

Question 1

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$F(s) = \frac{s+5}{(s+1)(s+3)}.$$

The basic pair $\mathcal{L}\{e^{-at}\} = 1/(s+a)$ holds for $s > -a$.

Tasks

1. Find the inverse by partial fractions and verify the decomposition.
2. Determine the full real convergence interval of the recovered function.
3. Prove that the inverse is positive for all $t \geq 0$. Does its negative partial-fraction coefficient contradict positivity?
4. Find its initial value, its unique maximum and its limit at infinity. Explain why a sum of decaying modes need not decrease from the start.

Question 1 – Solution

Strategy. Resolve the two poles, then factor the resulting time expression before deciding its sign or shape.

Step 1: Resolve and invert. Writing $F = A/(s+1) + B/(s+3)$ gives $A+B=1$ and $3A+B=5$. Thus $A=2$, $B=-1$, and

$$f(t) = 2e^{-t} - e^{-3t}.$$

Transforming these two terms gives $2/(s+1) - 1/(s+3)$, whose numerator is $2(s+3) - (s+1) = s+5$. This verifies the entire rational expression, not just its pole locations.

Step 2: Retain the actual domain. For $s > -1$, both exponential integrals converge absolutely. Since $f = e^{-t}(2 - e^{-2t}) \geq e^{-t}$ on $t \geq 0$, its weighted integral diverges for $s \leq -1$. Hence the exact real domain is $s > -1$; other algebraic values of F are not ordinary transform values.

Step 3: Determine the sign. The factor $2 - e^{-2t}$ lies in $[1, 2)$, so $f(t) > 0$ everywhere. The negative coefficient belongs to one term in a decomposition; it does not determine the sign of the whole function. Here the faster-decaying negative term never dominates the positive term.

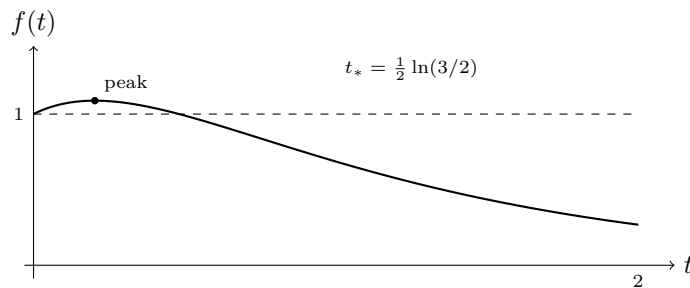
Step 4: Locate the peak. We have $f(0) = 1$, $f(t) \rightarrow 0$, and

$$f'(t) = e^{-t}(-2 + 3e^{-2t}).$$

The bracket decreases through zero once. Therefore f increases up to, then decreases after,

$$t_* = \frac{1}{2} \ln(3/2), \quad f(t_*) = \frac{4}{3} \sqrt{2/3}.$$

In particular $f'(0) = 1 > 0$: decay of each mode does not force monotonic decay of their signed combination. The graph marks the small but real overshoot.



Question 2

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{s^2 + 3s + 4}{(s + 1)^3}.$$

You may use $\mathcal{L}\{t^n e^{-t}\} = n!/(s + 1)^{n+1}$ for nonnegative integers n and $s > -1$.

Tasks

1. Find the complete partial-fraction expansion at the repeated pole.
2. Invert every term, explaining the factorial factors. Verify the transform of your answer.
3. A student includes only a term $C/(s + 1)^3$. Show why this cannot represent F , and identify the missing time terms.
4. Compute $f(0)$, $f'(0)$, $f''(0)$ and the exact convergence interval. Check the three initial quantities against the first three large- s coefficients of F .

Question 2 – Solution

Strategy. A pole of order three allows all three reciprocal powers, and each power has its own factorial normalization.

Step 1: Expand around the pole. Set $p = s + 1$. Then $s^2 + 3s + 4 = p^2 + p + 2$, so

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^3}.$$

Equivalently, multiplying by $(s+1)^3$ gives the polynomial identity $s^2 + 3s + 4 = (s+1)^2 + (s+1) + 2$.

Step 2: Apply the correctly normalized pairs. The inverse of $(s+1)^{-3}$ is $t^2 e^{-t}/2!$. Thus

$$f(t) = e^{-t}(1 + t + t^2).$$

Its three forward transforms are $1/(s+1)$, $1/(s+1)^2$ and $2/(s+1)^3$, confirming the result. Forgetting $2!$ would double the quadratic term incorrectly.

Step 3: Explain the incomplete ansatz. An expression $C/(s+1)^3$ has constant numerator after multiplication by the denominator. It cannot equal the nonconstant polynomial $s^2 + 3s + 4$ for every s . The proposed inverse $Ct^2 e^{-t}/2$ lacks both e^{-t} and te^{-t} ; no choice of C can repair that omission.

Step 4: Check initial data and convergence. Differentiation gives

$$f' = e^{-t}(t - t^2), \quad f'' = e^{-t}(1 - 3t + t^2),$$

so $f(0) = 1, f'(0) = 0, f''(0) = 1$. Independently, expansion at large positive s gives

$$F(s) = \frac{1}{s} + \frac{0}{s^2} + \frac{1}{s^3} + O(s^{-4}).$$

These are the initial-value coefficients obtained by repeated integration by parts; this polynomial-exponential function satisfies all needed derivative bounds. Its defining integral converges absolutely for $s > -1$. At $s = -1$ the integrand is the positive polynomial $1 + t + t^2$, and for smaller s it grows still faster. The exact domain is $s > -1$.

Question 3

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Recover the inverse of

$$F(s) = \frac{3s + 1}{s^2 + 4s + 13}.$$

Use the pairs for $e^{-at} \cos bt$ and $e^{-at} \sin bt$, whose numerators are $s + a$ and b , respectively, over $(s + a)^2 + b^2$.

Tasks

1. Complete the square and rewrite the numerator in terms of the shifted parameter.
2. Find and forward-check the inverse. Diagnose the candidate $e^{-2t}(3 \cos 3t + \frac{1}{3} \sin 3t)$.
3. Express the inverse as $Re^{-2t} \cos(3t + \phi)$ with $R > 0$ and $0 < \phi < \pi/2$. Find its first positive zero.
4. Give its exact real convergence interval and explain the roles of the decay envelope and oscillation frequency.

Question 3 – Solution

Strategy. Shift the numerator as well as the denominator; then distinguish phase, amplitude and exponential decay.

Step 1: Match the shifted pairs. The denominator is $(s+2)^2 + 9$, while $3s+1 = 3(s+2) - 5$. Therefore

$$F(s) = 3 \frac{s+2}{(s+2)^2 + 9} - \frac{5}{3} \frac{3}{(s+2)^2 + 9}.$$

Step 2: Invert and diagnose the sign error. The matching time function is

$$f(t) = e^{-2t} [3 \cos 3t - \frac{5}{3} \sin 3t].$$

Its forward transform reproduces $3(s+2) - 5 = 3s+1$. The proposed candidate instead has numerator $3(s+2) + 1 = 3s+7$. Completing the square without adjusting the numerator loses a constant contribution.

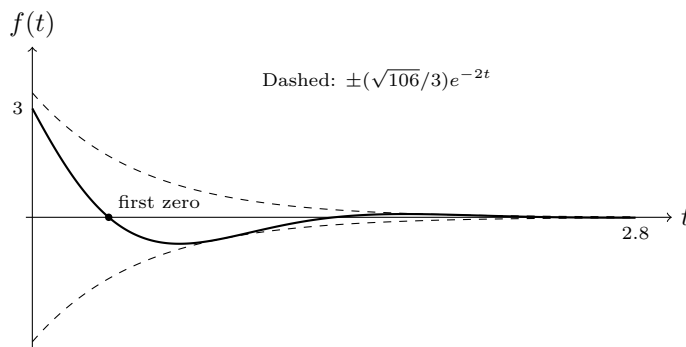
Step 3: Recover phase and the first zero. In $R \cos(3t + \phi) = R \cos \phi \cos 3t - R \sin \phi \sin 3t$, compare coefficients:

$$R = \frac{\sqrt{106}}{3}, \quad \phi = \arctan(5/9).$$

Both sine and cosine of ϕ are positive, so the specified quadrant removes any phase ambiguity. Starting at $f(0) = 3 > 0$, the first zero occurs when $3t + \phi = \pi/2$:

$$t_0 = \frac{\pi/2 - \arctan(5/9)}{3} = \frac{1}{3} \arctan(9/5).$$

Step 4: Separate envelope from oscillation. The bound $|f(t)| \leq R e^{-2t}$ gives absolute convergence for $s > -2$. At $s = -2$, the weighted signal is a nonzero sinusoid, whose primitive oscillates without a limit. For $s < -2$, positive fixed-length subintervals centered at successive cosine peaks have integrals growing exponentially, contradicting the Cauchy criterion. Thus $s > -2$ is exact. The decay rate is two and the angular frequency is three; neither gives the phase by itself. The graph shows both signed envelopes and the first zero.



Question 4

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Let

$$F(s) = \frac{5}{(s+1)(s^2+4)}.$$

The required basic pairs are e^{-t} , $\cos 2t$ and $\sin 2t$.

Tasks

1. Choose a complete real partial-fraction form and solve for its coefficients.
2. Find the inverse and verify its forward transform by recombining numerators.
3. Compute the first three initial quantities $f(0), f'(0), f''(0)$. Relate their cancellations to the leading large- s behavior of F .
4. Determine the exact real convergence interval. Does the inverse approach a limit as $t \rightarrow \infty$? Justify your answer using explicit sequences.

Question 4 – Solution

Strategy. An irreducible quadratic requires a linear numerator. The transient and persistent parts then have different long-time roles.

Step 1: Resolve the real factors. Use

$$F = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}.$$

Clearing denominators gives $5 = A(s^2+4) + (Bs+C)(s+1)$, so $A+B=0$, $B+C=0$, $4A+C=5$. Hence $A=1$, $B=-1$, $C=1$.

Step 2: Invert each numerator term. Since the sine pair has numerator two,

$$f(t) = e^{-t} - \cos 2t + \frac{1}{2} \sin 2t.$$

The forward numerator is $(s^2+4) - s(s+1) + (s+1) = 5$, giving the required transform. A constant numerator over the quadratic alone would have omitted the cosine contribution.

Step 3: Check the onset cancellations. Direct differentiation gives

$$f' = -e^{-t} + 2 \sin 2t + \cos 2t, \quad f'' = e^{-t} + 4 \cos 2t - 2 \sin 2t.$$

Therefore $f(0) = 0$, $f'(0) = 0$, $f''(0) = 5$. Also $F(s) = 5/s^3 + O(s^{-4})$: the $1/s$ and $1/s^2$ terms vanish. Repeated integration by parts connects precisely these coefficients to the displayed initial values. In time, the first nonzero Taylor term is $5t^2/2$.

Step 4: Identify the persistent oscillation. The exponential term decays, but $-\cos 2t + \frac{1}{2} \sin 2t$ does not. In fact,

$$f(n\pi) \longrightarrow -1, \quad f(n\pi + \pi/2) \longrightarrow 1.$$

Thus f has no limit at infinity, although it is bounded. Boundedness gives absolute convergence for $s > 0$. At $s = 0$, the primitive contains the nonconstant periodic term $-\sin 2R/2 + (1 - \cos 2R)/4$, so it has no limit. For $s < 0$, the persistent sinusoid has positive fixed-length intervals with exponentially growing weighted integrals; the decaying time term cannot cancel it on late such intervals. Consequently the exact domain is $s > 0$.

Question 5

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider the repeated quadratic factor

$$F(s) = \frac{1}{(s^2 + 4)^2}.$$

You may use $\mathcal{L}\{\sin 2t\} = 2/(s^2 + 4)$, $\mathcal{L}\{\cos 2t\} = s/(s^2 + 4)$ and $\mathcal{L}\{tg(t)\} = -G'(s)$ for $s > 0$.

Tasks

1. Derive the transform of $t \cos 2t$ by differentiating the cosine transform.
2. Find constants A, B such that the inverse is $A \sin 2t + Bt \cos 2t$. Verify the resulting rational numerator.
3. Show that the inverse is unbounded despite having no positive exponential factor. Give an explicit sequence witnessing growth.
4. Find its leading term near $t = 0$ and its exact real convergence interval. Compare the local term with the large- s scale $F(s) \sim s^{-4}$.

Question 5 – Solution

Strategy. A repeated quadratic can be handled with parameter differentiation and linear combinations, without assuming a product rule for inverses.

Step 1: Produce the repeated denominator. Differentiating the cosine pair gives

$$\mathcal{L}\{t \cos 2t\} = -\frac{d}{ds} \frac{s}{s^2 + 4} = \frac{s^2 - 4}{(s^2 + 4)^2}.$$

Polynomial factors times the damping kernel are integrable for $s > 0$, justifying the differentiation.

Step 2: Cancel the unwanted numerator. The transform of $A \sin 2t + Bt \cos 2t$ has numerator $2A(s^2 + 4) + B(s^2 - 4)$. Requiring it to equal one gives $2A + B = 0$, $8A - 4B = 1$, hence

$$f(t) = \frac{\sin 2t - 2t \cos 2t}{16}.$$

Substitution gives $(s^2 + 4)/8 - (s^2 - 4)/8 = 1$, verifying the numerator exactly. Inverting a product by multiplying inverses would not give this result.

Step 3: Exhibit growth, not just an envelope. The bound $|f(t)| \leq (1 + 2t)/16$ is only an upper bound. Actual unboundedness follows from

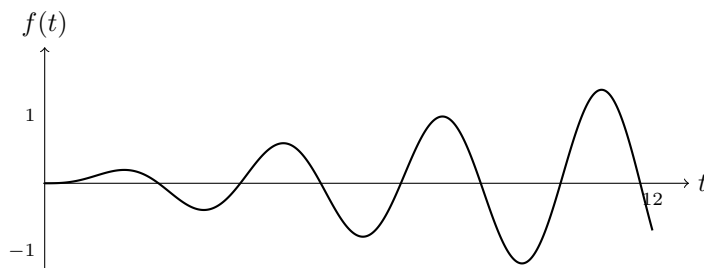
$$f(n\pi) = -n\pi/8 \longrightarrow -\infty.$$

The factor t produced by the repeated poles causes increasing oscillation size. The graph displays several oscillations at their actual scale.

Step 4: Check onset and convergence. Taylor expansion gives $\sin 2t - 2t \cos 2t = (8/3)t^3 + O(t^5)$, so $f(t) = t^3/6 + O(t^5)$. Since $\mathcal{L}\{t^3/6\} = s^{-4}$, the initial scale agrees with $F(s) \sim s^{-4}$. Absolute convergence holds for $s > 0$ by the linear envelope. At $s = 0$, direct integration gives

$$\int_0^R f(t) dt = \frac{1 - \cos 2R - R \sin 2R}{16},$$

which has no limit. For $s < 0$, intervals near successive positive peaks of $-\cos 2t$ have eventually positive integrand of order $te^{|s|t}$; their integrals fail the Cauchy criterion. Thus $s > 0$ is exact.



Question 6

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

For $a > 0$, let

$$F_a(s) = \frac{1}{(s+1)(s+a)}.$$

The two poles coincide when $a = 1$.

Tasks

1. Find the inverse for $a \neq 1$ and separately for $a = 1$. Explain why substitution into the unsimplified partial fractions fails at $a = 1$.
2. Prove the inverse is positive for every $t > 0$, and give its exact real convergence interval for all $a > 0$.
3. Derive the representation $f_a(t) = te^{-t} \int_0^1 e^{-(a-1)tv} dv$ directly from your formula, including the case $a = 1$.
4. Use the representation to prove uniform convergence $f_a \rightarrow f_1$ on the whole half-line as $a \rightarrow 1$. Obtain an explicit bound proportional to $|a - 1|$ for $1/2 \leq a \leq 3/2$.

Question 6 – Solution

Strategy. Combine the two large opposite coefficients before taking the coincident-pole limit. An integral representation controls the difference uniformly.

Step 1: Invert before and at collision. For $a \neq 1$, partial fractions give

$$F_a = \frac{1}{a-1} \left(\frac{1}{s+1} - \frac{1}{s+a} \right), \quad \boxed{f_a(t) = \frac{e^{-t} - e^{-at}}{a-1}}.$$

At $a = 1$, $F_1 = (s+1)^{-2}$, so $\boxed{f_1(t) = te^{-t}}$. Both forward transforms verify the original expression. The two individual partial-fraction coefficients diverge as $a \rightarrow 1$; their combined inverse has a finite limit.

Step 2: Check sign and the slowest tail. For $a > 1$, numerator and denominator of f_a are positive when $t > 0$; for $0 < a < 1$, both are negative. Also $te^{-t} > 0$. For unequal poles the slower exponential has a positive, nonzero coefficient, while the other term decays faster. Thus its tail is bounded below by a positive multiple of $e^{-\min(1,a)t}$. Together with direct absolute convergence, this proves the exact domain

$$\boxed{s > -\min(1, a)}.$$

At $a = 1$, the same statement follows from the positive polynomial-exponential tail.

Step 3: Remove the cancellation analytically. Integrating $e^{-(a-1)tv}$ in v gives

$$te^{-t} \int_0^1 e^{-(a-1)tv} dv = \frac{e^{-t} - e^{-at}}{a-1}$$

when $a \neq 1$ and $t > 0$. At $t = 0$ both sides extend to zero; at $a = 1$ the integral is one, yielding te^{-t} . This representation avoids subtracting nearly equal exponentials.

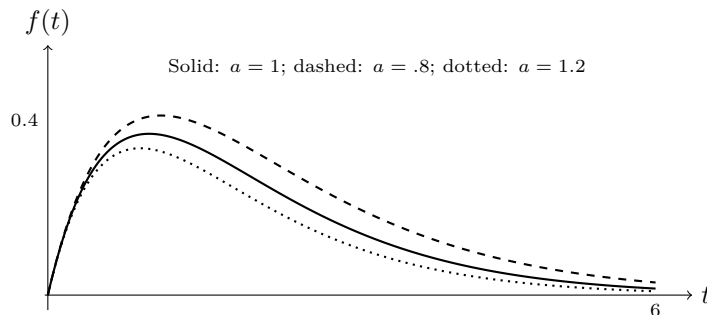
Step 4: Prove uniform half-line convergence. The mean value theorem gives $|e^{-x} - 1| \leq |x|e^{\max(0, -x)}$. With $m = \min(1, a)$, the representation yields

$$|f_a(t) - te^{-t}| \leq \frac{|a-1|}{2} t^2 e^{-mt}.$$

The maximum of $t^2 e^{-mt}$ on $t \geq 0$ occurs at $t = 2/m$ and equals $4/(m^2 e^2)$. Therefore, for $1/2 \leq a \leq 3/2$,

$$\boxed{\sup_{t \geq 0} |f_a(t) - f_1(t)| \leq \frac{8}{e^2} |a-1| \rightarrow 0}.$$

This is uniform on the entire half-line, stronger than checking each fixed time. The plotted nearby curves meet at the same zero initial value.



Question 7

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

A parameter-dependent expression is

$$F_\lambda(s) = \frac{s - 2 + \lambda}{(s - 2)(s + 1)}, \quad \lambda \in \mathbb{R}.$$

When a factor cancels, interpret its algebraic value at the canceled point by continuous extension.

Tasks

1. Find the inverse for every real λ and verify its transform.
2. Determine exactly which parameter values give a bounded inverse. State the full real convergence domain in every case.
3. Explain why the printed factor $s - 2$ alone does not establish a growing mode when $\lambda = 0$. What transform value exists at $s = 2$ in that case?
4. For $\lambda = 0.03$, find the time at which the growing and decaying contributions are equal. Explain how a small coefficient can change the eventual behavior.

Question 7 – Solution

Strategy. Residues, including their possible zeros, decide which modes are actually present. An unreduced denominator can mislead.

Step 1: Recover both mode coefficients. Write $F_\lambda = A/(s-2) + B/(s+1)$. Clearing denominators gives $A + B = 1$ and $A - 2B = -2 + \lambda$, so

$$f_\lambda(t) = \frac{\lambda}{3}e^{2t} + (1 - \frac{\lambda}{3})e^{-t}.$$

Forward transformation recombines to $(A+B)s + (A-2B) = s - 2 + \lambda$, as required, for sufficiently large s .

Step 2: Classify boundedness and domains. If $\lambda = 0$, the function is e^{-t} , bounded and decaying, with exact transform domain $s > -1$. If $\lambda \neq 0$, then $e^{-2t}f_\lambda(t) \rightarrow \lambda/3 \neq 0$. Thus it is unbounded and its weighted tail at $s \leq 2$ has an eventual fixed sign and nonintegrable magnitude. For $s > 2$, all remaining terms converge absolutely. Consequently

bounded exactly when $\lambda = 0$;

$$\begin{array}{ll} s > -1, & \lambda = 0, \\ s > 2, & \lambda \neq 0. \end{array}$$

This includes $\lambda = 3$, when the decaying term disappears.

Step 3: Interpret the canceled point. At $\lambda = 0$, cancellation gives $F_0 = 1/(s+1)$. The apparent pole at two was never a pole of the reduced transform. The actual integral at $s = 2$ is

$$\int_0^\infty e^{-2t}e^{-t} dt = 1/3.$$

The continuous algebraic extension agrees with it. The inverse is determined by the whole expression, not by a list of uncanceled denominator factors.

Step 4: Measure the delayed dominance. For $\lambda = .03$, the inverse is $.01e^{2t} + .99e^{-t}$. Equality of the contributions requires $e^{3t} = 99$, so

$$t_c = \frac{1}{3} \ln 99 \approx 1.5317.$$

The ratio of growing to decaying contributions is $e^{3t}/99$, strictly increasing through one. The initially small growing coefficient eventually dominates. At fixed finite time the inverse approaches e^{-t} as $\lambda \rightarrow 0$, but for any nonzero λ its eventual boundedness is different. That distinction does not justify discarding a small nonzero residue.

Question 8

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

An inverse is to be selected from the rational family

$$F_a(s) = \frac{as + 1}{(s + 1)^2}, \quad a \in \mathbb{R}.$$

The design requires a nonnegative continuous function with total integral one. Among such functions, one may also require it to be nonincreasing.

Tasks

1. Find the inverse for all a and verify its total integral and initial value.
2. Determine the exact set of a for which the inverse is nonnegative on $[0, \infty)$. Prove necessity as well as sufficiency.
3. Within that set, classify precisely when it is nonincreasing. For the other admissible parameters, locate the unique maximum.
4. The additional measurement $\int_0^\infty t f_a(t) dt = 3/2$ is exact. Determine a and decide whether all design requirements hold.

Question 8 – Solution

Strategy. Invert first, then turn global sign and monotonicity requirements into inequalities for a linear time factor.

Step 1: Separate the numerator at its pole. Since $as + 1 = a(s + 1) + (1 - a)$,

$$f_a(t) = e^{-t}[a + (1 - a)t].$$

Forward transformation gives $a/(s + 1) + (1 - a)/(s + 1)^2 = F_a$. The exact domain is $s > -1$ for every a : the nonzero linear or constant polynomial tail has eventual fixed sign and diverges at or below -1 . Thus $s = 0$ is allowed, and

$$\int_0^\infty f_a(t) dt = F_a(0) = 1, \quad f_a(0) = a.$$

Unit signed integral alone does not establish nonnegativity.

Step 2: Classify nonnegative inverses. Because $e^{-t} > 0$, the sign is that of $a + (1 - a)t$. Nonnegativity at zero requires $a \geq 0$. If $a > 1$, the negative slope makes the factor negative for sufficiently large t . Conversely, $0 \leq a \leq 1$ makes both its intercept and slope nonnegative. Therefore the exact admissible set is $0 \leq a \leq 1$.

Step 3: Distinguish monotone and peaked designs. For admissible a ,

$$f'_a(t) = e^{-t}[1 - 2a - (1 - a)t].$$

The bracket is nonincreasing in t , so it is nonpositive everywhere exactly when its initial value is nonpositive. Hence the nonincreasing designs have $1/2 \leq a \leq 1$. For $0 \leq a < 1/2$, the derivative changes from positive to negative at

$$t_* = \frac{1 - 2a}{1 - a}, \quad f_a(t_*) = (1 - a) \exp\left(-\frac{1 - 2a}{1 - a}\right).$$

This also covers $a = 0$, whose maximum occurs at one.

Step 4: Use the extra integral to select a design. Direct exponential moments give

$$\int_0^\infty t f_a(t) dt = a + 2(1 - a) = 2 - a.$$

Equating this to $3/2$ yields $a = 1/2$ and $f(t) = (1 + t)e^{-t}/2$. This is nonnegative, has unit integral, and is nonincreasing: $f'(t) = -te^{-t}/2 \leq 0$. Its derivative vanishes only at the initial endpoint. The measurement selects one member, rather than allowing the initial value to be prescribed independently.

Question 9

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

Consider

$$G(s) = \frac{s^2 + 2s + 2}{(s + 1)^2}.$$

Work only with ordinary integrals and continuous functions of exponential order. You may use that such a function has a global bound $|g(t)| \leq Me^{ct}$ on $t \geq 0$ for some $M > 0$ and real c .

Tasks

1. Divide the polynomials to separate G into its polynomial part and a strictly proper rational part.
2. Prove from the stated bound that every allowed ordinary transform tends to zero as $s \rightarrow \infty$. Decide whether G has an inverse in the stated class.
3. Find the inverse h of the strictly proper remainder and verify it directly. Explain why reporting h as the inverse of the whole G would be incorrect.
4. Define $k(t) = h(t)$ except that $k(1) = 7$. Do k and h have the same ordinary transform? Explain why this does not contradict uniqueness in the stated class.

Question 9 – Solution

Strategy. Check whether an ordinary inverse can exist before matching a rational remainder to a table. Keep the continuity hypothesis in any uniqueness claim.

Step 1: Divide before attempting inversion. Because $(s+1)^2 = s^2 + 2s + 1$,

$$G(s) = 1 + \frac{1}{(s+1)^2}.$$

The nonzero constant part cannot be silently discarded. In particular $G(s) \rightarrow 1$, rather than zero, at large positive s .

Step 2: Apply a necessary existence test. For an allowed g and $s > c$, absolute convergence and the global bound yield

$$|\mathcal{L}\{g\}(s)| \leq \int_0^\infty e^{-st} |g(t)| dt \leq M \int_0^\infty e^{-(s-c)t} dt = \frac{M}{s-c} \rightarrow 0.$$

A continuous function of exponential order admits this global bound by enlarging M on an initial compact interval. Since G has limit one, G has no inverse in the stated class. This conclusion concerns ordinary continuous functions under the explicit hypotheses.

Step 3: Invert only the actual remainder. The strictly proper part has inverse $h(t) = te^{-t}$. Direct integration by parts gives

$$\int_0^\infty te^{-(s+1)t} dt = \frac{1}{(s+1)^2}, \quad s > -1.$$

The integral diverges at or below -1 . Its transform is $G - 1$, not G . The missing constant does not arise from changing $h(0)$: changing one finite point value changes no ordinary integral.

Step 4: State uniqueness with its hypothesis. Here $h(1) = e^{-1} \neq 7$. Changing that one value leaves every finite-interval integral, and hence every convergent improper transform, unchanged. Thus k and h have identical transforms on $s > -1$. However, k is discontinuous at one, so it is not in the continuous class specified in the question. There is no contradiction: uniqueness within that class compares two continuous functions. The example separates equality of transforms from unrestricted pointwise equality. No generalized transform is needed to reach any of these conclusions.

Question 10

Use the ordinary one-sided Laplace transform for real s . Seek an inverse continuous on $t \geq 0$ and of exponential order; transforms agreeing for all sufficiently large s have at most one inverse in this class.

A transform has the form

$$F(s) = \frac{As^2 + Bs + C}{(s+1)(s+2)(s+3)}.$$

Its inverse satisfies $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 6$. Assume f is three times continuously differentiable and $f, f', f'', f^{(3)}$ are of exponential order, so repeated integration by parts is available.

Tasks

1. Derive the expansion $F(s) = f(0)/s + f'(0)/s^2 + f''(0)/s^3 + O(s^{-4})$. Use it to determine A, B, C .
2. Find the inverse by partial fractions and verify all three prescribed initial quantities.
3. Factor the inverse to prove its sign on $t \geq 0$ and determine its exact real convergence interval.
4. Find its sharp maximum value and the unique time at which it is attained. Explain why the initial data determine the numerator here but would not determine an arbitrary function.

Question 10 – Solution

Strategy. Initial data determine large-parameter coefficients. The fixed denominator converts those coefficients into a finite algebraic reconstruction.

Step 1: Recover the missing numerator. Three integrations by parts, with vanishing upper boundary terms for sufficiently large s , give

$$F(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{1}{s^3} \mathcal{L}\{f^{(3)}\}(s).$$

The last transform is $O(1/s)$ by an exponential bound. Since the denominator is $s^3 + 6s^2 + 11s + 6$, expansion of the rational expression gives

$$F(s) = \frac{A}{s} + \frac{B - 6A}{s^2} + \frac{C - 6B + 25A}{s^3} + O(s^{-4}).$$

The prescribed values imply $A = 0, B = 0, C = 6$.

Step 2: Resolve the three poles and check. Evaluation of the cleared identity at $s = -1, -2, -3$ gives coefficients 3, -6, 3. Hence

$$F = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3}, \quad f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}.$$

Their forward numerator is six. The initial derivative sums are $3 - 6 + 3 = 0$, $-3 + 12 - 9 = 0$, and $3 - 24 + 27 = 6$, verifying all data.

Step 3: Factor to determine the sign and tail. The useful form is $f(t) = 3e^{-t}(1 - e^{-t})^2$. It is zero at $t = 0$ and strictly positive for $t > 0$. Its tail is asymptotic to $3e^{-t}$, so the exact real convergence interval is $s > -1$: above it all terms converge; at or below it the eventually positive weighted tail has infinite integral.

Step 4: Find the sharp peak and scope of reconstruction. Put $u = e^{-t}$, so $0 < u \leq 1$ and $f = 3u(1 - u)^2$. Its derivative with respect to u is $3(1 - u)(1 - 3u)$, so the unique interior maximum occurs at $u = 1/3$:

$$t_* = \ln 3, \quad \max_{t \geq 0} f(t) = 4/9.$$

The endpoint values are zero (at $t = 0$ and as $t \rightarrow \infty$). The fixed three-parameter numerator made the reconstruction unique. Adding $t^3 e^{-t}$ to this inverse preserves all three initial data but changes its transform and leaves the specified rational family. Thus three initial quantities alone do not determine an arbitrary function.

