H_2(t)e^{-(t-2)} + 3H_3(t)e^{-(t-3)}.$$

Step 3: Verify the jumps and intervals. Every active exponential satisfies $y' + y = 0$ away from 2 and 3. The newly activated terms give

$$[y]_2 = \frac{3}{2}, \quad [y]_3 = 3, \quad y(0) = 2.$$

Equivalently, integrating the equation across either impulse makes the bounded y integral vanish as the interval shrinks, leaving the specified jump. Ordinary uniqueness on each interval and these jumps determine the whole response.

Step 4: Account for the initial stored response. Each shifted exponential of coefficient C has integral C . Hence

$$\int_0^\infty y(t) dt = 2 + \frac{3}{2} + 3 = \frac{13}{2}.$$

Integrating the impulsive equation over the half-line gives $y(\infty) - y(0) + \int_0^\infty y dt = 9/2$ and the same result. The derivative contribution is -2 , not zero. The positive nonzero e^{-t} tail gives exact real domain $s > -1$.

Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An oscillator moving through equilibrium receives an impulse:

$$y'' + 4y = 6\delta(t - a), \quad y(0) = 1, \quad y'(0) = 0, \quad a = \frac{\pi}{4}.$$

For unit mass, use $E = (y'^2 + 4y^2)/2$. Ordinary impulses in a second-order equation keep displacement continuous but may jump velocity.

Tasks

1. Derive the velocity jump and find $Y(s)$ and $y(t)$.
2. Compute the states immediately before and after the impulse and simplify the subsequent motion.
3. Find the energy change and derive the general energy-change formula for an impulse of strength J at this same time.
4. For $J > 0$, classify when the impulse lowers the energy, preserves it, or raises it. Identify the unique strength that brings this oscillator completely to rest.

Question 3 – Solution

Strategy. The impulse changes velocity, not displacement. Its work depends on the velocity immediately before the kick as well as its strength.

Step 1: Jump, transform and invert. Integrating across a gives $[y']_a = 6$, because the integral of $4y$ tends to zero. With $[y]_a = 0$, the transformed equation is

$$Y = \frac{s + 6e^{-as}}{s^2 + 4}, \quad \boxed{y(t) = \cos 2t + 3H_a(t) \sin(2(t-a))}.$$

The new term vanishes at activation, while its derivative starts at 6. Both pieces satisfy $y'' + 4y = 0$ away from a and retain the initial data.

Step 2: Evaluate the state at impact. Since $2a = \pi/2$,

$$(y(a-), y'(a-)) = (0, -2), \quad (y(a+), y'(a+)) = \boxed{(0, 4)}.$$

For $t = a + \tau$, the pre-existing response is $-\sin 2\tau$, so the total post-impact motion is $\boxed{y(a + \tau) = 2 \sin 2\tau}$. Thus the impulse reverses the direction and doubles the amplitude.

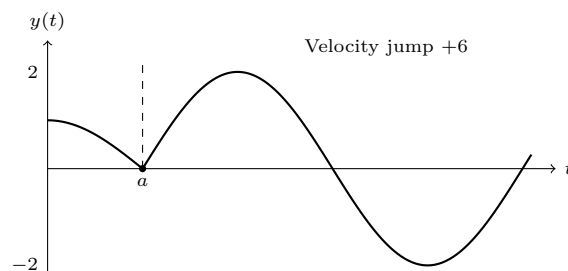
Step 3: Compute the energy transfer. The energies are $E_- = 2$ and $E_+ = 8$, giving $\boxed{\Delta E = 6}$. For any impulse J , continuity of position and $v_+ = v_- + J$ imply

$$\Delta E = \frac{1}{2}[(v_- + J)^2 - v_-^2] = Jv_- + \frac{1}{2}J^2.$$

At this impact time $v_- = -2$, hence

$$\boxed{\Delta E = \frac{1}{2}J(J - 4), \quad E_+ = \frac{1}{2}(J - 2)^2}.$$

Step 4: Classify the positive kicks. For $0 < J < 4$ the energy decreases; at $J = 4$ it is unchanged; for $J > 4$ it increases. A positive impulse can remove energy when it opposes the incoming velocity. Complete rest requires both post-impact state coordinates to vanish. Position is already zero here, so the unique strength is $\boxed{J = 2}$, which makes velocity zero. A zero velocity kick at a nonzero displacement would not in general leave an oscillator at rest. The plot shows the stated $J = 6$ case, with continuous displacement and a corner.



Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A critically damped system starts from rest and receives two impulses:

$$y'' + 2y' + y = \delta(t-1) + K\delta(t-b), \quad y(0) = y'(0) = 0, \quad b > 1.$$

You may choose b and K . The energy is $E = (y'^2 + y^2)/2$.

Tasks

1. Find $Y(s)$ and the response in terms of the age since each impulse.
2. Determine whether any finite $b > 1$ and K can make $y(t) = 0$ for every $t \geq b$.
3. For a fixed b , find the unique K minimizing the energy immediately after the second impulse, and find that minimum.
4. Derive the motion after this minimizing kick and explain why zero post-impact velocity does not imply complete rest.

Question 4 – Solution

Strategy. One impulse can adjust velocity only. Inspect the displacement at the chosen impact time before attempting to cancel the entire state.

Step 1: Find the impulse response. The zero-data transform is

$$Y = \frac{e^{-s} + Ke^{-bs}}{(s+1)^2}.$$

Since $\mathcal{L}\{te^{-t}\} = 1/(s+1)^2$, set $h(t) = te^{-t}$ for $t \geq 0$. Then

$$y = H_1(t)h(t-1) + KH_b(t)h(t-b).$$

Here $h(0) = 0$, $h'(0) = 1$, and $h'' + 2h' + h = 0$ for positive age. Thus displacement is continuous, and velocity jumps by 1 and K .

Step 2: Test the terminal displacement. Put $d = b - 1 > 0$. Immediately before the second kick,

$$y_b = de^{-d} > 0, \quad v_- = (1-d)e^{-d}.$$

The second impulse leaves y_b unchanged. A solution identically zero after b must have zero displacement there, so no finite $b > 1$ and K can produce complete rest. The strict positivity of $h(d)$ is the obstruction, not a shortage of algebraic manipulation.

Step 3: Minimize the post-impact energy. For fixed b ,

$$E_+(K) = \frac{1}{2}[(v_- + K)^2 + y_b^2].$$

This strictly convex quadratic has the unique minimizer and value

$$K_* = (d-1)e^{-d}, \quad E_{\min} = \frac{1}{2}d^2e^{-2d} > 0.$$

The minimizing kick makes $v_+ = 0$ but leaves positive stored spring energy.

Step 4: Follow the resulting motion. For $\tau = t - b \geq 0$, the critical homogeneous solution with state $(y_b, 0)$ is

$$y(b+\tau) = de^{-d}(1+\tau)e^{-\tau}.$$

Its derivative is $-de^{-d}\tau e^{-\tau}$: it starts horizontal and then moves strictly downward. Its acceleration at $\tau = 0+$ is $-y_b \neq 0$. Substitution verifies the equation and the terminal state. Thus the system eventually approaches rest, but is not at rest throughout the post-impact interval. The positive polynomial-exponential tail has transform domain $s > -1$.

Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a)dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

Two impulses act on an initially resting undamped oscillator:

$$y'' + \omega^2 y = I\delta(t-a) + K\delta(t-b), \quad y(0) = y'(0) = 0, \quad I, \omega > 0, \quad 0 < a < b.$$

Require complete rest after the second impulse.

Tasks

1. Find the transformed response and the time-domain solution.
2. Classify all delays $b-a$ and strengths K that meet the rest condition.
3. If K must be positive, find the earliest possible second impulse. Explain why the net impulse need not be zero.
4. Keep this earliest positive strength, but deliver it at $b+\eta$, where $|\eta| < \pi/\omega$. Find the residual amplitude and its leading behavior as $\eta \rightarrow 0$.

Question 5 – Solution

Strategy. The second kick must occur at a zero of displacement, then cancel the incoming velocity. Timing errors leave a measurable residual oscillation.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{Ie^{-as} + Ke^{-bs}}{s^2 + \omega^2}, \quad \boxed{y = \frac{I}{\omega}H_a(t) \sin \omega(t - a) + \frac{K}{\omega}H_b(t) \sin \omega(t - b).}$$

Each activated sine is zero initially, with derivative equal to its impulse strength. Thus y is continuous and $[y']_a = I$, $[y']_b = K$.

Step 2: Impose both state conditions. For $d = b - a$, the pre-impact state is

$$y(b-) = \frac{I}{\omega} \sin \omega d, \quad y'(b-) = I \cos \omega d.$$

Displacement cannot jump. Therefore complete rest requires and is ensured by

$$\boxed{d = \frac{n\pi}{\omega}, \quad K = (-1)^{n+1}I, \quad n = 1, 2, \dots}$$

After these jumps the homogeneous oscillator has zero data, so it remains zero. The corresponding output has compact support and a transform for every real s .

Step 3: Restrict the second impulse to be positive. Positive K requires odd n . The earliest choice is $b = a + \pi/\omega$, $K = I$. The net applied impulse is $2I$, yet the oscillator stops: during the intervening motion the restoring force also changes momentum. Indeed $\int_a^b \omega^2 y dt = 2I$, balancing the applied impulses in the integrated equation. Ignoring this force would give an incorrect momentum test.

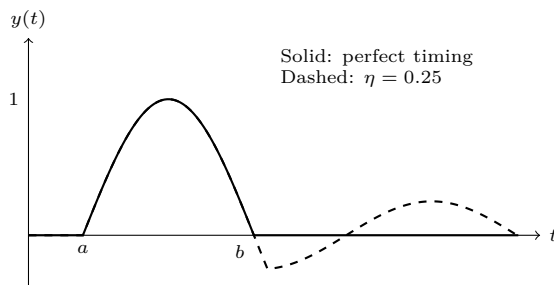
Step 4: Quantify a timing error. At the actual second kick the age is $\pi/\omega + \eta > 0$. The state immediately afterward is

$$y_+ = -\frac{I}{\omega} \sin \omega \eta, \quad v_+ = I(1 - \cos \omega \eta).$$

The subsequent homogeneous amplitude is

$$R = \sqrt{y_+^2 + (v_+/\omega)^2} = \boxed{\frac{2I}{\omega} |\sin(\omega\eta/2)|} = I|\eta| + O(|\eta|^3).$$

For every nonzero η in the stated interval this amplitude is positive, so the tail's exact real transform domain is $s > 0$. The plot compares perfect timing and $\eta = 0.25$ for $I = \omega = a = 1$.



Question 6

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A single unknown impulse drives a first-order system:

$$y' + \lambda y = J\delta(t-a), \quad y(0) = 0, \quad \lambda, J > 0, \quad 0 < a < 2.$$

Exact observations give $y(2) = 4$ and $y(3) = 2$. An additional measurement is the total response area $\int_0^\infty y(t) dt = 8$.

Tasks

1. Derive the transform and response for general λ, J, a .
2. Use only the two point observations to determine what is identifiable. Parameterize every admissible triple consistent with them.
3. Use the area measurement to recover all three parameters uniquely, and verify the delay restriction.
4. Check all measurements directly and explain why the extra area measurement resolves an ambiguity that another later point sample would not.

Question 6 – Solution

Strategy. Post-impulse samples determine a decay rate and an effective amplitude. The area determines the impulse strength independently of its delay.

Step 1: Solve the parameterized problem. The zero datum gives

$$Y = \frac{J e^{-as}}{s + \lambda}, \quad \boxed{y = J H_a(t) e^{-\lambda(t-a)}}.$$

The response has jump J at a , solves $y' + \lambda y = 0$ afterward, and has area J/λ . Its exact real transform domain is $s > -\lambda$.

Step 2: Identify the point-sample information. Both observations occur after the impulse. Taking their ratio gives $e^{-\lambda} = 1/2$, so $\lambda = \ln 2$. The first observation then gives $J 2^{a-2} = 4$, or

$$\boxed{\lambda = \ln 2, \quad J = 16 \cdot 2^{-a}, \quad 0 < a < 2.}$$

This is a full one-parameter family, not a unique impulse time. The strength ranges over $4 < J < 16$.

Step 3: Use the area and prove admissibility. The area condition now fixes $J/\ln 2 = 8$. Thus

$$\boxed{J = 8 \ln 2, \quad a = \frac{\ln(2/\ln 2)}{\ln 2}, \quad \lambda = \ln 2.}$$

Since $1/2 < \ln 2 < 1$, we have $1 < 2/\ln 2 < 4$. Taking logarithms to base 2 shows $0 < a < 2$. The area fixes J uniquely, and $J = 16 \cdot 2^{-a}$ is strictly decreasing in a , so this recovered triple is unique in the stated family.

Step 4: Verify and interpret. The recovered delay satisfies $2^a = 2/\ln 2$. Therefore

$$y(2) = (8 \ln 2) \frac{2/\ln 2}{4} = 4, \quad y(3) = \frac{1}{2} y(2) = 2, \quad \int_0^\infty y \, dt = \frac{8 \ln 2}{\ln 2} = 8.$$

For every member of the two-sample family, all later values are $y(t) = 16 \cdot 2^{-t}$ for $t \geq 2$. Thus another exact sample after 2 would either agree with every member or reject every member; it could not separate strength from delay. The total area includes the unsampled interval after the unknown impulse and supplies the missing information.

Question 7

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An unstable second-order system is given one corrective impulse:

$$y'' - y = J\delta(t-a), \quad y(0) = 1, \quad y'(0) = 0, \quad a > 0.$$

Choose J to make the response decay to zero after the impulse. A realized impulse may differ by a constant error Δ .

Tasks

1. Derive $Y(s)$ and $y(t)$ for arbitrary J , including the initial response.
2. Find the unique decaying choice J_* , its explicit tail and the global maximum of the selected response.
3. Derive the response error for strength $J_* + \Delta$ and a sharp bound on $|\Delta|$ ensuring error at most $\epsilon > 0$ throughout $a \leq t \leq a + L$, where $L > 0$.
4. Compare exact real transform domains for the selected and perturbed strengths. Explain the effect of cancellation on the system's stability.

Question 7 – Solution

Strategy. Cancel the growing mode's coefficient, then inspect the remaining mode and the error produced by imperfect cancellation.

Step 1: Transform and invert. For sufficiently large s ,

$$(s^2 - 1)Y - s = J e^{-as}, \quad Y = \frac{s + J e^{-as}}{s^2 - 1}, \quad \boxed{y = \cosh t + J H_a(t) \sinh(t - a)}.$$

The added term starts at zero with derivative J , so it preserves displacement and produces the correct velocity jump. Away from a both terms solve the homogeneous equation.

Step 2: Select and interpret the decaying mode. After a , the e^t coefficient is $(1 + J e^{-a})/2$. Decay requires

$$\boxed{J_* = -e^a, \quad y_*(t) = \begin{cases} \cosh t, & 0 \leq t < a, \\ \cosh a e^{-(t-a)}, & t \geq a. \end{cases}}$$

The pre-impact velocity is $\sinh a$ and the post-impact velocity is $-\cosh a$; their difference is $-e^a$, as required. The response increases strictly before a and decreases strictly after it. Its unique global maximum is $\boxed{y_*(a) = \cosh a}$.

Step 3: Quantify the sensitivity. Linearity or direct subtraction gives

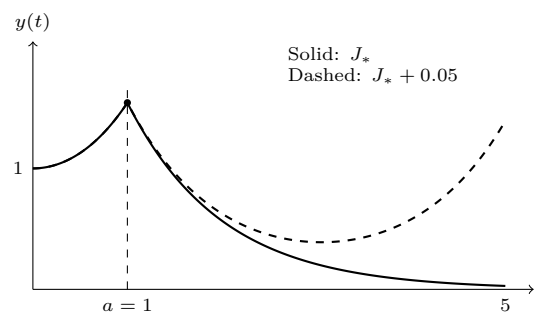
$$\boxed{y(t) - y_*(t) = \Delta H_a(t) \sinh(t - a)}.$$

Because $\sinh \tau$ increases strictly on $[0, L]$, the desired error bound is equivalent to

$$\boxed{|\Delta| \leq \frac{\epsilon}{\sinh L}}.$$

This is necessary as well as sufficient, and equality attains the bound at $a + L$. The selected solution still approaches zero only asymptotically; the impulse does not put it at the zero state.

Step 4: Check convergence and stability. The selected tail is a positive multiple of e^{-t} , so its exact real transform domain is $\boxed{s > -1}$. The apparent singularity at $s = 1$ in its transform is removable. For every $\Delta \neq 0$, the error contains a nonzero growing mode and the actual domain becomes $\boxed{s > 1}$. The differential equation's unstable homogeneous mode is unchanged. The plot uses $a = 1$, with a dashed curve for $\Delta = 0.05$.



Question 8

Interpret the equation through ordinary motion away from the impulse and its jump conditions. For this question only, the initial-time transform convention is specified below; write $[v]_0 = v(0^+) - v(0^-)$ for a jump.

Use a *full-mass initial impulse* convention: the transform starts at 0^- , $\mathcal{L}\{\delta(t)\} = 1$, and

$$\mathcal{L}\{y'\} = sY - y(0^-), \quad \mathcal{L}\{y''\} = s^2Y - sy(0^-) - y'(0^-).$$

Consider

$$2y'' + 4y' + 2y = J\delta(t), \quad y(0^-) = 1, \quad y'(0^-) = 2.$$

These are pre-impact data, not post-impact data.

Tasks

1. Find $y(0^+)$ and $y'(0^+)$ by the impulse jump conditions.
2. Use the stated 0^- convention to find $Y(s)$ and the response for $t \geq 0$.
3. Reproduce the same response by solving a regular homogeneous IVP from 0^+ . Explain why keeping the impulse again would double-count it.
4. Classify every J for which the post-impact response is nonnegative and nonincreasing for all $t \geq 0$. Describe the boundary cases.

Question 8 – Solution

Strategy. An impulse at the initial time requires a declared convention. Either transform pre-impact data with the impulse or restart after the jump.

Step 1: Compute the post-impact data. There is no displacement jump under an ordinary force impulse. Integrating the equation across the origin gives $2[y']_0 = J$; the damping contribution is $4[y]_0 = 0$. Thus

$$y(0^+) = 1, \quad y'(0^+) = 2 + \frac{J}{2}.$$

The factor 2 is the mass and must not be omitted.

Step 2: Transform the pre-impact formulation. The stipulated derivative rules give

$$2(s^2Y - s - 2) + 4(sY - 1) + 2Y = J,$$

so

$$Y = \frac{2s + 8 + J}{2(s + 1)^2} = \frac{1}{s + 1} + \frac{3 + J/2}{(s + 1)^2}.$$

Writing $c = 3 + J/2$, the inverse is $y(t) = e^{-t}(1 + ct)$, $t \geq 0$. It has $y(0^+) = 1$ and $y'(0^+) = c - 1 = 2 + J/2$, exactly as the jump calculation.

Step 3: Check the equivalent restart. For $t > 0$ the equation is $y'' + 2y' + y = 0$. Using the post-impact data $1, 2 + J/2$ gives the critical solution $e^{-t}[1 + (3 + J/2)t]$ again. This restart has no remaining initial impulse: its effect is already encoded in the velocity. Inserting $J\delta(t)$ again with those data would add a second velocity change $J/2$.

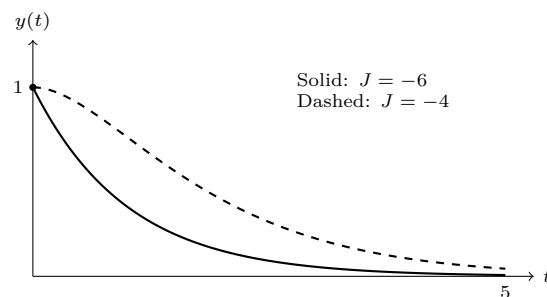
Step 4: Classify positivity and monotonicity. Since the exponential is positive, nonnegativity for every $t \geq 0$ requires and is ensured by $c \geq 0$. Also

$$y' = e^{-t}(c - 1 - ct).$$

For $c \geq 0$, this is nonpositive for all $t \geq 0$ exactly when $c \leq 1$. Therefore

$$0 \leq c \leq 1 \iff -6 \leq J \leq -4.$$

At $J = -6$, $y = e^{-t}$; at $J = -4$, $y = (1 + t)e^{-t}$ starts horizontal and then decreases strictly. For $J < -6$ the response eventually becomes negative, and for $J > -4$ it initially increases. All these nonzero polynomial-exponential responses have exact real transform domain $s > -1$.



Question 9

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

An actuator may use ordinary impulses at 1 and 3, with no other force:

$$y'' + 2y' + y = J_1\delta(t-1) + J_3\delta(t-3), \quad y(0) = y'(0) = 0.$$

The prescribed response has the form

$$y(t) = \begin{cases} 0, & 0 \leq t < 1, \\ (t-1)e^{-(t-1)}, & 1 \leq t < 3, \\ Ce^{-(t-3)}, & t \geq 3. \end{cases}$$

An ordinary impulse in this equation can jump velocity but not displacement.

Tasks

1. Determine the only possible value of C under the permitted actuator.
2. Recover J_1, J_3 from the one-sided states, verifying that no ordinary forcing is needed between impulses.
3. Find $Y(s)$, invert it, and confirm the prescribed tail. Decide whether choosing $C = 0$ could shut the motion off instantaneously with an ordinary impulse.
4. Verify the total response area from the time formula and the integrated impulsive equation, and state the exact real transform domain.

Question 9 – Solution

Strategy. Infer the required impulses from velocity jumps after enforcing displacement continuity. Then independently check the recovered forcing by inversion.

Step 1: Enforce continuity at the second impulse. The target displacement just before 3 is $2e^{-2}$, while the right-hand value is C . Hence the permitted actuator requires

$$\boxed{C = 2e^{-2}}.$$

The first junction is already continuous because $(t-1)e^{-(t-1)}$ vanishes at 1.

Step 2: Recover the impulse strengths. For $h(\tau) = \tau e^{-\tau}$, $h'(0) = 1$ and $h'(2) = -e^{-2}$. Thus the first velocity jump is 1. At the second switch the new tail has velocity $-C = -2e^{-2}$, so

$$\boxed{J_1 = 1, \quad J_3 = -2e^{-2} - (-e^{-2}) = -e^{-2}}.$$

Because y is continuous, integrating the equation across a junction gives $J_j = [y']_j$. Direct differentiation of each piece gives $y'' + 2y' + y = 0$ on all three open intervals.

Step 3: Verify by transformation and inversion. The recovered input gives

$$Y = \frac{e^{-s} - e^{-2}e^{-3s}}{(s+1)^2}, \quad y = H_1(t)(t-1)e^{-(t-1)} - e^{-2}H_3(t)(t-3)e^{-(t-3)}.$$

For $t = 3 + \tau$, the last expression reduces to

$$e^{-2}[(\tau+2) - \tau]e^{-\tau} = 2e^{-2}e^{-\tau},$$

which is the prescribed tail. Setting $C = 0$ instead would require a jump of $-2e^{-2}$ in displacement at 3, forbidden for the stated impulse-only actuator. Canceling velocity cannot instantaneously erase nonzero displacement.

Step 4: Check the area balance. Direct integration gives

$$\int_0^\infty y \, dt = \int_0^2 \tau e^{-\tau} \, d\tau + 2e^{-2} \int_0^\infty e^{-\tau} \, d\tau = 1 - 3e^{-2} + 2e^{-2} = \boxed{1 - e^{-2}}.$$

The integrated derivative terms are zero when their impulse jumps are included: initial and limiting displacement and velocity are zero. Thus $\int y = J_1 + J_3 = 1 - e^{-2}$ independently confirms the result. The nonzero e^{-t} tail gives exact real domain $\boxed{s > -1}$.

Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. The unit impulse satisfies $\int \phi(t)\delta(t-a) dt = \phi(a)$ for continuous ϕ near a . Interpret equations between impulses and through their jump conditions; use right-hand values at jumps. Write $[v]_a = v(a^+) - v(a^-)$ for a jump.

A leaky system receives a sequence of equal positive impulses:

$$y' + \kappa y = I \sum_{n=1}^{\infty} \delta(t - nT), \quad y(0) = 0, \quad I, \kappa, T > 0.$$

The first impulse is at T , not at zero. Use right-hand response values at each impulse.

Tasks

1. Derive $Y(s)$ by a geometric series and give the locally finite time-domain sum.
2. For $x_n = y(nT^+)$ and $x_0 = 0$, derive and solve the recurrence for post-impulse values.
3. Find the attracting periodic profile and the exact transient error. State its peak, pre-impulse trough and cycle mean, distinguishing a left limit from an attained value.
4. For $I = \kappa = 1$, classify all T for which the steady peak is at most 2 while the steady cycle mean is at least 1. State whether $y(t)$ has a single long-time limit.

Question 10 – Solution

Strategy. Each impulse adds I to the state, while the gap between impulses multiplies it by an exponential. The cycle map determines the limiting profile.

Step 1: Sum the transforms and invert. For $s > 0$, the impulse series converges geometrically:

$$(s + \kappa)Y = I \sum_{n=1}^{\infty} e^{-nTs} = \frac{Ie^{-Ts}}{1 - e^{-Ts}}, \quad Y = \frac{Ie^{-Ts}}{(s + \kappa)(1 - e^{-Ts})}.$$

Thus

$$y(t) = I \sum_{n=1}^{\infty} H_{nT}(t) e^{-\kappa(t-nT)}.$$

Only finitely many terms are active at each time. The formula gives zero before T , jumps of I , and $y' + \kappa y = 0$ between impulses.

Step 2: Solve the post-impulse recurrence. Put $q = e^{-\kappa T} \in (0, 1)$. Decay followed by a jump gives

$$x_n = qx_{n-1} + I, \quad x_n = \frac{I(1 - q^n)}{1 - q}, \quad n \geq 0.$$

The steady post-impulse level is $P = I/(1 - q)$.

Step 3: Construct the profile and compare. For $t = nT + \tau$, $0 \leq \tau < T$, define $y_{\text{per}}(t) = Pe^{-\kappa\tau}$. Then $y(t) = x_n e^{-\kappa\tau}$, including $n = 0$, and

$$y_{\text{per}}(t) - y(t) = Pe^{-\kappa t}.$$

The profile's peak P is attained just after each impulse. Its pre-impulse trough is the left limit qP ; under the right-hand convention this infimum is not attained within a cycle. Its mean is

$$\frac{1}{T} \int_0^T Pe^{-\kappa\tau} d\tau = \frac{I}{\kappa T}.$$

The jump $P - qP = I$ is preserved in the limit, not smoothed away.

Step 4: Design a feasible spacing and interpret. With $I = \kappa = 1$, the peak constraint gives $1/(1 - e^{-T}) \leq 2$, equivalent to $T \geq \ln 2$. The mean constraint gives $1/T \geq 1$, equivalent to $T \leq 1$. Therefore

$$\ln 2 \leq T \leq 1.$$

There is no single long-time limit: samples at nT tend to P , while samples at $nT + T/2$ tend to the different value $Pe^{-\kappa T/2}$. Both responses have exact real transform domain $s > 0$ because their persistent positive cycles prevent convergence at or below zero. The plot uses $I = \kappa = T = 1$ and marks the jump endpoints explicitly.

