

Question 1

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A signal follows the schedule

$$f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ -1, & 1 \leq t < 3, \\ 4, & 3 \leq t < 5, \\ 0, & t \geq 5. \end{cases}$$

Tasks

1. Write f as an initial level plus a linear combination of u_1, u_3, u_5 . Explain how to choose the coefficients.
2. Find its transform first for $s > 0$, then determine its full real convergence set.
3. Compute $F(0)$ and $F'(0)$ directly from the finite time intervals. Check the removable singularity in your formula.
4. A student uses the new levels themselves as the step coefficients. Diagnose the error by finding the resulting levels after each switch.

Question 1 – Solution

Strategy. A step adds a change to the existing level; it does not replace that level. Compact support must be assessed after the changes are combined.

Step 1: Encode the changes. The changes at times one, three and five are $-3, +5, -4$. Therefore

$$f(t) = 2 - 3u_1(t) + 5u_3(t) - 4u_5(t).$$

The convention $u_a(a) = 1$ gives the prescribed value at every switch. Successively subtracting adjacent levels determines these coefficients uniquely for the specified switch times.

Step 2: Transform and check support. For $s > 0$, $\mathcal{L}\{u_a\} = e^{-as}/s$ by direct integration from a to infinity. Thus

$$F(s) = \frac{2 - 3e^{-s} + 5e^{-3s} - 4e^{-5s}}{s} \quad (s \neq 0).$$

Although the separate step integrals require $s > 0$, the actual signal vanishes after five. Integrating on its three finite intervals proves the same formula for every real $s \neq 0$; the original integral exists for every real s .

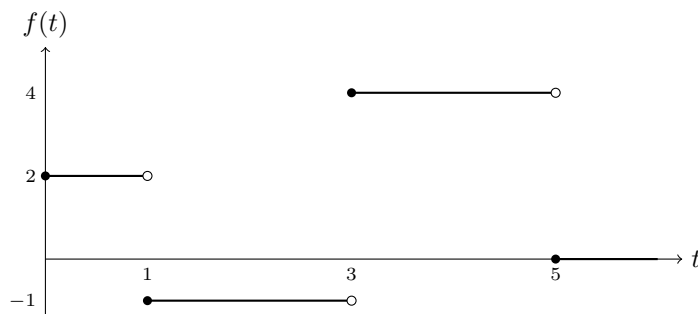
Step 3: Evaluate the removable value and first moment. Directly,

$$F(0) = 2(1) - 1(2) + 4(2) = \boxed{8},$$

$$F'(0) = -\int_0^5 t f(t) dt = -[1 - 4 + 32] = \boxed{-29}.$$

Finite support justifies differentiation under the integral. Expanding the numerator gives $8s - 29s^2 + O(s^3)$, so $F(s) = 8 - 29s + O(s^2)$, confirming both results. Zero is removable, not a divergent transform point.

Step 4: Diagnose replacement versus addition. The mistaken expression $2 - u_1 + 4u_3 + 0u_5$ produces levels 2, 1, 5, 5. It neither reaches the intended levels nor shuts off. Each step must carry the difference between the new and old levels. The graph uses open and filled endpoints to record the specified right-hand values.



Question 2

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Compare the switched-on and delayed signals

$$p(t) = t^2 u_2(t), \quad q(t) = (t - 2)^2 u_2(t).$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{t\} = 1/s^2$ and $\mathcal{L}\{t^2\} = 2/s^3$ for $s > 0$.

Tasks

1. Derive the shift formula for $u_a(t)g(t - a)$ by substitution in the defining integral.
2. Find the transforms of p and q . Rewrite the switched-on polynomial in its local time before using the formula.
3. A student assigns $2e^{-2s}/s^3$ to both functions. Identify the missing terms and check the values immediately after the switch.
4. Give the exact real convergence domains. Determine the leading large- s behavior of $e^{2s}P(s)$ and $e^{2s}Q(s)$ and relate it to the local onset.

Question 2 – Solution

Strategy. The factor e^{-as} shifts the time origin of the entire accompanying inverse, not just its domain of activity.

Step 1: Derive the delayed-shape rule. For parameters where the integral converges, substitute $v = t - a$:

$$\int_a^\infty e^{-st} g(t-a) dt = e^{-as} \int_0^\infty e^{-sv} g(v) dv.$$

Hence $\boxed{\mathcal{L}\{u_a(t)g(t-a)\} = e^{-as}G(s)}$. The effective shape is a function of elapsed time since the switch.

Step 2: Rewrite the unshifted polynomial. For $t \geq 2$, write $t = v + 2$. Then p has local shape $(v+2)^2 = v^2 + 4v + 4$, whereas q has local shape v^2 . Therefore

$$\boxed{P(s) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right), \quad Q(s) = \frac{2e^{-2s}}{s^3}, \quad s > 0.}$$

Direct substitution in each defining integral verifies these expressions.

Step 3: Identify the omitted local terms. The difference is

$$p - q = [4(t-2) + 4]u_2(t), \quad P - Q = e^{-2s}(4/s^2 + 4/s).$$

At the switch, $p(2) = 4$ while $q(2) = 0$, equal to their respective right limits. The switched-on signal jumps from zero to four; the delayed square starts continuously at zero with zero right slope. The student's expression describes only the latter.

Step 4: Check domains and onset scales. Both signals have nonnegative quadratic tails, so they converge absolutely for $s > 0$ and diverge for $s \leq 0$. Their exact domain is $\boxed{s > 0}$. Removing the delay factor gives

$$e^{2s}P(s) \sim 4/s, \quad e^{2s}Q(s) = 2/s^3 \quad (s \rightarrow \infty).$$

The constant local onset of p produces the $1/s$ term; the quadratic local onset of q produces $2/s^3$. A common switch time does not imply a common transformed shape.

Question 3

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

An oscillator is observed only during the window

$$f(t) = \sin t [u_{\pi/2}(t) - u_{3\pi/2}(t)].$$

Use $\mathcal{L}\{\cos t\} = s/(s^2 + 1)$ for $s > 0$.

Tasks

1. Write the exact piecewise function, including its values at both window endpoints.
2. Express each switched sine using its local clock and derive $F(s)$ for $s > 0$.
3. Determine the full real convergence set, $F(0)$ and $F'(0)$. Verify them by direct finite-interval integration.
4. Explain why $F(s) > 0$ for $s > 0$ even though the signal has both signs and zero unweighted integral. Use the window symmetry to justify the sign.

Question 3 – Solution

Strategy. Re-express the phase at each switch separately. The second switch occurs at a different oscillator phase.

Step 1: Identify the window and endpoint values. The function equals $\sin t$ on $[\pi/2, 3\pi/2]$ and zero elsewhere. In particular $f(\pi/2) = 1$, while $f(3\pi/2) = 0$. At the latter time the left limit is -1 ; the graph distinguishes it from the assigned value.

Step 2: Shift the two phases correctly. With $a = \pi/2$ and $b = 3\pi/2$, we have $\sin(v + a) = \cos v$ and $\sin(v + b) = -\cos v$. Therefore

$$f = u_a(t) \cos(t - a) + u_b(t) \cos(t - b),$$

$$F(s) = \frac{s(e^{-\pi s/2} + e^{-3\pi s/2})}{s^2 + 1}.$$

The plus sign in the transform reflects the phase reversal at shutoff. After b , the two shifted cosine terms cancel exactly, as the original window requires.

Step 3: Use finite support to evaluate zero. The defining integral has finite support, so it converges for every real s ; finite-interval integration gives the displayed expression on the whole real line. Directly,

$$F(0) = \int_a^b \sin t \, dt = 0,$$

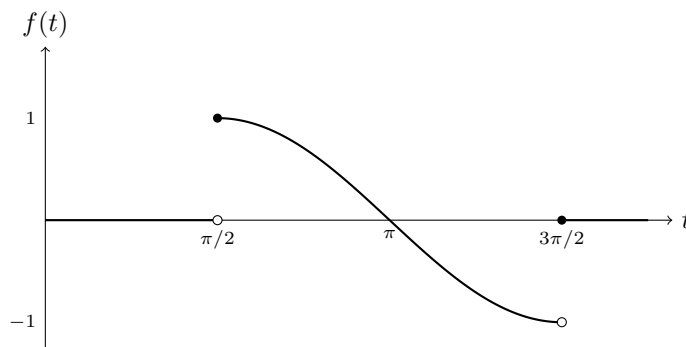
$$F'(0) = - \int_a^b t \sin t \, dt = -[-t \cos t + \sin t]_a^b = \boxed{2}.$$

The formula also gives $F(0) = 0$ and expands as $2s + O(s^2)$, verifying the derivative.

Step 4: Pair the equal and opposite lobes. Write $t = \pi + x$ with $-\pi/2 \leq x \leq \pi/2$. Pair x with $-x$. For $0 < x < \pi/2$, the earlier value is $\sin x > 0$ and the later value is $-\sin x$. Their weighted contribution is

$$e^{-\pi s} \sin x [e^{sx} - e^{-sx}] > 0 \quad (s > 0).$$

Integrating these pairs proves strict positivity, despite cancellation at $s = 0$. This also agrees with the positive numerator and denominator of the formula for $s > 0$.



Question 4

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Let $r_a(t) = (t - a)u_a(t)$. A continuous triangular signal is

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ t - 1, & 1 \leq t < 3, \\ (7 - t)/2, & 3 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$$

Tasks

1. Express f as a linear combination of r_1, r_3, r_7 by tracking slope changes.
2. Find $F(s)$ using shifted ramp transforms, and establish its full real convergence set.
3. Explain algebraically why the ramp combination is zero after time seven. State the two cancellation conditions for a general sum $\sum c_j r_{a_j}(t)$ to vanish after its last switch.
4. Compute the total area and the first moment. Confirm the values from the expansion of F at zero.

Question 4 – Solution

Strategy. A ramp changes slope rather than level. Cancel both the final slope and the final intercept to create compact support.

Step 1: Encode slope changes. The slopes are 0, 1, $-1/2$, 0, giving changes 1, $-3/2$, $1/2$. Thus

$$f = r_1 - \frac{3}{2}r_3 + \frac{1}{2}r_7.$$

Each ramp is zero at its own switch, so the combination is continuous. At the peak time three its value is two; it then decreases to zero at seven.

Step 2: Transform the ramps. For $s > 0$, substitution gives $\mathcal{L}\{r_a\} = e^{-as}/s^2$. Hence

$$F(s) = \frac{e^{-s} - \frac{3}{2}e^{-3s} + \frac{1}{2}e^{-7s}}{s^2} \quad (s \neq 0).$$

The actual function has finite support, so its integral exists for all real s . Direct integration of the two linear pieces extends the formula to all $s \neq 0$; zero is removable.

Step 3: Cancel the tail, not just the slope. For $t \geq 7$, the ramp sum is

$$(1 - \frac{3}{2} + \frac{1}{2})t - [1 - \frac{3}{2}(3) + \frac{1}{2}(7)] = 0.$$

In general, after the last switch a finite ramp sum equals $t \sum c_j - \sum c_j a_j$. It vanishes identically there exactly when

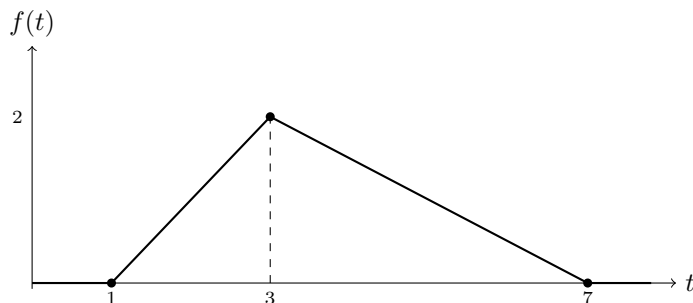
$$\sum c_j = 0, \quad \sum c_j a_j = 0.$$

Canceling only the slope could leave a nonzero constant tail and a different convergence domain.

Step 4: Check area and moment. The triangle has base six and height two, so $F(0) = 6$. Direct integration gives

$$\int_1^3 t(t-1) dt = \frac{14}{3}, \quad \frac{1}{2} \int_3^7 t(7-t) dt = \frac{52}{3}.$$

Thus the first moment is $\frac{22}{3}$ and $F'(0) = -22$. Expanding the numerator of F gives $6s^2 - 22s^3 + O(s^4)$, agreeing with both results. The horizontal time coordinate of the area centroid is $22/6 = 11/3$, consistent with the three triangle vertices.



Question 5

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Find a piecewise continuous inverse of

$$F(s) = (e^{-2s} - e^{-5s}) \frac{s+3}{(s+1)^2 + 4}.$$

Use the step convention above to specify values at switches. Basic shifted sine and cosine pairs may be used.

Tasks

1. Invert the rational factor, then apply the two delays correctly.
2. Write the result on $0 \leq t < 2$, $2 \leq t < 5$ and $t \geq 5$. Find the jumps at two and five.
3. Does the inverse vanish after five? Compare it with genuinely cutting off the first delayed signal at five.
4. Determine the exact real convergence interval and verify the transform on it.

Question 5 – Solution

Strategy. Subtracting a delayed copy restarts that copy's clock. It generally does not switch off the earlier copy.

Step 1: Invert the common shape. Since $s + 3 = (s + 1) + 2$, the rational factor is the transform of

$$g(v) = e^{-v}(\cos 2v + \sin 2v), \quad v \geq 0.$$

Thus one inverse with the stipulated endpoint convention is

$$f(t) = u_2(t)g(t-2) - u_5(t)g(t-5).$$

Step 2: Inspect the clocks and jumps. The piecewise values are zero before two, $g(t-2)$ on $[2, 5)$, and $g(t-2) - g(t-5)$ on $[5, \infty)$. As $g(0) = 1$, the jumps are $\boxed{+1 \text{ at } 2, -1 \text{ at } 5}$. In particular $f(2) = 1$ and $f(5) = g(3) - 1$, while the left limit at five is $g(3)$.

Step 3: Distinguish subtraction from shutoff. At $t = 5$, $|g(3)| \leq \sqrt{2}e^{-3} < 1$, so $g(3) - 1 \neq 0$. In fact the final piece is a nonzero exponentially decaying sinusoid, not zero. A genuinely truncated signal is instead

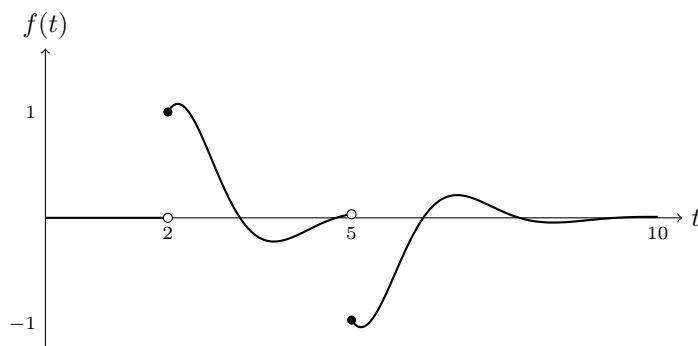
$$h(t) = [u_2(t) - u_5(t)]g(t-2).$$

Its shutoff term has the same global argument $t-2$. To transform that term at time five, its local shape must be $g(v+3)$, not $g(v)$. The graph shows the actual continuing tail of f .

Step 4: Verify the transform and domain. On $t \geq 5$, put $v = t - 5$. The tail is

$$e^{-v}\{[e^{-3}(\cos 6 + \sin 6) - 1]\cos 2v + [e^{-3}(\cos 6 - \sin 6) - 1]\sin 2v\}.$$

The cosine coefficient is nonzero by the preceding bound. Hence absolute convergence holds for $s > -1$; at $s = -1$ a nonzero undamped sinusoid has no convergent primitive, and below it fixed-sign lobe intervals violate the Cauchy criterion. The exact domain is $\boxed{s > -1}$. On that domain each delayed copy converges and transforms to $e^{-2s}G(s)$ or $e^{-5s}G(s)$ by substitution, proving the given expression. The truly truncated h would instead converge for every real s .



Question 6

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A function has a deliberately exceptional point value:

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 7, & t = 1, \\ e^{-(t-1)}, & t > 1. \end{cases}$$

Compare it with $g(t) = u_1(t)e^{-(t-1)}$.

Tasks

1. Compare f and g pointwise, including their left and right limits at one.
2. Find their ordinary transforms and exact convergence intervals.
3. Give a pointwise formula for f using g and the indicator of the singleton $\{1\}$. Explain why a finite sum of right-continuous step terms times continuous functions cannot reproduce this exceptional point value.
4. If a step convention instead assigns value $1/2$ at the switch, what changes in the formula for g and what stays the same in its transform? State precisely the uniqueness conclusion justified here.

Question 6 – Solution

Strategy. Separate equality at every point from equality except at isolated points. An ordinary integral cannot recover the latter point values.

Step 1: Locate the discrepancy. For $t \neq 1$, $f(t) = g(t)$. At one, $g(1) = 1$ but $f(1) = 7$. Both functions have left limit zero and right limit one. Thus g is right-continuous, whereas the chosen value of f agrees with neither limit.

Step 2: Transform the interval contribution. A single changed value contributes nothing to an integral. Substituting $v = t - 1$ in the tail gives, for both functions,

$$F(s) = G(s) = \frac{e^{-s}}{s+1}, \quad s > -1.$$

At $s = -1$, the weighted tail is the positive constant e ; below -1 it grows exponentially. Thus the stated convergence interval is exact. The isolated value seven cannot change this conclusion.

Step 3: Give a pointwise correction and prove the limitation. If $\mathbf{1}_{\{1\}}$ is one only at $t = 1$ and zero elsewhere, then

$$f(t) = g(t) + 6\mathbf{1}_{\{1\}}(t).$$

Each u_a under our convention is right-continuous. Multiplying it by a continuous function preserves right-continuity, and finite sums preserve that property. Such a sum must equal its right limit at one, so it cannot have the specified value seven while its right limit is one. This is a representational obstruction, not a transform-calculation error.

Step 4: Change the convention without changing the integral. With switch value $1/2$, the corresponding expression for g equals $1/2$ at one and is unchanged elsewhere. Its ordinary transform is still $e^{-s}/(s+1)$. A pointwise correction to recover f would now be $(13/2)\mathbf{1}_{\{1\}}$. Thus transform data alone do not specify the value at an isolated discontinuity. Here the right-hand continuous representative is fixed by the chosen convention; an unrestricted pointwise uniqueness claim would be false. The usual uniqueness statement at continuity points is consistent with all these examples.

Question 7

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

Let $f(t) = u_2(t)[1 + (t - 2)]$. Define its ordinary derivative away from the switch as $h(t) = 0$ for $t < 2$ and $h(t) = 1$ for $t > 2$; its value at two is irrelevant to integration. A student asserts $H(s) = sF(s) - f(0)$.

Tasks

1. Calculate $F(s)$ and $H(s)$ directly for $s > 0$, and test the assertion.
2. Apply integration by parts separately on $[0, 2)$ and $(2, R]$ to identify the missing contribution.
3. For a piecewise C^1 function with finitely many jumps $J_j = f(c_j+) - f(c_j-)$ at positive times, derive the corresponding formula for the transform of its ordinary derivative on the smooth pieces. State the needed boundary assumptions.
4. Apply your formula to the continuous delayed ramp $k(t) = (t - 2)u_2(t)$. Explain why the usual derivative rule works for k despite its corner.

Question 7 – Solution

Strategy. Integration by parts across smooth pieces leaves internal boundary terms. A jump is different from a continuous corner.

Step 1: Compute both ordinary integrals. The local shape of f is $1 + v$, and the local shape of h is one. Thus

$$F(s) = e^{-2s}(1/s + 1/s^2), \quad H(s) = e^{-2s}/s, \quad s > 0.$$

Since $f(0) = 0$, the student's expression is $sF = e^{-2s}(1 + 1/s)$, larger than H by $\boxed{e^{-2s}}$. Both actual integrals have exact domain $s > 0$ because their tails are a positive linear function and a positive constant, respectively.

Step 2: Retain the internal boundary. Only the second piece contributes. For $R > 2$, integration by parts gives

$$\int_2^R e^{-st} f'(t) dt = e^{-sR} f(R) - e^{-2s} f(2+) + s \int_2^R e^{-st} f(t) dt.$$

Here $f(2+) = 1$. The upper term vanishes for $s > 0$, leaving $H = sF - e^{-2s}$. The omitted term records the jump from zero to one, not a value assigned to h at the single point two.

Step 3: Sum the piecewise boundary terms. At an internal time c_j , the left piece contributes $e^{-sc_j} f(c_j-)$ and the right piece contributes $-e^{-sc_j} f(c_j+)$. Their sum is $-J_j e^{-sc_j}$. Consequently

$$\boxed{\mathcal{L}\{f'_{\text{pw}}\}(s) = sF(s) - f(0+) - \sum_j J_j e^{-sc_j}.$$

Use finite one-sided limits, convergence of the piecewise function and derivative integrals, and $e^{-sR} f(R) \rightarrow 0$. Exponential-order bounds on the pieces and their derivatives suffice for large s . If f is right-continuous at zero, $f(0+) = f(0)$. The formula concerns ordinary piecewise derivatives only.

Step 4: Check a continuous corner. For k , both one-sided values at two are zero, so its jump is zero. We have $K = e^{-2s}/s^2$ and its ordinary piecewise derivative is u_2 except possibly at two. Thus

$$\mathcal{L}\{k'_{\text{pw}}\} = e^{-2s}/s = sK - k(0).$$

A corner changes slopes but introduces no internal value jump. Splitting the integral proves the usual rule here without pretending the derivative exists at the corner.

Question 8

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

A three-stage pulse is one on $[0, 1)$, has unknown height B on $[1, 2)$ and unknown height C on $[2, 3)$, and is zero for $t \geq 3$. It must satisfy

$$\int_0^\infty f(t) dt = 0, \quad \int_0^\infty t f(t) dt = 0.$$

Tasks

1. Determine B, C and prove that these heights are uniquely determined.
2. Write the resulting pulse using steps and derive its transform in factored form.
3. Find the full real convergence set, the first nonzero term of $F(s)$ near zero, and the second time moment.
4. Explain why the two vanishing moments do not make the pulse or its transform identically zero. Determine the sign of $F(s)$ for $s > 0$.

Question 8 – Solution

Strategy. Moment constraints form a small linear system. The step differences then expose cancellation in the transform numerator.

Step 1: Solve the two moment equations. The unit-width intervals give

$$1 + B + C = 0, \quad \frac{1}{2} + \frac{3}{2}B + \frac{5}{2}C = 0.$$

Subtracting the first equation from twice the second yields $2B + 4C = 0$. Combining this with $B + C = -1$ gives $\boxed{B = -2, C = 1}$. The coefficient determinant is $5/2 - 3/2 = 1 \neq 0$, so the solution is unique.

Step 2: Encode changes and factor the transform. The levels 1, -2, 1, 0 have jumps $-3, +3, -1$. Hence

$$f(t) = 1 - 3u_1(t) + 3u_2(t) - u_3(t),$$

$$\boxed{F(s) = \frac{1 - 3e^{-s} + 3e^{-2s} - e^{-3s}}{s} = \frac{(1 - e^{-s})^3}{s} \quad (s \neq 0).}$$

This follows first by transforming steps for $s > 0$, or directly on the three finite intervals for any $s \neq 0$.

Step 3: Interpret the removable zero. Compact support gives convergence for every real s . Since $1 - e^{-s} = s + O(s^2)$,

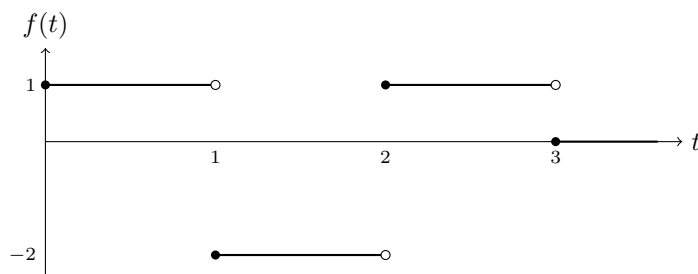
$$\boxed{F(s) = s^2 + O(s^3), \quad F(0) = F'(0) = 0, \quad F''(0) = 2.}$$

Directly, the second moment is

$$\int_0^3 t^2 f(t) dt = \frac{1}{3} - 2\left(\frac{7}{3}\right) + \frac{19}{3} = \boxed{2},$$

consistent with differentiating the finite-support transform twice.

Step 4: Separate two constraints from identity. The pulse is one throughout its first interval, so it is not zero. For $s > 0$, both $1 - e^{-s}$ and s are positive, giving $\boxed{F(s) > 0}$. The zero area and first moment eliminate only the constant and linear terms near $s = 0$. They do not eliminate all weighted integrals. The second moment already witnesses the remaining information. The graph makes the signed cancellation visible without replacing it by a pointwise sign claim.



Question 9

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

For an integer $n \geq 2$, define a right-continuous staircase

$$f_n(t) = \frac{k}{n} \quad \left(\frac{k}{n} \leq t < \frac{k+1}{n}, \quad k = 0, \dots, n-1\right), \quad f_n(t) = 0 \quad (t \geq 1).$$

The target is $r(t) = t$ on $0 \leq t < 1$ and $r(t) = 0$ for $t \geq 1$.

Tasks

1. Express f_n using steps and derive a finite-sum formula for its transform.
2. Find $F_n(0)$ and the target transform $R(s)$, including $R(0)$. Give the full real convergence sets.
3. For $s > 0$, derive a bound on $0 \leq R(s) - F_n(s)$ from the pointwise staircase error.
4. Using that bound, find the least integer n that certifies $R(s) - F_n(s) \leq 0.01$ simultaneously for every $s \geq 1$. Distinguish this certificate from an exact minimal-error calculation.

Question 9 – Solution

Strategy. Encode each upward increment and the final reset. A pointwise error bound becomes an integral certificate valid for a whole range of parameters.

Step 1: Include the final shutoff. There are increments $1/n$ at k/n , $1 \leq k \leq n-1$, followed by a drop $(n-1)/n$ at one. Hence

$$f_n(t) = \frac{1}{n} \sum_{k=1}^{n-1} u_{k/n}(t) - \frac{n-1}{n} u_1(t),$$

$$F_n(s) = \frac{\sum_{k=1}^{n-1} e^{-ks/n} - (n-1)e^{-s}}{ns} \quad (s \neq 0).$$

Transforming steps gives the formula for $s > 0$; finite-interval integration gives it for all $s \neq 0$.

Step 2: Evaluate area and the target. Each stair has width $1/n$, so

$$F_n(0) = \frac{1}{n^2} \sum_{k=0}^{n-1} k = \frac{n-1}{2n}.$$

Integration by parts on $[0, 1]$ gives

$$R(s) = \frac{1 - (1+s)e^{-s}}{s^2} \quad (s \neq 0), \quad R(0) = \frac{1}{2}.$$

Both original functions have finite support, so both transform domains are the whole real line. Their apparent singularities at zero are removable.

Step 3: Convert pointwise error to weighted error. On each half-open stair, $0 \leq r - f_n < 1/n$, and both functions vanish from one onward. For $s > 0$,

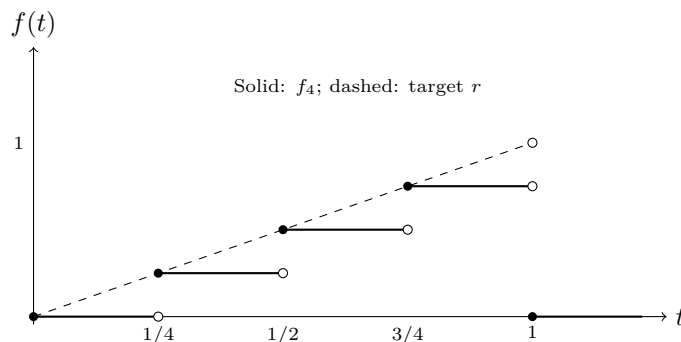
$$0 \leq R(s) - F_n(s) \leq \frac{1}{n} \int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{ns}.$$

At zero the actual error is $1/(2n)$, while the continuous limiting bound is $1/n$. The bound is sufficient, not an equality.

Step 4: Certify a simultaneous tolerance. The function $b(s) = \int_0^1 e^{-st} dt$ is decreasing for $s > 0$, since its derivative is $-\int_0^1 te^{-st} dt < 0$. Its largest value on $s \geq 1$ is $1 - e^{-1}$. Thus the certificate requires

$$n \geq 100(1 - e^{-1}) \approx 63.2121, \quad \boxed{n = 64}.$$

For $n = 63$ this bound exceeds .01 at $s = 1$; that does not prove the actual error exceeds .01. The graph shows $n = 4$ only to display the staircase geometry; it is not the certified design.



Question 10

For $a \geq 0$, let $u_a(t) = 0$ for $t < a$ and $u_a(t) = 1$ for $t \geq a$. Use ordinary one-sided Laplace integrals for real s ; a value at one isolated point does not change an integral.

An unknown rectangular pulse has height $A > 0$ on $[a, b)$ and is zero elsewhere, where $0 \leq a < b$. Let F be its transform and set $H(s) = sF(s)$. Exact data are

$$H(1) = \frac{3}{8}, \quad H(2) = \frac{15}{64}, \quad H(3) = \frac{63}{512}.$$

Tasks

1. Write the pulse using steps and derive $H(s)$ in terms of A, a, b .
2. Set $x = e^{-a}$ and $y = e^{-b}$. Use the ratios $H(2)/H(1)$ and $H(3)/H(1)$ to determine $x + y$ and xy .
3. Recover A, a, b , prove uniqueness within the stated pulse family, and check all three data.
4. Find the pulse area and full real transform domain. Explain, with a construction or a dimension argument, why these three data do not determine an arbitrary piecewise continuous signal.

Question 10 – Solution

Strategy. Consecutive exponential measurements determine symmetric functions of the two endpoints. The ordering of the endpoints then removes the root ambiguity.

Step 1: Encode the window. The pulse is $f = A(u_a - u_b)$. Integrating on its finite support gives

$$F(s) = A \frac{e^{-as} - e^{-bs}}{s} \quad (s \neq 0), \quad H(s) = A(e^{-as} - e^{-bs}).$$

With $x = e^{-a}$ and $y = e^{-b}$, the restrictions become $0 < y < x \leq 1$.

Step 2: Recover the two symmetric quantities. Since $H(1) = A(x - y) > 0$,

$$\frac{H(2)}{H(1)} = x + y = \frac{5}{8}, \quad \frac{H(3)}{H(1)} = x^2 + xy + y^2 = \frac{21}{64}.$$

Therefore $xy = (5/8)^2 - 21/64 = \boxed{1/16}$. The numbers x, y are the roots of $z^2 - (5/8)z + 1/16 = 0$.

Step 3: Order the roots and find the height. The roots are $1/2$ and $1/8$. The condition $x > y$ gives

$$a = \ln 2, \quad b = \ln 8, \quad A = \frac{3/8}{1/2 - 1/8} = 1.$$

Both endpoint restrictions hold. Direct checks yield $x - y = 3/8$, $x^2 - y^2 = 15/64$ and $x^3 - y^3 = 63/512$. The two roots, their ordering and the nonzero first measurement fix all three parameters, proving uniqueness within this family.

Step 4: State what the data do and do not determine. The area is $\boxed{F(0) = A(b - a) = \ln 4}$.

Finite support gives convergence for every real s , with zero removable in the displayed formula. To see the limitation of three measurements, take four disjoint unit pulses on $[4 + j, 5 + j)$, $j = 0, 1, 2, 3$. Requiring their linear combination to have zero transform at $s = 1, 2, 3$ gives three homogeneous linear equations in four coefficients. A nonzero coefficient vector exists; disjoint supports make its signal nonzero. Adding it preserves all three data but changes the original signal. This signed perturbation need not belong to the rectangular family, so it does not contradict the uniqueness just proved.

