

Question 1

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A first-order system starts above zero and receives a finite pulse:

$$y' + 2y = 3[H_1(t) - H_3(t)], \quad y(0) = 1, \quad t \geq 0.$$

Tasks

1. Find $Y(s)$ and invert it in step-function form, retaining the initial response.
2. Write the solution on the three time intervals and verify the equation and matching values.
3. Find the jumps in y' at both switches. Explain whether y jumps.
4. Locate the global maximum on $[0, \infty)$, prove it is global, and state the exact real transform domain.

Question 1 – Solution

Strategy. Keep the initial response separate from the two delayed step responses, then compare the three monotonic pieces.

Step 1: Transform and invert. The initial term gives $(s+2)Y - 1 = 3(e^{-s} - e^{-3s})/s$, hence

$$Y = \frac{1}{s+2} + \frac{3(e^{-s} - e^{-3s})}{s(s+2)}, \quad s > 0.$$

Since $\mathcal{L}^{-1}\{1/[s(s+2)]\} = (1 - e^{-2t})/2$,

$$y = e^{-2t} + \frac{3}{2}H_1(t)(1 - e^{-2(t-1)}) - \frac{3}{2}H_3(t)(1 - e^{-2(t-3)}).$$

Step 2: Match the pieces. Set $M = \frac{3}{2} + (e^{-2} - \frac{3}{2})e^{-4}$. The same solution is

$$y(t) = \begin{cases} e^{-2t}, & 0 \leq t < 1, \\ \frac{3}{2} + (e^{-2} - \frac{3}{2})e^{-2(t-1)}, & 1 \leq t < 3, \\ Me^{-2(t-3)}, & t \geq 3. \end{cases}$$

The matching values are $y(1) = e^{-2}$ and $y(3) = M$. Each piece satisfies $y' + 2y = 0, 3, 0$, respectively, and $y(0) = 1$. These matching conditions also give uniqueness interval by interval.

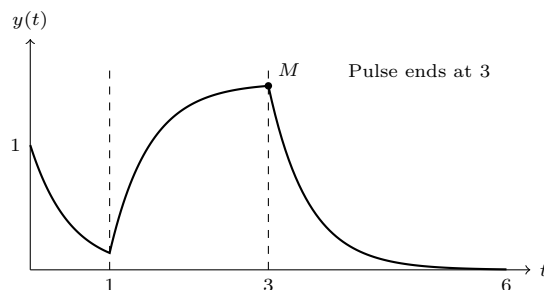
Step 3: Identify the corners. Writing $[v]_a = v(a+) - v(a-)$, the equation and continuity imply

$$[y']_1 = 3, \quad [y']_3 = -3, \quad [y]_1 = [y]_3 = 0.$$

The step factors multiply responses that vanish at their activation times, so no jump in y is introduced.

Step 4: Compare all possible maxima. The solution decreases on $[0, 1]$, increases on $[1, 3]$ toward $3/2$, then decreases to zero. Moreover $M > \frac{3}{2} - \frac{3}{2}e^{-4} > 1$, since $e^4 > 3$. Thus the unique global maximum is

$y(3) = M$. Its nonzero tail is exactly a positive multiple of e^{-2t} , so the exact real transform domain is $s > -2$; the displayed formula extends removably across $s = 0$.



Question 2

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A ramp is switched on without resetting its clock, then switched off:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} t, & 1 \leq t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

A proposed input transform is $(e^{-s} - e^{-3s})/s^2$.

Tasks

1. Explain the error in the proposed transform and derive the correct $F(s)$.
2. Find $Y(s)$ and invert it, rewriting each active input in its own delayed time.
3. Give the piecewise solution, its maximum, and the two jumps in y' .
4. Compute $\int_0^\infty y(t) dt$ directly and from the differential equation. Explain the apparent singularities in the transform formula.

Question 2 – Solution

Strategy. A window does not reset the ramp's clock. At each endpoint rewrite t as delayed time plus the endpoint.

Step 1: Shift the whole input. The input is $H_1(t)[(t-1)+1] - H_3(t)[(t-3)+3]$, so

$$F = e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right) - e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right).$$

The proposed expression instead transforms $H_1(t)(t-1) - H_3(t)(t-3)$, which has a constant tail of 2 after 3. It describes neither the given ramp window nor its shutoff.

Step 2: Invert the response. The zero datum gives $Y = F/(s+1)$. For a delayed input $\tau + c$, the zero-start response solves $r'_c + r_c = \tau + c$, and is

$$r_c(\tau) = \tau + c - 1 + (1 - c)e^{-\tau}.$$

Its transform is $(1/s^2 + c/s)/(s+1)$, as direct transformation confirms. In particular $r_1 = \tau$ and $r_3 = \tau + 2 - 2e^{-\tau}$. Thus

$$y = H_1(t)(t-1) - H_3(t)[(t-3) + 2 - 2e^{-(t-3)}].$$

Step 3: Read the intervals and switches. Simplifying after the second switch gives

$$y(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 3, \\ 2e^{-(t-3)}, & t \geq 3. \end{cases}$$

The pieces meet at 0 and 2, and satisfy $y' + y = 0, t, 0$. Their slopes are $0, 1, -2e^{-(t-3)}$, so $[y']_1 = 1$ and $[y']_3 = -3$. The unique global maximum is $y(3) = 2$, by increase during the window and strict decay afterward.

Step 4: Check the total response. Directly,

$$\int_0^\infty y \, dt = \int_1^3 (t-1) \, dt + \int_3^\infty 2e^{-(t-3)} \, dt = 2 + 2 = 4.$$

Integrating the equation gives the same answer because $y(\infty) - y(0) = 0$ and $\int_1^3 t \, dt = 4$. The compact input has a transform for every real s , with its apparent $s = 0$ poles canceling. The output's nonzero exponential tail gives exact domain $s > -1$, and the removable value $Y(0) = 4$ agrees with the area.

Question 3

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An undamped oscillator receives a rectangular force of selectable duration:

$$y'' + \omega^2 y = F[1 - H_T(t)], \quad y(0) = y'(0) = 0, \quad F > 0, \omega > 0, T > 0.$$

Use the energy $E = (y'^2 + \omega^2 y^2)/2$.

Tasks

1. Derive $Y(s)$ and the delayed step-response formula for y .
2. Find the displacement and velocity at shutoff, and the residual oscillation amplitude.
3. Classify every duration T for which the system remains at rest after shutoff; explain why $y(T) = 0$ alone is generally insufficient.
4. Compute the residual energy and classify the exact real transform domain, including the rest-producing durations.

Question 3 – Solution

Strategy. Removing the force subtracts a delayed step response. The two surviving state coordinates determine whether motion persists.

Step 1: Transform and invert. For $s > 0$,

$$Y = \frac{F(1 - e^{-Ts})}{s(s^2 + \omega^2)}, \quad \boxed{y = \frac{F}{\omega^2} [1 - \cos \omega t - H_T(t)(1 - \cos \omega(t - T))].}$$

Before T this has zero initial data and satisfies the forced equation. The subtracted response and its first derivative vanish at activation, so y, y' match at T ; after T the equation is homogeneous.

Step 2: Recover the shutoff state and amplitude. The matching state is

$$y(T) = \frac{F}{\omega^2}(1 - \cos \omega T), \quad y'(T) = \frac{F}{\omega} \sin \omega T.$$

For $t \geq T$, subtraction and the cosine difference identity give

$$y(t) = \frac{2F}{\omega^2} \sin(\omega T/2) \sin(\omega(t - T/2)).$$

Its amplitude is therefore $\boxed{R = 2F|\sin(\omega T/2)|/\omega^2}.$

Step 3: Classify complete return to rest. A homogeneous oscillator remains zero exactly when both state coordinates are zero. Here that happens precisely for

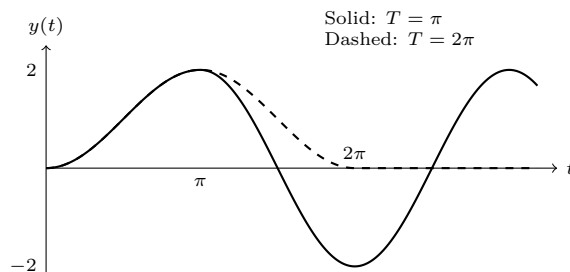
$$\boxed{T = \frac{2\pi n}{\omega}, \quad n = 1, 2, \dots}$$

In this particular pulse family, $y(T) = 0$ forces $\cos \omega T = 1$ and hence also $y'(T) = 0$. For a general forcing history, zero displacement alone is insufficient: a nonzero velocity launches a new oscillation.

Step 4: Compute energy and convergence. After shutoff,

$$\boxed{E = \frac{1}{2}\omega^2 R^2 = \frac{2F^2}{\omega^2} \sin^2(\omega T/2).}$$

If $R > 0$, the oscillatory tail has an ordinary Laplace transform exactly for $s > 0$: at zero its primitive does not converge, and negative s gives growing integrals over fixed sign intervals. If $R = 0$, the output has compact support, so its transform exists for every real s . Apparent poles in the displayed formula then cancel. The graph compares $F = \omega = 1$ at two durations.



Question 4

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A critically damped system receives a unit pulse:

$$y'' + 2y' + y = 1 - H_2(t), \quad y(0) = y'(0) = 0.$$

The force is already zero for $t > 2$.

Tasks

1. Find $Y(s)$ and derive the unit-step response needed to invert it.
2. Write the post-shutoff response explicitly and verify continuity of y and y' at 2.
3. Find the unique global peak time and height. Prove whether the peak occurs before or after shutoff.
4. Determine the jump in acceleration at shutoff and the total area under y . State the exact real transform domain.

Question 4 – Solution

Strategy. A decaying stable system can continue moving upward after its input stops. Differentiate the actual tail to locate the peak.

Step 1: Transform the pulse. The zero data give

$$Y = \frac{1 - e^{-2s}}{s(s+1)^2}, \quad \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Thus $r(t) = 1 - (1+t)e^{-t}$ is the unit-step response, with $r(0) = r'(0) = 0$ and $r'' + 2r' + r = 1$. The solution is $y = r(t) - H_2(t)r(t-2)$.

Step 2: Simplify the tail and match. For $t \geq 2$,

$$y(t) = e^{-t}[(e^2 - 1)t - (e^2 + 1)].$$

At 2 this gives $1 - 3e^{-2} = r(2)$; its derivative there is $2e^{-2} = r'(2)$. Hence y, y' are continuous, with positive velocity even after the forcing is removed.

Step 3: Locate and certify the peak. For $0 < t < 2$, $y' = te^{-t} > 0$. On the tail,

$$y' = e^{-t}[2e^2 - (e^2 - 1)t].$$

This derivative changes sign exactly once, at

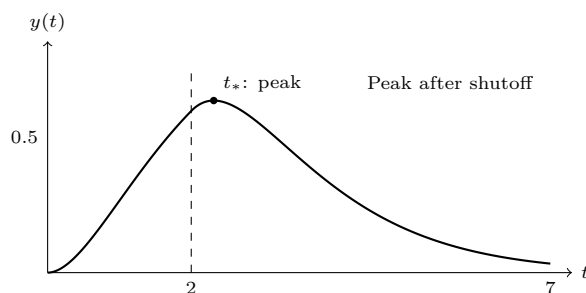
$$t_* = \frac{2e^2}{e^2 - 1} > 2, \quad y(t_*) = (e^2 - 1)e^{-t_*}.$$

Thus the peak is unique and global. The tail remains positive because its linear factor is already positive at 2 and has positive slope, while $y(t) \rightarrow 0$. Damping does not imply an immediate drop in displacement at shutoff.

Step 4: Check the switch, area and domain. Since y, y' are continuous, we have $[y'']_2 = -1$. Both y, y' start at zero and tend to zero. Integrating the equation over the half-line, including the matching boundary terms, yields

$$\int_0^\infty y \, dt = \int_0^2 1 \, dt = 2.$$

The nonzero te^{-t} tail gives exact domain $s > -1$. The $s = 0$ singularity in the transform expression is removable, with $Y(0) = 2$.



Question 5

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A resonant input is windowed without restarting its phase:

$$y'' + y = \sin t [H_a(t) - H_b(t)], \quad y(0) = y'(0) = 0, \quad a = \frac{\pi}{2}, \quad b = \frac{3\pi}{2}.$$

Tasks

1. Rewrite the forcing in delayed coordinates and derive its transform. Pay particular attention to the sign at the second switch.
2. Find and invert $Y(s)$, deriving the required repeated-quadratic transform pair.
3. Find the exact motion after b and its displacement, velocity and energy at shutoff.
4. Determine the acceleration jumps and the exact real transform domain. Explain why a zero displacement at shutoff does not mean rest.

Question 5 – Solution

Strategy. The two endpoints have opposite sine phases. Rewriting both phases first prevents a sign error in the delayed inverse.

Step 1: Shift the phases. For delayed time τ , $\sin(\tau + a) = \cos \tau$ and $\sin(\tau + b) = -\cos \tau$. Therefore

$$f = H_a(t) \cos(t - a) + H_b(t) \cos(t - b), \quad F = \frac{s(e^{-as} + e^{-bs})}{s^2 + 1}.$$

The plus sign at b cancels the continuing negative-phase input; it does not add a second physical forcing window.

Step 2: Derive and invert the response pair. The zero-data equation gives

$$Y = \frac{s(e^{-as} + e^{-bs})}{(s^2 + 1)^2}, \quad s > 0.$$

Differentiating $\mathcal{L}\{\sin t\} = 1/(s^2 + 1)$ with respect to s gives $\mathcal{L}\{t \sin t/2\} = s/(s^2 + 1)^2$. With $q(\tau) = \tau \sin \tau/2$,

$$\boxed{y = H_a(t)q(t - a) + H_b(t)q(t - b)}.$$

Here $q(0) = q'(0) = 0$ and $q'' + q = \cos \tau$, directly verifying the equation, the zero initial data and both matching conditions.

Step 3: Evaluate the residual motion. For $t = b + v$, $v \geq 0$, the two ages are $v + \pi$ and v . Hence

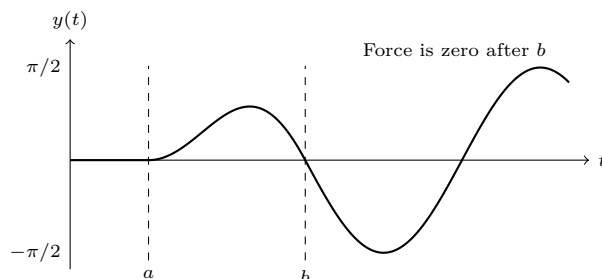
$$y(b + v) = q(v + \pi) + q(v) = \boxed{-\frac{\pi}{2} \sin v}.$$

In particular $y(b) = 0$, $y'(b) = -\pi/2$, and the post-shutoff energy is

$$\boxed{E = \frac{1}{2}(y'^2 + y^2) = \frac{\pi^2}{8}}.$$

The pulse ends precisely at a zero crossing with nonzero velocity, so the system continues oscillating with amplitude $\pi/2$.

Step 4: Inspect the switches and convergence. The input jumps from 0 to 1 at a , and from -1 to 0 at b . Continuity of y gives $\boxed{[y'']_a = [y'']_b = 1}$. There are no jumps in y or y' . The nonzero sinusoidal tail has exact real transform domain $\boxed{s > 0}$: its undamped primitive fails to converge at zero, and exponential growth of fixed-sign tail integrals rules out negative s .



Question 6

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Choose two constant force levels to bring an oscillator exactly to rest:

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0, \quad f(t) = \begin{cases} A, & 0 \leq t < a, \\ B, & a \leq t < T, \\ 0, & t \geq T, \end{cases} \quad a = \frac{\pi}{4}, \quad T = \frac{\pi}{2}.$$

Require $y(t) = 0$ for every $t \geq T$. The levels A, B may have either sign.

Tasks

1. Express f using steps and derive $Y(s)$, including the initial response.
2. Invert $Y(s)$ and derive the two linear equations that the terminal conditions impose on A, B .
3. Solve for both force levels and prove that they give a unique motion which remains at rest after T .
4. Decide whether nonnegative levels could meet the target. Describe the smoothness at both switches and the exact real transform domain of the designed output.

Question 6 – Solution

Strategy. Exact rest requires two terminal state conditions. Use the step response to convert them into two equations for the two force levels.

Step 1: Transform with the initial displacement. The level changes give $f = A + (B - A)H_a(t) - BH_T(t)$, so

$$(s^2 + 1)Y - s = \frac{A + (B - A)e^{-as} - Be^{-Ts}}{s}.$$

Writing $r(t) = 1 - \cos t$, the inverse is

$$y = \cos t + Ar(t) + (B - A)H_a(t)r(t - a) - BH_T(t)r(t - T).$$

The unforced term $\cos t$ supplies the initial data. Since $r(0) = r'(0) = 0$, each added response preserves y, y' at activation, and $r'' + r = 1$ verifies the forcing on each open interval.

Step 2: Impose both terminal conditions. Set $c = \cos a = \sin a = 1/\sqrt{2}$. At $T = \pi/2$ the last response and its derivative still vanish. The terminal values are

$$y(T) = cA + (1 - c)B, \quad y'(T) = -1 + (1 - c)A + cB.$$

Thus rest requires

$$cA + (1 - c)B = 0, \quad (1 - c)A + cB = 1.$$

Step 3: Solve and prove complete rest. The determinant is $c^2 - (1 - c)^2 = 2c - 1 = \sqrt{2} - 1 > 0$. Consequently

$$A = -\frac{1 - c}{2c - 1} = \boxed{-\frac{1}{\sqrt{2}}}, \quad B = \frac{c}{2c - 1} = \boxed{1 + \frac{1}{\sqrt{2}}}.$$

These are the unique levels in the stated two-stage family. The formula satisfies the original IVP and has zero displacement and velocity at T . For $t > T$ the forcing is zero, so uniqueness of the homogeneous oscillator with these terminal data implies $y(t) = 0$ there. Matching gives a unique solution on the entire half-line.

Step 4: Check feasibility and regularity. Both weights in $cA + (1 - c)B = 0$ are positive. If $A, B \geq 0$, that equation forces $A = B = 0$, contradicting the velocity equation. A negative first level is therefore necessary within this family. The motion is C^1 , while the acceleration jumps are

$$[y'']_a = B - A = 1 + \sqrt{2}, \quad [y'']_T = -B = -1 - \frac{1}{\sqrt{2}}.$$

After T the output is identically zero, so its transform exists for $\boxed{\text{every real } s}$, despite the apparent denominators in Y .

Question 7

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An unstable first-order system is to be stopped by a two-stage input:

$$y' - y = f_B(t), \quad y(0) = 1, \quad f_B(t) = \begin{cases} -1, & 0 \leq t < 1, \\ B, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

Choose $B = B_*$ so that $y(t) = 0$ for all $t \geq 2$. A realized level may instead be $B_* + \delta$, where δ is a constant error.

Tasks

1. Find $Y(s)$ and invert it using the unstable system's step response.
2. Determine B_* and the resulting piecewise output. Verify the stopping condition.
3. Derive the exact output error for $t \geq 2$ and the sharp bound on $|\delta|$ that ensures $|y(t)| \leq 0.01$ for every $2 \leq t \leq 5$.
4. Compare the exact real transform domains for perfect and imperfect levels. Explain why exact cancellation does not make the system stable.

Question 7 – Solution

Strategy. A finite input can cancel one unstable trajectory exactly. Keep the cancellation coefficient explicit to measure its sensitivity.

Step 1: Transform and invert. Here $f_B = -1 + (B + 1)H_1(t) - BH_2(t)$ and

$$Y = \frac{1 + [-1 + (B + 1)e^{-s} - Be^{-2s}]/s}{s - 1}, \quad s > 1.$$

The zero-start response to a unit step for $y' - y = 1$ is $r(t) = e^t - 1$. Adding the initial response gives

$$y = e^t - r(t) + (B + 1)H_1(t)r(t - 1) - BH_2(t)r(t - 2).$$

This formula satisfies $y(0) = 1$ and is continuous at both switches.

Step 2: Select the exact cancellation. Before 1, $y = 1$. On $[1, 2]$, $y = (B + 1)e^{t-1} - B$, so $y(2) = e + B(e - 1)$. Consequently

$$B_* = -\frac{e}{e - 1}, \quad y_*(t) = \begin{cases} 1, & 0 \leq t < 1, \\ (e - e^{t-1})/(e - 1), & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

Direct substitution gives $y'_* - y_* = -1, B_*, 0$ on the three intervals, and the values match at 1 and 2. The zero terminal state remains zero under the homogeneous equation after 2.

Step 3: Quantify the error sharply. The error is zero before 1, then equals $\delta(e^{t-1} - 1)$ until 2. After 2 the homogeneous equation amplifies its terminal value:

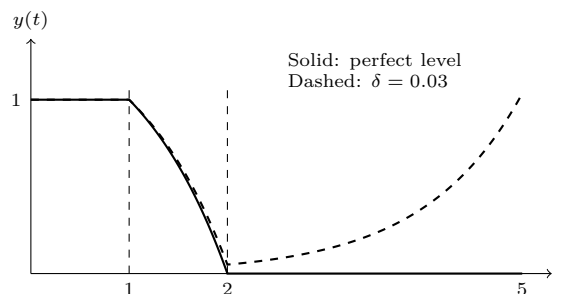
$$y(t) - y_*(t) = \delta(e - 1)e^{t-2}, \quad t \geq 2.$$

Its magnitude increases on $[2, 5]$ when $\delta \neq 0$. Thus the required necessary and sufficient tolerance is

$$|\delta| \leq \frac{0.01 e^{-3}}{e - 1}.$$

Equality attains the limit at $t = 5$.

Step 4: Distinguish trajectory cancellation from stability. The perfect output has compact support and a transform for every real s . For any $\delta \neq 0$, a nonzero e^t tail remains and the exact real domain is $s > 1$. At the perfect setting the apparent singularity at $s = 1$ is removable, as is the $s = 0$ expression after cancellations. Small level errors eventually grow without bound; selecting one compact response does not change the unstable homogeneous equation.



Question 8

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

An initially unexcited system is driven by one unknown positive step:

$$y' + 2y = AH_a(t), \quad y(0) = 0, \quad A > 0, \quad a > 0.$$

Exact measurements give $y(1) = 0$, $y(2) = 1$, and $y(3) = 2$. A fourth proposed measurement is $y(4) = 3$.

Tasks

1. Derive the transformed and time-domain response for arbitrary A, a .
2. Use positivity and the measurements to locate the switch interval, then determine A and a exactly.
3. Verify that the recovered parameters satisfy all three original measurements and are unique in this one-step family.
4. Test the fourth measurement and find the limiting response. Explain the scope of the identification.

Question 8 – Solution

Strategy. The zero measurement locates the delay. Once the input is on, equally spaced samples obey a simple exponential approach to the same level.

Step 1: Solve the parameterized IVP. For $s > 0$,

$$Y = \frac{Ae^{-as}}{s(s+2)}, \quad \boxed{y(t) = \frac{A}{2}H_a(t)(1 - e^{-2(t-a)}).$$

The response is zero through a , positive for $t > a$, and continuous at a . Its derivative jumps by A , as the equation requires.

Step 2: Locate and recover the switch. Since $A > 0$, the first two measurements imply $1 \leq a < 2$. Put $r = e^{-2}$ and $C = A/2$. On each full unit interval after activation,

$$y(t+1) = ry(t) + C(1-r).$$

Thus $2 = r + C(1-r)$, and

$$\boxed{A = \frac{2(2-r)}{1-r}}.$$

Using $y(2) = 1$ gives $e^{-2(2-a)} = 1 - 1/C = 1/(2-r)$, hence

$$\boxed{a = 2 - \frac{1}{2} \ln(2 - e^{-2})}.$$

Step 3: Verify admissibility and uniqueness. Because $1 < 2 - r < 2 < e^2$, one has $0 < \ln(2 - r) < 2$, so $1 < a < 2$. Also $C = (2 - r)/(1 - r) > 0$. The recovered values give $y(1) = 0$ and

$$y(2) = C \left(1 - \frac{1}{2-r} \right) = 1, \quad y(3) = r + C(1-r) = 2.$$

The sample recurrence fixes C uniquely. Then the exponential expression for $y(2)$ is strictly monotone in a , fixing the delay uniquely. Direct substitution into the response verifies the IVP on both sides of the switch.

Step 4: Test the extra datum and interpret. The same recurrence predicts

$$\boxed{y(4) = 2r + C(1-r) = 2 + e^{-2} < 3.}$$

The fourth datum is inconsistent with the stated model. The limiting level is $\boxed{C = (2 - e^{-2})/(1 - e^{-2})}$, approached from below after activation. Its positive constant tail gives exact real transform domain $s > 0$. Uniqueness here concerns the specified single positive step with known system coefficient; finitely many samples do not identify an arbitrary forcing history.

Question 9

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

Consider two qualitatively different input changes:

$$y'' + 3y' + 2y = H_1(t) + (t - 2)H_2(t), \quad y(0) = y'(0) = 0.$$

One claim says that the jump at 1 makes y jump; another says that because the ramp is continuous at 2, the solution must be smooth to every order there.

Tasks

1. Find $Y(s)$ and invert it explicitly using the step and ramp responses.
2. Verify the original IVP and determine the jumps in y, y', y'', y''' at 1 and 2.
3. Derive the leading small- τ behavior of each newly activated response. Relate the orders of onset to the jump calculations.
4. Evaluate both claims and explain whether a different assigned value of $H_a(a)$ changes the response.

Question 9 – Solution

Strategy. A second-order system smooths a step more than the input itself, but only by a finite number of derivatives. Check one-sided limits directly.

Step 1: Transform and invert. With zero data,

$$Y = \frac{e^{-s}}{s(s+1)(s+2)} + \frac{e^{-2s}}{s^2(s+1)(s+2)}.$$

Partial fractions give the two zero-start responses

$$r(\tau) = \frac{1}{2} - e^{-\tau} + \frac{1}{2}e^{-2\tau}, \quad p(\tau) = \frac{\tau}{2} - \frac{3}{4} + e^{-\tau} - \frac{1}{4}e^{-2\tau},$$

so $y = H_1(t)r(t-1) + H_2(t)p(t-2)$. Their transforms are the two unshifted rational factors displayed above.

Step 2: Verify the equation and all jumps. Direct differentiation gives $r'' + 3r' + 2r = 1$ and $p'' + 3p' + 2p = \tau$. Moreover $r(0) = r'(0) = p(0) = p'(0) = 0$, verifying the initial data and C^1 matching. For the next derivatives,

$$r''(0) = 1, \quad r'''(0) = -3, \quad p''(0) = 0, \quad p'''(0) = 1.$$

The previously active response is smooth at the next switch. Thus

switch	$[y]$	$[y']$	$[y'']$	$[y''']$
1	0	0	1	-3
2	0	0	0	1

These are jumps of one-sided ordinary derivatives, not assigned values at the switch.

Step 3: Relate onset orders to regularity. Taylor expansion of the newly added responses gives

$$r(\tau) = \frac{1}{2}\tau^2 - \frac{1}{2}\tau^3 + O(\tau^4), \quad p(\tau) = \frac{1}{6}\tau^3 + O(\tau^4), \quad \tau \downarrow 0.$$

Joining the first to zero preserves one derivative but creates a second- derivative jump. Joining the second to zero preserves two derivatives but creates a third-derivative jump. These leading coefficients reproduce the jump sizes above.

Step 4: Resolve both claims and the convention. The first claim is false: y and y' stay continuous at 1, while y'' jumps. The second is also false: y is C^2 near 2 but not C^3 there. Changing an input at one isolated time does not change its Laplace integral or the solution obtained by continuous state matching. At switches the equation is interpreted through its one-sided limits, not by demanding an unavailable classical derivative. Away from them the solution is smooth. The growing linear tail gives exact real transform domain $s > 0$.

Question 10

Let $H_a(t) = 0$ for $t < a$ and $H_a(t) = 1$ for $t \geq a$. Use one-sided Laplace transforms. Solutions are continuous (and have continuous first derivative for second-order equations); satisfy the equation away from switches and use one-sided derivatives there. Isolated input values do not change the solution.

A unit input is switched on and off repeatedly:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} 1, & 2k \leq t < 2k+1, \\ 0, & 2k+1 \leq t < 2k+2, \end{cases} \quad k = 0, 1, 2, \dots$$

Use only step functions and a geometric series; no periodic-transform formula is needed.

Tasks

1. Represent f by a locally finite sum of steps, derive $F(s)$ for $s > 0$, and obtain $Y(s)$ and a delayed-response series.
2. For $x_k = y(2k)$, derive and solve a recurrence for the samples.
3. Find the attracting two-periodic response explicitly on one cycle and prove the exact transient error for all $t \geq 0$.
4. Find its cycle minimum, maximum and mean. Does $y(t)$ have a limit? Find the earliest $T \geq 0$ for which the error from the periodic response is at most 0.01 for every $t \geq T$.

Question 10 – Solution

Strategy. The step series gives the transient solution, while the cycle map identifies a periodic state and its exact attraction rate.

Step 1: Transform the repeated switches. At each fixed time only finitely many terms are active:

$$f = \sum_{k=0}^{\infty} (H_{2k}(t) - H_{2k+1}(t)), \quad F = \frac{1 - e^{-s}}{s} \sum_{k=0}^{\infty} e^{-2ks} = \frac{1}{s(1 + e^{-s})}, \quad s > 0.$$

Integration of the nonnegative disjoint pulses justifies the sum. With $r(t) = 1 - e^{-t}$ for $t \geq 0$,

$$Y = \frac{1}{s(s+1)(1+e^{-s})}, \quad y = \sum_{k=0}^{\infty} [H_{2k}(t)r(t-2k) - H_{2k+1}(t)r(t-2k-1)].$$

This time series is locally finite; each activated response starts at zero and satisfies the required first-order equation on its active intervals.

Step 2: Solve the cycle recurrence. During the on-half, $y(2k+1) = 1 + (x_k - 1)e^{-1}$. During the off-half this value is multiplied by e^{-1} , so

$$x_{k+1} = e^{-2}x_k + e^{-1}(1 - e^{-1}), \quad \boxed{x_k = m(1 - e^{-2k}), \quad m = \frac{1}{e+1}},$$

using $x_0 = 0$. The cycle map has the unique fixed point m .

Step 3: Build the periodic state and exact error. Write $t = 2k + \tau$, $0 \leq \tau < 2$, and put $M = e/(e+1)$. Then

$$y_{\text{per}}(t) = \begin{cases} 1 - (1 - m)e^{-\tau}, & 0 \leq \tau < 1, \\ Me^{-(\tau-1)}, & 1 \leq \tau < 2. \end{cases}$$

The pieces match at M , and the value at the end of the cycle is m . Both solutions have the same forcing, so their difference satisfies $d' + d = 0$ with $d(0) = m$. Continuity propagates this identity across every switch:

$$\boxed{y_{\text{per}}(t) - y(t) = me^{-t}, \quad t \geq 0.}$$

Step 4: Interpret the limiting cycle and tolerance. The periodic minimum is m and maximum is M . Integrating its equation over a cycle gives $\int_0^2 y_{\text{per}} dt = 1$, so its mean is $1/2$. The even and odd integer samples of y tend to the different values m, M ; there is no single limit. The error decreases strictly, giving the sharp time

$$\boxed{T = \ln\left(\frac{100}{e+1}\right)}.$$

Both responses have exact real transform domain $s > 0$, since their positive persistent cycles prevent convergence at or below zero.

