

Question 1

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the regular IVP

$$(1+t)y' + y = 1, \quad y(0) = 0, \quad t \geq 0.$$

An answer in terms of a convergent parameter integral is acceptable for Y .

Tasks

1. Derive the differential equation satisfied by Y . Differentiate the entire transform of y' when handling ty' .
2. Solve that equation using an integrating factor and a condition at $s = \infty$ that a bounded time function must satisfy.
3. Identify the inverse and verify the original IVP. Check your parameter-integral expression directly against its forward transform.
4. State the exact real transform domain and derive the leading large- s behavior from the initial slope.

Question 1 – Solution

Strategy. Multiplication by time makes a differential equation in the transform variable. Its integration constant must be selected using transform behavior.

Step 1: Differentiate the full derivative transform. Here $\mathcal{L}\{y'\} = sY$, so $\mathcal{L}\{ty'\} = -(Y + sY')$. The transformed equation is

$$sY - Y - sY' + Y = 1/s, \quad \boxed{Y' - Y = -1/s^2}, \quad s > 0.$$

Treating $\mathcal{L}\{ty'\}$ as only $-sY'$ would introduce an erroneous extra term.

Step 2: Select the integration constant. Multiply by e^{-s} . A bounded function has $Y = O(1/s)$ as $s \rightarrow \infty$, hence $e^{-s}Y \rightarrow 0$. Integration from s to infinity gives

$$\boxed{Y(s) = e^s \int_s^\infty \frac{e^{-u}}{u^2} du.}$$

The other homogeneous term is Ce^s , incompatible with that bound unless $C = 0$. The inverse identified below is bounded, so this selection is self-consistent and verified rather than assumed without a check.

Step 3: Identify and forward-check the inverse. The original left side is $[(1+t)y]'$, giving $(1+t)y = t$ and

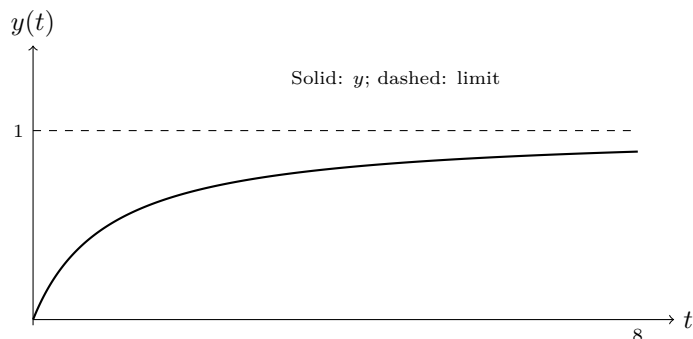
$$\boxed{y(t) = \frac{t}{1+t}.}$$

Its derivative is $(1+t)^{-2}$, so $(1+t)y' + y = 1$ and $y(0) = 0$. For $s > 0$, substitution $u = s(1+t)$ gives $\mathcal{L}\{(1+t)^{-1}\} = e^s \int_s^\infty e^{-u}/u du$. Thus the forward transform is $1/s - e^s \int_s^\infty e^{-u}/u du$, equal to the boxed parameter integral by integration by parts. The regular first-order IVP is unique.

Step 4: Check domain and onset. Since $y \rightarrow 1$ and is nonnegative, its exact domain is $\boxed{s > 0}$. At the origin $y'(0) = 1$, and the substitution $v = st$ gives

$$s^2 Y(s) = \int_0^\infty e^{-v} \frac{v}{1+v/s} dv \rightarrow 1.$$

The integrand is dominated by ve^{-v} , so dominated convergence justifies the limit. Thus $Y \sim 1/s^2$, matching the zero initial value and unit initial slope.



Question 2

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve

$$y' + 2ty = 0, \quad y(0) = 1, \quad t \geq 0.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-v^2} dv = \sqrt{\pi}/2$.

Tasks

1. Derive the first-order equation for Y and solve it using an integrating factor.
2. Select the admissible transform branch by its behavior as $s \rightarrow \infty$.
3. Identify the time solution, verify the IVP, and prove that the Gaussian-integral expression is its forward transform for every real s .
4. Evaluate $Y(0)$ and prove $1/s - 2/s^3 \leq Y(s) \leq 1/s$ for $s > 0$. Interpret the resulting initial-value limit.

Question 2 – Solution

Strategy. The variable damping gives a growing homogeneous solution in transform space; a genuine transform selects the complementary integral tail.

Step 1: Transform and apply the integrating factor. The equation becomes $sY - 1 - 2Y' = 0$, or

$$Y' - \frac{s}{2}Y = -\frac{1}{2}, \quad (e^{-s^2/4}Y)' = -\frac{1}{2}e^{-s^2/4}.$$

Step 2: Impose the transform condition at infinity. For a bounded candidate, $Y = O(1/s)$, so $e^{-s^2/4}Y \rightarrow 0$. Integrating to infinity yields

$$Y(s) = \frac{1}{2}e^{s^2/4} \int_s^\infty e^{-u^2/4} du = e^{s^2/4} \int_{s/2}^\infty e^{-v^2} dv.$$

Any additional $Ce^{s^2/4}$ violates the bound. The recovered inverse will verify boundedness and the entire transform calculation.

Step 3: Identify the inverse and its full domain. Separation in time gives $y = e^{-t^2}$, with $y' = -2te^{-t^2}$ and $y(0) = 1$. Completing the square directly gives

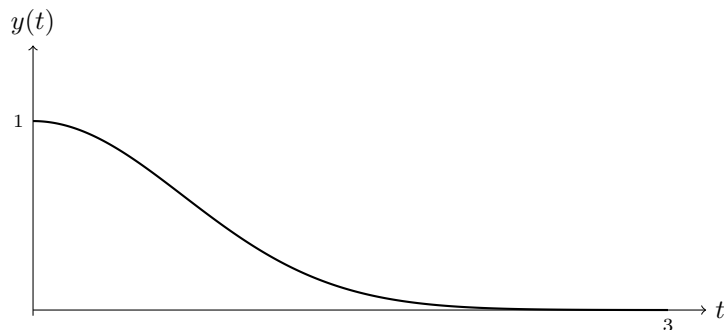
$$\int_0^\infty e^{-st-t^2} dt = e^{s^2/4} \int_{s/2}^\infty e^{-v^2} dv.$$

The Gaussian tail is integrable for every real s , so this is the exact real domain. This direct verification also justifies the selected transformed branch. Uniqueness follows from the regular first-order equation.

Step 4: Check the area and large-parameter bound. At zero, $Y(0) = \sqrt{\pi}/2$. For $t \geq 0$, the elementary inequality $e^{-x} \geq 1 - x$ gives $1 - t^2 \leq e^{-t^2} \leq 1$. Integrating with the positive kernel yields

$$\frac{1}{s} - \frac{2}{s^3} \leq Y(s) \leq \frac{1}{s}, \quad s > 0.$$

The lower bound may be negative for small s but remains valid. Multiplying by s and taking $s \rightarrow \infty$ gives $sY(s) \rightarrow 1 = y(0)$. The graph shows the actual rapidly decaying time solution.



Question 3

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Investigate the validity of the Laplace method for

$$y' - 2ty = 0, \quad y(0) = 1, \quad t \geq 0.$$

Do not assume that every global smooth IVP solution has an ordinary Laplace transform.

Tasks

1. Solve the time-domain IVP and determine whether its ordinary transform exists for any real s .
2. Formally apply the transform rules and solve the resulting equation for a candidate Y , without yet claiming it is a transform.
3. Show that every member of this formal family satisfies $Y(s) \rightarrow 0$ and $sY(s) \rightarrow 1$ as $s \rightarrow \infty$.
4. Explain why neither those limits nor solving the formal equation validates the calculation. Identify the failed hypothesis and contrast it with the sign-reversed damping equation.

Question 3 – Solution

Strategy. Check the actual time solution first. Necessary transform limits can hold for functions of s that are not transforms of the IVP solution.

Step 1: Establish nonexistence of an ordinary transform. Separation and the initial datum give $y = e^{t^2}$, which is smooth and global. For any fixed real s , however, $t^2 - st \geq t^2/2$ for sufficiently large t . Thus

$$\int_0^\infty e^{-st} y(t) dt = \int_0^\infty e^{t^2-st} dt = \infty.$$

The exact real transform domain is empty.

Step 2: Solve the formal transformed equation. Pretending the rules apply gives $sY - 1 + 2Y' = 0$, so

$$Y' + \frac{s}{2}Y = \frac{1}{2}, \quad Y_C(s) = e^{-s^2/4} \left[C + \frac{1}{2} \int_0^s e^{u^2/4} du \right].$$

This is a valid family of solutions to the differential equation in s . It is not yet a family of Laplace transforms.

Step 3: Check the deceptively correct limits. As $s \rightarrow \infty$, l'Hopital's rule gives

$$\frac{\int_0^s e^{u^2/4} du}{2e^{s^2/4}/s} \rightarrow 1,$$

because the derivative ratio is $1/(1 - 2/s^2) \rightarrow 1$. Hence the integral contribution to Y_C is asymptotic to $1/s$, while $Ce^{-s^2/4}$ decays faster. Every real C therefore satisfies

$$Y_C(s) \rightarrow 0, \quad sY_C(s) \rightarrow 1.$$

These two necessary-looking conditions do not even select a unique formal branch here.

Step 4: Identify the invalid step. The ordinary integrals defining Y , the transform of ty , and the derivative transform do not converge for any real s . The time solution is not of exponential order, and direct growth proves failure, not merely absence of a sufficient hypothesis. Applying transform identities to these divergent integrals was unjustified. The limits in Step3 cannot repair that step. In contrast, changing the sign to $y' + 2ty = 0$ gives e^{-t^2} , whose transform exists for every real s and can be verified directly. A differential equation in transform space is useful only after its relationship to convergent time integrals is established.

Question 4

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider a singular initial point:

$$ty' + 2y = e^{-t}, \quad y(0) = \frac{1}{2}.$$

Seek a solution continuously differentiable on $[0, \infty)$ and satisfying the equation also at zero.

Tasks

1. Check compatibility at zero and determine whether boundedness near zero leaves a free integration constant in the time equation.
2. Derive and solve the equation for Y , selecting a branch that tends to zero at large positive s .
3. Identify and verify the inverse, including its limiting value and derivative at zero.
4. Find the full real convergence set and $Y(0)$. Explain why a finite transform value at zero need not imply a finite derivative there.

Question 4 – Solution

Strategy. A vanishing leading coefficient imposes compatibility. Boundedness at the singular endpoint can replace a free constant that would exist away from it.

Step 1: Inspect the singular endpoint. At zero, the equation requires $2y(0) = 1$, so the given datum is compatible. For $t > 0$, multiplying by t gives $(t^2y)' = te^{-t}$. Its general integral has an additive constant in t^2y . Boundedness of y forces that constant to be zero, hence

$$y(t) = \frac{1 - (1+t)e^{-t}}{t^2} \quad (t > 0).$$

This proves uniqueness among bounded solutions near zero, without invoking a regular-IVP theorem at a singular point.

Step 2: Solve in transform space. The full rule $\mathcal{L}\{ty'\} = -Y - sY'$ gives $Y - sY' = 1/(s+1)$. Therefore

$$(Y/s)' = -\frac{1}{s^2(s+1)}, \quad Y = s \int_s^\infty \frac{du}{u^2(u+1)} = \boxed{1 - s \ln \frac{s+1}{s}}, \quad s > 0.$$

The omitted homogeneous term is Cs and cannot tend to zero. The inverse below verifies boundedness and transform existence.

Step 3: Verify the endpoint and forward transform. A nonsingular representation is $y(t) = \int_0^1 ue^{-ut} du$, including at zero. Thus $y(0) = 1/2$, $y'(0) = -1/3$, and differentiation verifies the original equation (or use $(t^2y)' = te^{-t}$ for $t > 0$ and compatibility at zero). For $s > 0$, integrating first in time gives

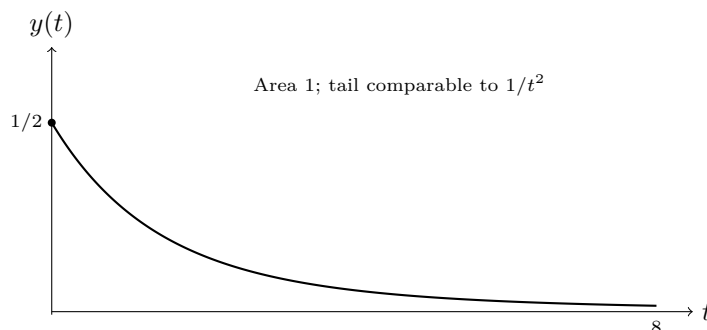
$$Y(s) = \int_0^1 \frac{u}{s+u} du = 1 - s \ln \frac{s+1}{s}.$$

The finite absolute double integral justifies interchange.

Step 4: Check the boundary rather than just the formula. The function is positive and $y(t) \sim 1/t^2$. Therefore the exact real convergence set is $\boxed{[0, \infty)}$, with divergence for negative s . At zero the same

nonnegative double integral gives $\boxed{Y(0) = \int_0^1 1 du = 1}$. But $ty(t) \sim 1/t$, so its first moment diverges.

Indeed $Y'(s) = -\ln[(s+1)/s] + 1/(s+1) \rightarrow -\infty$ as $s \downarrow 0$. A finite boundary area does not imply a finite boundary moment.



Question 5

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Seek a twice continuously differentiable solution of

$$ty'' + 2y' + ty = 0, \quad y(0) = 1,$$

including at $t = 0$. When checking the transform boundary, you may use the estimate $|\int_R^M e^{-st} \sin t/t dt| \leq 2/R$ for $M > R > 0$, uniformly for $s \geq 0$, obtained by integration by parts.

Tasks

1. Determine the initial slope forced by the equation at zero, then derive the differential equation for Y .
2. Solve for $Y(s)$ on $s > 0$ using its decay at infinity and identify the inverse.
3. Use $w(t) = ty(t)$ to verify the equation and prove uniqueness in the stated smooth class. Explain why the initial slope disappeared from the transformed equation.
4. Find the exact real transform domain, including the ordinary boundary value at zero. Use the supplied uniform tail estimate to justify passage to that value.

Question 5 – Solution

Strategy. The singular equation fixes a missing initial slope. Transform differentiation removes that slope algebraically, so compatibility must be checked separately.

Step 1: Recover compatibility and transform. At zero the equation requires $y'(0) = 0$. Since

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] = -2sY - s^2Y' + 1,$$

adding $2\mathcal{L}\{y'\}$ and $\mathcal{L}\{ty\}$ gives

$$(s^2 + 1)Y' = -1.$$

Step 2: Select and identify the inverse. The condition $Y \rightarrow 0$ as $s \rightarrow \infty$ gives

$$Y(s) = \int_s^\infty \frac{du}{1+u^2} = \arctan(1/s), \quad s > 0.$$

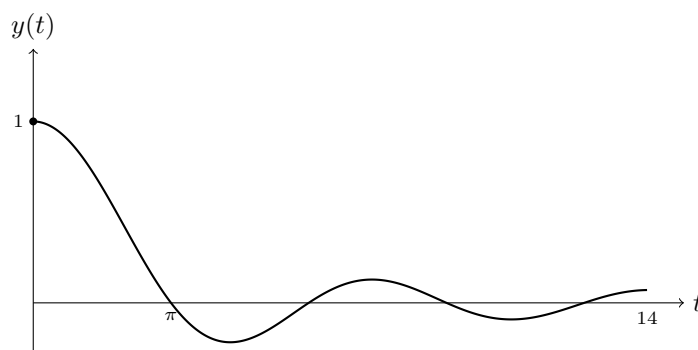
The time function $y(t) = \sin t/t$ ($t > 0$), $y(0) = 1$ has representation $y = \int_0^1 \cos(ut) du$. Transforming this integral for $s > 0$ yields $\int_0^1 s/(s^2 + u^2) du = \arctan(1/s)$. Absolute double integrability justifies the interchange.

Step 3: Verify smoothness and uniqueness. The expansion $y = 1 - t^2/6 + O(t^4)$ supplies a smooth extension with slope zero. Set $w = ty$. Then $w'' + w = ty'' + 2y' + ty = 0$, with $w(0) = 0$, $w'(0) = 1$. The regular oscillator IVP has the unique solution $w = \sin t$, proving the original equation and uniqueness of its smooth extension. The term $-y'(0)$ is constant in s , so it disappears when differentiating the transform of y'' ; arbitrary initial slopes are nevertheless not admissible.

Step 4: Establish the boundary value. The supplied estimate makes the tails uniformly small for $s \geq 0$. On each fixed finite interval the damped integrals converge to the undamped one as $s \downarrow 0$. Taking the interval long first and then taking $s \downarrow 0$ proves

$$Y(0) = \int_0^\infty \frac{\sin t}{t} dt = \pi/2.$$

The boundary integral is conditional: intervals where $|\sin t| \geq 1/2$ give a divergent harmonic lower bound for its absolute integral. For $s < 0$, fixed-length positive lobe intervals have weighted integrals growing like $e^{|s|t}/t$, violating the Cauchy criterion. The exact real domain is $[0, \infty)$.



Question 6

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Solve the regular second-order IVP

$$(1+t)y'' + y' = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

A convergent parameter integral may be retained in the formula for Y .

Tasks

1. Derive the differential equation for Y , including all initial contributions.
2. Solve it with an integrating factor and explain the condition that removes its homogeneous term.
3. Identify and verify the time solution, then check the parameter-integral transform directly.
4. Prove $0 < Y(s) < 1/s^2$ for $s > 0$ and determine the exact real convergence interval. Explain why a growing time solution can still be transformable.

Question 6 – Solution

Strategy. Keep the initial derivative term from the unmultiplied second derivative. The coefficient $1 + t$ never vanishes on the time domain.

Step 1: Transform with both data. The three contributions are

$$\mathcal{L}\{ty''\} = -2sY - s^2Y', \quad \mathcal{L}\{y''\} = s^2Y - 1, \quad \mathcal{L}\{y'\} = sY.$$

Hence $-s^2Y' + (s^2 - s)Y = 1$, or

$$Y' - (1 - 1/s)Y = -1/s^2, \quad s > 0.$$

Step 2: Select the parameter integral. The integrating factor is se^{-s} , so $(se^{-s}Y)' = -e^{-s}/s$. An exponential-order continuous time solution has $Y = O(1/s)$ for large s , and thus $se^{-s}Y \rightarrow 0$. Therefore

$$Y(s) = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u} du.$$

An additional term Ce^s/s violates the large-parameter bound unless $C = 0$. Existence and the growth condition are verified by the recovered solution.

Step 3: Identify and check the inverse. In time, the left side is $[(1+t)y']'$. Integrating with the data gives $y' = 1/(1+t)$ and

$$y(t) = \ln(1+t).$$

It has $y(0) = 0$, $y'(0) = 1$, $y'' = -1/(1+t)^2$, and zero equation residual. The coefficient is regular, so uniqueness applies. Integrating its transform by parts gives

$$Y(s) = \frac{1}{s} \int_0^\infty \frac{e^{-st}}{1+t} dt = \frac{e^s}{s} \int_s^\infty \frac{e^{-u}}{u} du,$$

where the upper boundary vanishes for $s > 0$.

Step 4: Compare growth rates and domains. For $t > 0$, $0 < \ln(1+t) < t$. Multiplying by the positive kernel and integrating gives

$$0 < Y(s) < \frac{1}{s^2}, \quad s > 0.$$

The exact domain is $s > 0$: logarithmic growth is dominated by the exponential kernel above zero, whereas its positive time integral diverges at zero and for negative parameters. Unboundedness by itself does not prevent a Laplace transform; growth relative to the kernel is what matters.

Question 7

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Consider the local IVP

$$(t-1)y' + y = 0, \quad y(0) = 1.$$

For $0 < T < 1$, define the finite transform $F_T(s) = \int_0^T e^{-st}y(t) dt$.

Tasks

1. Solve the IVP up to its first singularity and decide whether a global classical solution on $[0, \infty)$ exists.
2. Formally applying the infinite-transform rules gives $Y' + Y = 1/s$. Identify why this equation cannot be used as an ordinary transform solution of the IVP.
3. Derive the correct differential equation for F_T by retaining the upper boundary term during integration by parts.
4. Check that finite-transform equation directly, including its value at $s = 0$. Explain why a finite cutoff repairs the calculation but does not remove the singularity.

Question 7 – Solution

Strategy. An interior zero of the leading coefficient can stop the classical solution. A finite transform before that point must retain its endpoint term.

Step 1: Locate the obstruction in time. Since $[(t-1)y]' = 0$, the datum gives $(t-1)y = -1$. Thus

$$y(t) = \frac{1}{1-t}, \quad 0 \leq t < 1.$$

This diverges as $t \uparrow 1$ and admits no continuous classical continuation through one. There is no global classical IVP solution on the full half-line. The integral up to one already diverges for every real s , because its positive weight stays bounded below near one while $1/(1-t)$ has a logarithmic divergence.

Step 2: Diagnose the formal infinite equation. The formal substitutions $\mathcal{L}\{ty'\} = -Y - sY'$ and $\mathcal{L}\{y'\} = sY - 1$ would give $Y' + Y = 1/s$. But the required infinite time integrals do not exist and no global classical solution satisfies the starting hypotheses. Solving this formal equation cannot provide an ordinary transform of the local IVP solution.

Step 3: Retain the actual cutoff boundary. For $T < 1$, integration by parts gives

$$0 = [e^{-st}(t-1)y(t)]_0^T + s \int_0^T e^{-st}(t-1)y(t) dt.$$

The first term is $1 - e^{-sT}$, using $(T-1)y(T) = -1$. Since $\int_0^T te^{-st}y = -F'_T$, the result is

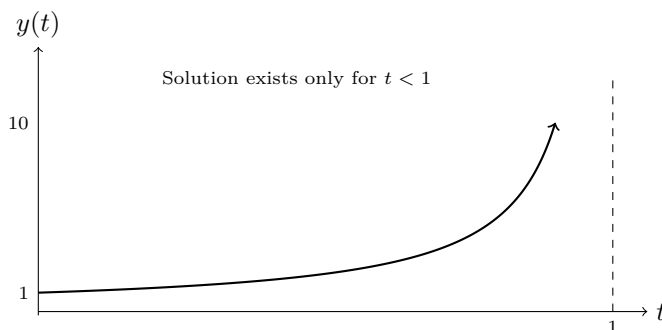
$$F'_T(s) + F_T(s) = \frac{1 - e^{-sT}}{s} \quad (s \neq 0).$$

Discarding the upper endpoint would lose the term $-e^{-sT}/s$.

Step 4: Verify the finite identity and its limits. Direct differentiation on the compact interval gives

$$F'_T + F_T = \int_0^T (1-t)e^{-st}y(t) dt = \int_0^T e^{-st} dt.$$

This is the stated expression for $s \neq 0$ and equals $\boxed{T}$ at $s = 0$. Also $F_T(0) = -\ln(1-T)$ and $F'_T(0) = T + \ln(1-T)$, whose sum is T . Every finite F_T exists for all real s , but it diverges as $T \uparrow 1$ for each fixed s . A valid finite-window calculation does not supply a missing continuation through the pole. The plot stops short of the singular time.



Question 8

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Classify the data for which a twice continuously differentiable solution on $[0, \infty)$ can satisfy

$$t^2 y'' + 2ty' - 2y = 0, \quad y(0) = a, \quad y'(0) = b.$$

For a smooth exponential-order function with $a = 0$, you may use $\lim_{s \rightarrow \infty} s^2 Y(s) = b$.

Tasks

1. Find the compatibility condition at zero and solve the time equation for $t > 0$ using trial powers t^m .
2. Derive the second-order differential equation for Y , accounting for the effect of two parameter derivatives on the initial terms.
3. Solve the transformed equation using trial powers s^m . Determine what the condition $Y \rightarrow 0$ fixes and what initial information is still needed.
4. Classify all admissible pairs (a, b) , verify the resulting solutions, and state their exact transform domains. Is specifying only $y(0)$ enough for uniqueness?

Question 8 – Solution

Strategy. Singular coefficients can erase initial data from the transformed equation. Recover admissibility and the remaining amplitude separately.

Step 1: Check compatibility and time powers. At zero, the equation requires $\boxed{a = 0}$. For $t > 0$, trial powers give $m(m-1) + 2m - 2 = (m-1)(m+2) = 0$. The two independent solutions are t and t^{-2} , so $y = Ct + Dt^{-2}$. They span the solutions on $(0, \infty)$ because the normalized equation is regular there. Continuity at zero forces $D = 0$; the initial slope then sets $C = b$.

Step 2: Transform the polynomial coefficients. For sufficiently large s ,

$$\mathcal{L}\{t^2 y''\} = \frac{d^2}{ds^2}[s^2 Y - sa - b] = 2Y + 4sY' + s^2 Y'',$$

while $2\mathcal{L}\{ty'\} = -2Y - 2sY'$. Adding $-2Y$ gives

$$\boxed{s^2 Y'' + 2sY' - 2Y = 0.}$$

Both a and b vanished under differentiation; their disappearance is not permission to ignore the time-domain data.

Step 3: Select the transform and its amplitude. Trial powers of s have exponents one and -2 , yielding $Y = As + B/s^2$ on $s > 0$. A genuine transform of an exponential-order continuous function tends to zero at large s , so $A = 0$. This still leaves B arbitrary. The supplied initial-slope limit gives $\boxed{B = b}$ and hence $Y = b/s^2$. The initial-value limit $sY \rightarrow 0$ also agrees with the necessary datum $a = 0$.

Step 4: Verify and classify uniqueness. All and only the data $\boxed{a = 0, b \in \mathbb{R}}$ are admissible, with unique solution $\boxed{y = bt}$. Substitution gives $0 + 2tb - 2bt = 0$, and both data hold. For $b \neq 0$, the exact domain is $s > 0$; the signed linear tail diverges at or below zero. For $b = 0$, the zero function has transform zero for every real s . Specifying only $y(0) = 0$ leaves the entire family bt , so it does not determine a unique solution at this singular point.

Question 9

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

An unknown parameter $\alpha > 0$ appears in

$$(1+t)y' + \alpha y = 0, \quad y(0) = 1.$$

An exact measurement gives $\int_0^\infty y(t) dt = 1/2$. A second proposed measurement is $\int_0^\infty ty(t) dt = 1/3$.

Tasks

1. Derive the differential equation for Y and solve it as a convergent parameter integral for $s > 0$.
2. Identify and verify the time solution for general $\alpha > 0$.
3. Determine which parameters admit a finite area, then recover α from the first measurement.
4. Test the proposed first moment for consistency, and state the exact real transform domain of the recovered solution.

Question 9 – Solution

Strategy. Solve the parameter family before fitting measurements. A boundary transform value is meaningful only when the actual time integral converges.

Step 1: Derive and solve the transform equation. Using $\mathcal{L}\{y'\} = sY - 1$ and $\mathcal{L}\{ty'\} = -Y - sY'$ gives

$$-sY' + (s + \alpha - 1)Y = 1.$$

The integrating factor for the divided equation is $e^{-s}s^{1-\alpha}$, so its product with Y has derivative $-e^{-s}s^{-\alpha}$. The transform bound at infinity selects

$$Y(s) = e^s s^{\alpha-1} \int_s^\infty e^{-u} u^{-\alpha} du, \quad s > 0.$$

The alternative homogeneous term $Ce^s s^{\alpha-1}$ violates decay as $s \rightarrow \infty$.

Step 2: Verify the parameter family in time. Separation gives $y(t) = (1+t)^{-\alpha}$. Its derivative is $-\alpha(1+t)^{-\alpha-1}$, so the equation and initial value hold. The coefficient $1+t$ is nonzero on $t \geq 0$, establishing regular-IVP uniqueness. Substitution $u = s(1+t)$ in the forward integral gives exactly the parameter expression above, verifying its inversion.

Step 3: Use the convergent area to identify the parameter. The positive power tail has finite area exactly when $\alpha > 1$, and then

$$\int_0^\infty (1+t)^{-\alpha} dt = \frac{1}{\alpha-1}.$$

Equating this to $1/2$ gives the unique value $\alpha = 3$. This uses convergence of the integral, not a formal substitution into an expression initially derived only for $s > 0$.

Step 4: Test the second datum rather than refit. A finite first moment requires $\alpha > 2$. Substituting $x = 1+t$ gives

$$\int_0^\infty \frac{t}{(1+t)^\alpha} dt = \frac{1}{\alpha-2} - \frac{1}{\alpha-1} = \frac{1}{(\alpha-1)(\alpha-2)}.$$

At $\alpha = 3$, it is $1/2$, not $1/3$, so the proposed data are inconsistent with this family. The recovered positive tail is $(1+t)^{-3}$; its exact transform domain is $[0, \infty)$. Positive s gives exponential damping, zero gives the finite area $1/2$, and negative s makes the weighted power tail diverge.

Question 10

Use ordinary one-sided Laplace integrals for real s . Where justified, write $Y(s) = \mathcal{L}\{y\}(s)$ and use $\mathcal{L}\{ty\} = -Y'(s)$. Check existence and initial compatibility before treating a formal solution in s as a transform.

Choose constants A, B so that a twice continuously differentiable solution on $[0, \infty)$ satisfies

$$ty' + (t+2)y = (At+B)e^{-t}, \quad y(0) = 1, \quad y'(0) = 0.$$

The equation must hold at the singular initial point as well as for $t > 0$.

Tasks

1. Derive the differential equation for Y without inserting unsupported free initial terms.
2. Solve it for general A, B and select the branch that can be an ordinary transform.
3. Use the recovered time solution and the two initial conditions to determine A, B . Verify the resulting equation at and away from zero.
4. Prove uniqueness in the stated smooth class, determine the exact transform domain, and describe the sign and monotonicity of the selected response.

Question 10 – Solution

Strategy. The data lost from the transform equation return as compatibility conditions on the forcing. Boundedness removes the singular time-domain homogeneous mode.

Step 1: Transform the time multipliers. The three left-hand terms transform to $-Y - sY'$, $-Y'$ and $2Y$. Thus

$$-(s+1)Y' + Y = \frac{A}{(s+1)^2} + \frac{B}{s+1}.$$

The initial value in $sY - y(0)$ differentiates to zero in $\mathcal{L}\{ty'\}$. It must still be imposed on the inverse, not added back as a fictitious term.

Step 2: Solve and remove the inadmissible branch. Writing $p = s + 1$, divide by p^2 to obtain

$$(Y/p)' = -A/p^4 - B/p^3.$$

Integration gives

$$Y = \frac{A/3}{(s+1)^2} + \frac{B/2}{s+1} + C(s+1).$$

Decay requires $C = 0$. The remaining terms invert to $y = e^{-t}(At/3 + B/2)$, whose transform can now be verified directly.

Step 3: Recover the forcing constants and verify. The initial value requires $B/2 = 1$, and the initial slope requires $A/3 - B/2 = 0$. Hence

$$A = 3, \quad B = 2, \quad y = (1+t)e^{-t}.$$

Its derivative is $-te^{-t}$, so

$$ty' + (t+2)y = e^{-t}[-t^2 + (t+2)(1+t)] = (3t+2)e^{-t}.$$

At zero the equation gives $2y(0) = B$, and both specified data hold. The smooth extension also has $y''(0) = -1$.

Step 4: Prove uniqueness and check the response. For $t > 0$, multiplying the original first-order equation by te^t gives $(t^2e^ty)' = At^2 + Bt$. Thus every local solution differs from the selected one by Ke^{-t}/t^2 . This mode is unbounded at zero unless $K = 0$, proving uniqueness in the stated smooth class. The selected function is positive, with $y' < 0$ for $t > 0$, and decreases from one to zero after an initially horizontal tangent. Its positive polynomial-exponential tail gives exact transform domain $s > -1$. The parameter restrictions arise from compatibility, not from a regular existence theorem where the leading coefficient vanishes.

