

### Question 1

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

Consider the IVP

$$y' + 2y = 3e^{-t}, \quad y(0) = 1.$$

A student writes  $Y(s) = 3/[(s+1)(s+2)]$ .

#### Tasks

1. Derive the correct transformed equation, and identify which IVP the student actually solved.
2. Find the correct inverse using partial fractions.
3. Check the original differential equation and initial value directly. State the exact real convergence interval of  $Y$ .
4. Find the unique maximum of  $y$  and its time. Explain why a decaying forcing does not force the response to decrease from the start.

## Question 1 – Solution

**Strategy.** The derivative transform carries the initial value into the algebra. Keep that contribution before resolving the two poles.

**Step 1: Include the initial term.** Using  $\mathcal{L}\{y'\} = sY - y(0)$  gives

$$(s+2)Y - 1 = \frac{3}{s+1}, \quad Y = \frac{1}{s+2} + \frac{3}{(s+1)(s+2)}.$$

The student omitted the first term. Their expression solves the same equation with initial value zero, not one.

**Step 2: Resolve and invert.** Combining the terms yields

$$Y = \frac{s+4}{(s+1)(s+2)} = \frac{3}{s+1} - \frac{2}{s+2}, \quad \boxed{y(t) = 3e^{-t} - 2e^{-2t}}.$$

Each reciprocal factor matches an elementary exponential transform.

**Step 3: Verify the IVP and domain.** We have  $y(0) = 3 - 2 = 1$  and  $y' = -3e^{-t} + 4e^{-2t}$ . Consequently

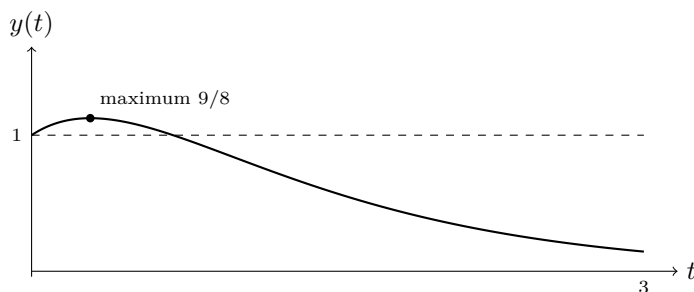
$$y' + 2y = (-3 + 6)e^{-t} + (4 - 4)e^{-2t} = 3e^{-t}.$$

The coefficients and forcing are continuous, so the usual uniqueness theorem for this linear IVP makes the verified solution the unique one. Its tail is asymptotic to  $3e^{-t}$  and is positive; hence its exact real transform domain is  $\boxed{s > -1}$ . The student instead obtains  $3e^{-t} - 3e^{-2t}$ , whose initial value is zero.

**Step 4: Locate the overshoot.** Since  $y' = e^{-t}(-3 + 4e^{-t})$ , the derivative is positive before and negative after  $e^{-t} = 3/4$ . Thus

$$\boxed{t_* = \ln(4/3), \quad y(t_*) = 9/8.}$$

Here  $y'(0) = 1$  although the forcing decreases. The input initially exceeds the loss term  $2y$ , so the state first rises. It later decays to zero. The graph marks the overshoot above the initial level one.



### Question 2

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

Solve

$$y'' + 3y' + 2y = 4, \quad y(0) = 1, \quad y'(0) = -1.$$

Call the solution with the same initial data and zero forcing the initial-data response. Call the solution with forcing four and zero initial data the forced response.

#### Tasks

1. Derive  $Y$  and split it into the two responses defined above.
2. Invert both parts and obtain the complete solution.
3. Verify both initial data and the original equation. Check that the two parts satisfy their respective data and forcing.
4. Find the limiting value and the exact real transform domain. Explain why solving only the forced response would not solve the stated IVP.

## Question 2 – Solution

**Strategy.** Linearity separates the numerator contribution of initial data from that of the input, but the final answer needs both.

**Step 1: Transform all derivative terms.** The derivative transforms are  $s^2Y - s + 1$  and  $sY - 1$ . Thus

$$(s^2 + 3s + 2)Y - s - 2 = 4/s,$$

$$Y = \underbrace{\frac{s+2}{(s+1)(s+2)}}_{Y_{\text{initial}}} + \underbrace{\frac{4}{s(s+1)(s+2)}}_{Y_{\text{forced}}}.$$

The first part simplifies to  $1/(s+1)$ ; the second resolves as  $2/s - 4/(s+1) + 2/(s+2)$ .

**Step 2: Invert and combine.** Hence

$$y_{\text{initial}} = e^{-t}, \quad y_{\text{forced}} = 2 - 4e^{-t} + 2e^{-2t},$$

$$\boxed{y(t) = 2 - 3e^{-t} + 2e^{-2t}}.$$

The initial-data response satisfies the homogeneous equation, and the constant term in the forced response supplies the constant forcing.

**Step 3: Verify each contribution.** For the sum,  $y' = 3e^{-t} - 4e^{-2t}$  and  $y'' = -3e^{-t} + 8e^{-2t}$ . Then

$$y'' + 3y' + 2y = 4 + (-3 + 9 - 6)e^{-t} + (8 - 12 + 4)e^{-2t} = 4.$$

Also  $y(0) = 1$  and  $y'(0) = -1$ . The initial-data part has values 1, -1 and zero equation residual. The forced part has values 0, 0 and residual four. These checks establish the intended decomposition; uniqueness of the linear IVP establishes the final solution.

**Step 4: Interpret the steady value and domain.** The exponentials vanish, so  $\boxed{y(t) \rightarrow 2}$ . This agrees with the constant balance  $2y = 4$ . Its ordinary transform converges exactly for  $s > 0$ : boundedness gives convergence above zero, while the eventually positive nonzero constant tail causes divergence at or below zero. Although the initial-data response has the wider domain  $s > -1$ , the complete response does not. Reporting only the forced response gets the limiting value right but misses both prescribed initial data.

**Question 3**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

Consider a repeated-root operator with matching exponential forcing:

$$y'' + 2y' + y = e^{-t}, \quad y(0) = 2, \quad y'(0) = -2.$$

**Tasks**

1. Find  $Y$  and identify the order of the pole produced by the forcing.
2. Invert  $Y$ , keeping the factorial normalization of the repeated pole.
3. Verify the equation and both initial data directly. You may simplify the residual by setting  $y = e^{-t}v$ .
4. Determine whether  $y$  decreases to zero and whether  $e^t y$  stays bounded. Explain why these conclusions are compatible and state the transform domain.

### Question 3 – Solution

**Strategy.** A repeated operator root and a matching input pole combine. Pole order controls the polynomial factor, not necessarily growth of the full time response.

**Step 1: Form the transformed equation.** The initial terms give

$$(s^2Y - 2s + 2) + 2(sY - 2) + Y = \frac{1}{s+1},$$

so

$$(s+1)^2Y = 2(s+1) + \frac{1}{s+1}, \quad Y = \frac{2}{s+1} + \frac{1}{(s+1)^3}.$$

The forcing contributes a third-order pole at  $s = -1$ , although the differential equation has order two.

**Step 2: Invert the repeated pole correctly.** Since  $\mathcal{L}\{t^2e^{-t}\} = 2/(s+1)^3$ ,

$$y(t) = e^{-t}(2 + t^2/2).$$

The factor  $1/2$  is essential. Transforming the two terms back gives exactly the displayed  $Y$ .

**Step 3: Check the residual and data.** With  $v = 2 + t^2/2$ , the product rule gives  $y' = e^{-t}(v' - v)$  and  $y'' = e^{-t}(v'' - 2v' + v)$ . Therefore

$$y'' + 2y' + y = e^{-t}v'' = e^{-t}.$$

At zero,  $v(0) = 2$ ,  $v'(0) = 0$ , so  $y(0) = 2$  and  $y'(0) = -2$ . These checks and linear-IVP uniqueness verify the solution.

**Step 4: Distinguish a scaled response from the actual response.** We have

$$y' = e^{-t}(-2 + t - t^2/2) = -\frac{1}{2}e^{-t}[(t-1)^2 + 3] < 0.$$

Thus  $y$  decreases strictly and tends to zero, since exponential decay dominates its quadratic factor. Meanwhile  $e^ty = 2 + t^2/2 \rightarrow \infty$ . Dividing out the decaying envelope changes the quantity being studied; there is no contradiction. The positive polynomial-exponential tail gives the exact domain  $s > -1$ , with divergence at and below the boundary.

**Question 4**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

An undamped oscillator starts at rest:

$$y'' + 4y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 0.$$

Define its energy by  $E(t) = [y'(t)^2 + 4y(t)^2]/2$ .

**Tasks**

1. Derive  $Y$  and invert it. You may use  $\mathcal{L}\{t \cos 2t\} = (s^2 - 4)/(s^2 + 4)^2$ .
2. Check the equation and both initial data directly.
3. Prove that the response is unbounded by evaluating a suitable explicit time sequence. State its exact real transform domain.
4. Derive the energy identity and calculate  $E(n\pi)$  for positive integers  $n$ . Explain why bounded forcing can supply unbounded accumulated energy.

## Question 4 – Solution

**Strategy.** Matching forcing and natural frequencies produce repeated transform poles. Explicit values and the work identity distinguish real growth from a mere upper bound.

**Step 1: Transform and invert.** Zero initial data give

$$(s^2 + 4)Y = \frac{2}{s^2 + 4}, \quad Y = \frac{2}{(s^2 + 4)^2}.$$

Combining the sine pair with the supplied time-cosine pair yields

$$y(t) = \frac{\sin 2t - 2t \cos 2t}{8}.$$

Indeed the forward numerator is  $(s^2 + 4)/4 - (s^2 - 4)/4 = 2$ .

**Step 2: Verify the IVP.** Direct differentiation gives

$$y' = \frac{t}{2} \sin 2t, \quad y'' = \frac{1}{2} \sin 2t + t \cos 2t.$$

Adding  $4y = \frac{1}{2} \sin 2t - t \cos 2t$  gives  $\sin 2t$ . Both initial values are zero, proving the claimed solution by uniqueness.

**Step 3: Exhibit unbounded values and the domain.** At  $t = n\pi$ ,

$$y(n\pi) = -n\pi/4 \longrightarrow -\infty.$$

The actual response is therefore unbounded; a growing envelope alone would not prove that. Its linear envelope gives absolute transform convergence for  $s > 0$ . At zero, its finite integral is  $[1 - \cos 2R - R \sin 2R]/8$ , which has no limit. Below zero, intervals around successive peaks of the  $t \cos 2t$  term violate the Cauchy criterion. Hence the exact domain is  $s > 0$ .

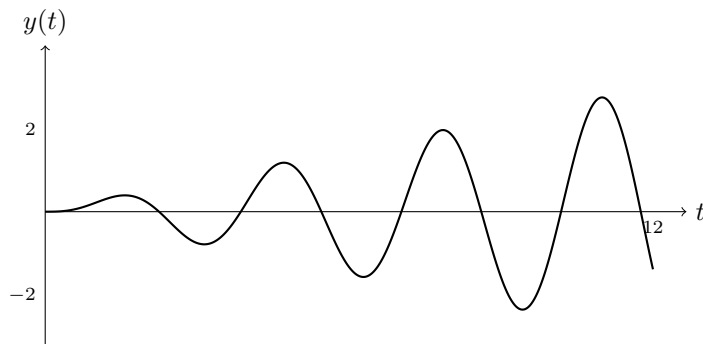
**Step 4: Account for the energy input.** Multiplying the equation by  $y'$  gives

$$E' = y'(y'' + 4y) = y' \sin 2t = \frac{t}{2} \sin^2 2t \geq 0.$$

At  $n\pi$ , the velocity is zero, so

$$E(n\pi) = 2y(n\pi)^2 = n^2 \pi^2 / 8.$$

Although the forcing amplitude is at most one, its work accumulates over an increasing time interval. Resonance makes that accumulated work unbounded. The graph shows the growing displacement oscillations, not the energy itself.



**Question 5**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

An unstable equation has adjustable initial velocity:

$$y'' - y = e^{-2t}, \quad y(0) = 1, \quad y'(0) = v.$$

**Tasks**

1. Derive and invert  $Y$  for arbitrary real  $v$ .
2. Find the unique  $v$  that gives a bounded solution, and verify the resulting IVP.
3. If the chosen velocity is perturbed by  $\delta$ , derive the exact error in the solution. Determine the largest allowed  $|\delta|$  that guarantees an error at most 0.01 for every  $0 \leq t \leq 5$ .
4. State the exact transform domains for the bounded choice and for every other  $v$ . Explain why the uncanceled operator factor alone does not determine the response domain.

## Question 5 – Solution

**Strategy.** Initial data control the coefficient of the unstable mode. Exact cancellation can give a bounded trajectory but does not remove sensitivity to an initial error.

**Step 1: Transform and resolve the modes.** The transformed equation is

$$(s^2 - 1)Y = s + v + \frac{1}{s + 2}.$$

Partial fractions give

$$Y = \frac{v/2 + 2/3}{s - 1} + \frac{-v/2}{s + 1} + \frac{1/3}{s + 2},$$

$$y(t) = (v/2 + 2/3)e^t - (v/2)e^{-t} + \frac{1}{3}e^{-2t}.$$

Recombining the fractions reproduces the transformed equation.

**Step 2: Cancel the growing coefficient.** Boundedness requires  $v/2 + 2/3 = 0$ , so the unique choice is

$$v = -4/3, \quad y_b(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-2t}.$$

Then  $y_b(0) = 1$ ,  $y'_b(0) = -2/3 - 2/3 = -4/3$ , and  $y''_b - y_b = e^{-2t}$ . If the growing coefficient is nonzero, it eventually dominates both decaying terms, so no other velocity works.

**Step 3: Quantify the initial-error tolerance.** Replacing  $v$  by  $-4/3 + \delta$  changes the solution by

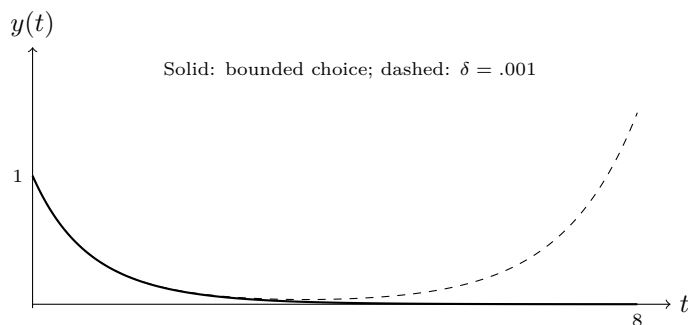
$$y - y_b = \frac{\delta}{2}(e^t - e^{-t}) = \delta \sinh t.$$

Since  $\sinh t$  increases on  $t \geq 0$ , the maximum absolute error on  $[0, 5]$  is  $|\delta| \sinh 5$ . The exact, necessary and sufficient tolerance is

$$|\delta| \leq \frac{0.01}{\sinh 5} \approx 1.3477 \times 10^{-4}.$$

Equality is permitted and attains the error limit at time five. The graph compares the bounded choice with a small positive velocity error.

**Step 4: Check the actual tails.** For the bounded choice, the nonzero slow tail is  $(2/3)e^{-t}$ , so the exact domain is  $s > -1$ . For every other  $v$ , a nonzero  $e^t$  term gives exact domain  $s > 1$ . The operator factor  $s - 1$  cancels from the selected response numerator. Its presence in the operator does not mean it survives in every solution; the initial data determine the residue.



**Question 6**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

Two parameters are unknown in the IVP

$$y'' + 2y' + 5y = Ae^{-t}, \quad y(0) = 0, \quad y'(0) = v.$$

Exact measurements are  $e^{\pi/2}y(\pi/2) = 2$  and  $e^{\pi/4}y(\pi/4) = 2$ .

**Tasks**

1. Solve by Laplace transforms in terms of  $A, v$ .
2. Use the two measurements to recover  $A, v$  and prove uniqueness of this parameter recovery.
3. Verify the calibrated equation and both initial data. State the transform domain of the calibrated solution.
4. Would measurements taken only at  $t = n\pi$ , for positive integers  $n$ , identify these parameters? Justify your answer from the exact response, not from a count of samples.

## Question 6 – Solution

**Strategy.** Solve once with symbolic parameters, then evaluate the resulting basis functions at the measurement times.

**Step 1: Transform with unknown initial velocity.** The transformed equation is

$$[(s+1)^2 + 4]Y = v + \frac{A}{s+1}.$$

Writing  $p = s + 1$  and using  $1/[p(p^2 + 4)] = (1/4)[1/p - p/(p^2 + 4)]$  gives

$$y(t) = e^{-t} \left[ \frac{v}{2} \sin 2t + \frac{A}{4} (1 - \cos 2t) \right].$$

Transforming the sine and constant-minus-cosine terms verifies the algebraic equation for sufficiently large  $s$ .

**Step 2: Recover the parameters.** At  $t = \pi/2$ , the local sine is zero and the cosine is  $-1$ , so  $A/2 = 2$ , giving  $A = 4$ . At  $t = \pi/4$ , the sine is one and the cosine is zero, so  $v/2 + A/4 = 2$ , giving  $v = 2$ . Thus

$$A = 4, \quad v = 2, \quad y = e^{-t}(1 - \cos 2t + \sin 2t).$$

The first measurement fixes  $A$  independently, then the second fixes  $v$  with a nonzero coefficient. This proves uniqueness for the specified two-parameter model.

**Step 3: Verify calibration and the IVP.** Set  $w = 1 - \cos 2t + \sin 2t$ , so  $y = e^{-t}w$ . The product rule gives

$$y'' + 2y' + 5y = e^{-t}(w'' + 4w) = 4e^{-t}.$$

Also  $w(0) = 0$  and  $w'(0) = 2$ , so  $y(0) = 0$  and  $y'(0) = 2$ . The exact transform domain is  $s > -1$ . At its boundary the weighted signal is the nonzero periodic function  $w$ , with nonzero mean; below it fixed-sign lobe intervals fail the Cauchy criterion. Above it the exponential envelope gives absolute convergence.

**Step 4: Identify blind sampling times.** At every  $t = n\pi$ , both  $\sin 2t = 0$  and  $1 - \cos 2t = 0$ . Therefore

$$y(n\pi) = 0 \quad \text{for every } A, v.$$

Even infinitely many such samples carry no information about either parameter. The issue is where the response basis vanishes, not merely how many measurements are collected. The two given times avoid this degeneracy.

**Question 7**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

The data are prescribed at time one, not at zero:

$$y'' + y = t, \quad y(1) = 2, \quad y'(1) = 0, \quad t \geq 1.$$

Use a new time origin before applying a one-sided transform.

**Tasks**

1. Define  $\tau = t - 1$  and  $z(\tau) = y(1 + \tau)$ . Derive the transformed IVP for  $z$ .
2. Find  $Z(s) = \mathcal{L}_\tau\{z\}(s)$  and invert it.
3. Convert back to  $t$  and verify the equation and the data at time one.
4. A student treats the supplied values as  $y(0), y'(0)$  and transforms the forcing as  $1/s^2$ . Explain both changes needed to repair that calculation. State the exact domain of  $Z$ .

## Question 7 – Solution

**Strategy.** Shift the independent variable and the forcing together. A Laplace transform beginning at the new origin must use data at that origin.

**Step 1: Move the initial time.** With  $\tau = t - 1$ , the derivatives have the same form but the forcing becomes  $\tau + 1$ . Thus

$$z'' + z = \tau + 1, \quad z(0) = 2, \quad z'(0) = 0.$$

Writing  $Z = \mathcal{L}_\tau\{z\}$  gives

$$(s^2 + 1)Z - 2s = \frac{1}{s^2} + \frac{1}{s}.$$

**Step 2: Resolve and invert.** A useful decomposition is

$$Z = \frac{2s + 1/s^2 + 1/s}{s^2 + 1} = \frac{1}{s^2} + \frac{1}{s} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}.$$

Therefore  $\boxed{z(\tau) = \tau + 1 + \cos \tau - \sin \tau}$ . Transforming each term back checks the decomposition, including the constant part of the shifted forcing.

**Step 3: Return to the original clock.** The solution on the requested domain is

$$\boxed{y(t) = t + \cos(t - 1) - \sin(t - 1), \quad t \geq 1.}$$

Its first derivative is  $1 - \sin(t - 1) - \cos(t - 1)$  and its second is  $-\cos(t - 1) + \sin(t - 1)$ . Hence  $y'' + y = t$ , while  $y(1) = 2$  and  $y'(1) = 0$ . These are the actual specified data, so uniqueness on  $[1, \infty)$  verifies the result.

**Step 4: Diagnose the two clock errors.** The values two and zero belong to  $z(0), z'(0)$ , not to values of  $y$  at an unspecified earlier time. Once that clock change is made, the forcing is  $\tau + 1$  and its transform is  $1/s^2 + 1/s$ , not just  $1/s^2$ . The transform being inverted is  $Z$ , defined using  $\tau \geq 0$ . Since  $z(\tau) = \tau + O(1)$  is eventually positive, it has exact domain  $\boxed{s > 0}$ . No extension of  $y$  to times before one was needed.

**Question 8**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

An accumulated quantity enters an IVP:

$$y'(t) + y(t) + \int_0^t y(u) du = 1, \quad y(0) = 0.$$

You may use  $\mathcal{L}\{\int_0^t y(u) du\} = Y(s)/s$  where the integrals converge.

**Tasks**

1. Solve directly by Laplace transforms, including the integral term.
2. Find an explicit expression for  $I(t) = \int_0^t y(u) du$  and verify the original equation.
3. Differentiate the original equation and determine the extra initial condition needed for the resulting second-order equation. Explain why omitting that condition enlarges the solution set.
4. Find the first positive zero of  $y$ , its limit, the limit of  $I$ , and the exact transform domain of  $Y$ . Explain why a constant right-hand side does not force a nonzero limiting  $y$  here.

### Question 8 – Solution

**Strategy.** Transform the accumulated state as well as the instantaneous state. Verify the original integral equation after any differentiation.

**Step 1: Transform and complete the square.** For sufficiently large  $s$ , the initial value gives

$$sY + Y + Y/s = 1/s, \quad Y = \frac{1}{s^2 + s + 1}.$$

Let  $b = \sqrt{3}/2$ . Completing the square gives

$$y(t) = \frac{2}{\sqrt{3}} e^{-t/2} \sin(bt).$$

This inverts  $Y = 1/[(s + 1/2)^2 + b^2]$  with the required sine normalization.

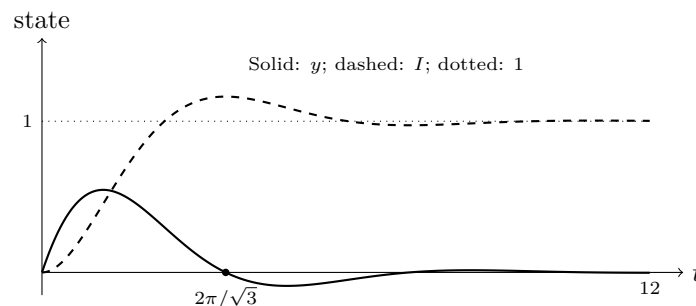
**Step 2: Reconstruct the integral and verify.** Integration, or differentiation of the following expression, gives

$$I(t) = 1 - e^{-t/2} [\cos(bt) + \frac{1}{\sqrt{3}} \sin(bt)].$$

It satisfies  $I(0) = 0$  and  $I' = y$ . Also  $y' = e^{-t/2} [\cos(bt) - \sin(bt)/\sqrt{3}]$ . Substitution yields  $y' + y + I = 1$  identically, and  $y(0) = 0$ . This checks the undifferentiated problem.

**Step 3: Retain the compatibility condition.** Differentiation gives  $y'' + y' + y = 0$ . Evaluating the original equation at zero gives  $y'(0) = 1$ . Thus the equivalent second-order IVP has data  $y(0) = 0, y'(0) = 1$ . If only  $y(0) = 0$  is retained, a free initial slope remains; it sets the constant value of  $y' + y + I$ , which need not be one. Conversely, a solution of the differentiated equation with both correct data makes that quantity constant and equal to one, proving equivalence and uniqueness.

**Step 4: Interpret the limiting balance.** The first positive zero is  $t = \pi/b = 2\pi/\sqrt{3}$ . The function changes sign there and tends to zero, while  $I(t) \rightarrow 1$ . The accumulated state supplies the limiting balance, rather than a nonzero instantaneous state. The exact domain of  $Y$  is  $s > -1/2$ , from the damped sine and its nonconvergent boundary integral. The integral state's transform needs  $s > 0$  because  $I \rightarrow 1$ ; the initial derivation used this common domain. The wider domain of  $Y$  follows from the recovered function itself.



### Question 9

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

Use Laplace transforms for the third-order IVP

$$y^{(3)} + 3y'' + 3y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

#### Tasks

1. Derive the transformed equation, displaying the contributions of all three initial data.
2. Invert the result at the repeated pole, and check all three data.
3. Verify the third-order differential equation directly, using  $y = e^{-t}v$  if useful.
4. Prove positivity and strict decrease for  $t > 0$ . Compute the total integral and exact transform domain, and explain why the initial flatness does not make the solution constant.

## Question 9 – Solution

**Strategy.** Higher derivatives introduce more initial terms. A repeated transform pole packages the polynomial-exponential solution compactly.

**Step 1: Transform the derivative hierarchy.** The general third-derivative rule is  $\mathcal{L}\{y^{(3)}\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$ . Here the first three derivatives transform to  $sY - 1$ ,  $s^2Y - s$  and  $s^3Y - s^2$ . Hence

$$(s^3Y - s^2) + 3(s^2Y - s) + 3(sY - 1) + Y = 0,$$

$$Y = \frac{s^2 + 3s + 3}{(s + 1)^3}.$$

Even the zero initial values must be placed in the correct derivative formulas before simplifying.

**Step 2: Expand at the repeated pole.** Writing  $p = s + 1$  makes the numerator  $p^2 + p + 1$ , so

$$Y = \frac{1}{s + 1} + \frac{1}{(s + 1)^2} + \frac{1}{(s + 1)^3}, \quad \boxed{y = e^{-t}(1 + t + t^2/2)}.$$

We have  $y' = -t^2e^{-t}/2$  and  $y'' = e^{-t}(t^2/2 - t)$ . Thus the three initial values are exactly 1, 0, 0.

**Step 3: Verify the operator identity.** For  $y = e^{-t}v$ , repeated product rules give

$$y^{(3)} + 3y'' + 3y' + y = e^{-t}v^{(3)}.$$

Here  $v = 1 + t + t^2/2$  has third derivative zero. This verifies the equation everywhere. The linear third-order IVP with all three data has a unique solution.

**Step 4: Check shape, area and domain.** The polynomial factor is positive on  $t \geq 0$ , and  $y' < 0$  for every  $t > 0$ , so  $y$  is positive and strictly decreasing after its initially flat point. It tends to zero. Direct exponential moments give

$$\boxed{\int_0^\infty y(t) dt = 1 + 1 + 1 = 3 = Y(0)}.$$

The exact transform domain is  $s > -1$ ; at or below  $-1$  the positive polynomial tail makes the integral diverge. The data  $y'(0) = y''(0) = 0$  describe only the initial point. The equation gives  $y^{(3)}(0) = -1$ , so the first change is cubic rather than absent.

**Question 10**

Use one-sided Laplace transforms and retain all initial-value terms. Write  $Y(s) = \mathcal{L}\{y\}(s)$ , with real  $s$  sufficiently large during the transformation. Unless stated otherwise, solve on  $t \geq 0$ .

An input  $q(t)$  is to produce a prescribed smooth rise in

$$y'' + 3y' + 2y = q(t), \quad y(0) = 0, \quad y'(0) = 0.$$

The target family is  $y_k(t) = 1 - (1 + kt)e^{-kt}$ , with  $k > 0$ . Allowed inputs have the form  $q(t) = 2 + Ce^{-kt}$ , with no term involving  $te^{-kt}$ .

**Tasks**

1. Find the target transform  $Y_k(s)$  and derive the input transform required by the IVP.
2. Invert that input transform and determine exactly which  $k > 0$  satisfy the allowed input form.
3. An additional requirement is  $y(0) = 4$ . Select  $k, C$  and verify the complete IVP with the resulting input.
4. Prove that the selected response increases to one without overshoot. Explain why the required input is unique once the full target trajectory is fixed.

### Question 10 – Solution

**Strategy.** Transform the target first, then use the differential operator to recover the necessary input. Check the permitted input family before selecting parameters.

**Step 1: Transform the prescribed trajectory.** The exponential and time-exponential pairs give

$$Y_k = \frac{1}{s} - \frac{1}{s+k} - \frac{k}{(s+k)^2} = \boxed{\frac{k^2}{s(s+k)^2}}.$$

All targets have  $y_k(0) = y'_k(0) = 0$ , so their required input transform is

$$Q_k = (s^2 + 3s + 2)Y_k = \frac{k^2(s+1)(s+2)}{s(s+k)^2}.$$

**Step 2: Recover the input and impose the restriction.** Partial fractions, verified by clearing denominators, give

$$Q_k = \frac{2}{s} + \frac{k^2 - 2}{s+k} - \frac{k(k-1)(k-2)}{(s+k)^2},$$

$$q_k(t) = 2 + (k^2 - 2)e^{-kt} - k(k-1)(k-2)te^{-kt}.$$

For  $k > 0$ , the last term vanishes exactly for  $\boxed{k=1 \text{ or } k=2}$ . Their respective values of  $C$  are  $-1$  and  $2$ . A nonzero  $te^{-kt}$  coefficient cannot be absorbed into a constant multiple of  $e^{-kt}$  on an interval.

**Step 3: Select and verify the desired acceleration.** Differentiation gives  $y'_k = k^2te^{-kt}$  and  $y''_k = k^2(1 - kt)e^{-kt}$ , so  $y''_k(0) = k^2$ . The added requirement selects

$$\boxed{k=2, \quad C=2, \quad q=2+2e^{-2t}, \quad y=1-(1+2t)e^{-2t}}.$$

Here  $y' = 4te^{-2t}$ ,  $y'' = (4 - 8t)e^{-2t}$ , and  $y'' + 3y' + 2y = 2 + 2e^{-2t}$ . The initial values are  $0, 0$  and the acceleration is four, as required. Both  $Y$  and  $Q$  have exact domain  $s > 0$  because their original functions approach positive constants.

**Step 4: Prove the rise and uniqueness of the input.** Since  $y'(t) > 0$  for  $t > 0$ , the response increases from zero. Also  $1 - y = (1 + 2t)e^{-2t} > 0$  for every finite  $t$ , and this difference tends to zero. Thus  $\boxed{0 < y(t) < 1 \ (t > 0), \quad y(t) \rightarrow 1}$  with no overshoot. Once the entire twice-differentiable target is fixed, the equation forces  $q = y'' + 3y' + 2y$  pointwise. The input is unique, rather than merely one successful choice; linear-IVP uniqueness then verifies that it produces this trajectory.

