

Question 1

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $f(t) = \sin^2(3t)$. A student proposes

$$\mathcal{L}\{\sin^2(3t)\}(s) = [\mathcal{L}\{\sin 3t\}(s)]^2.$$

You may use $\mathcal{L}\{1\} = 1/s$, $\mathcal{L}\{\cos bt\} = s/(s^2 + b^2)$ and $\mathcal{L}\{\sin bt\} = b/(s^2 + b^2)$ for $s > 0$.

Tasks

1. Find the correct transform using a trigonometric identity and linearity.
2. Determine the full real convergence set and compare the correct result with the student's proposed expression.
3. Use $|\sin x| \leq |x|$ to prove a direct upper bound for the transform when $s > 0$.
4. Use $u = st$ in the defining integral to derive $\lim_{s \rightarrow \infty} s^3 F(s)$ independently of the rational formula. Explain how this limit diagnoses the proposed product rule.

Question 1 – Solution

Strategy. Linearity applies to sums and scalar multiples, not to pointwise products. A large-parameter check can expose the distinction.

Step 1: Rewrite the function before transforming. Since $\sin^2(3t) = (1 - \cos 6t)/2$, linearity gives

$$F(s) = \frac{1}{2s} - \frac{s}{2(s^2 + 36)} = \boxed{\frac{18}{s(s^2 + 36)}}, \quad s > 0.$$

Step 2: Check convergence and the proposed rule. For $s > 0$, boundedness of f gives absolute convergence. For $s \leq 0$, the weight is at least one and $\sin^2(3t)$ has positive integral on each period, so the nonnegative improper integral diverges. Thus the real convergence set is exactly $(0, \infty)$. The proposed expression is

$$\frac{9}{(s^2 + 9)^2},$$

which is not the correct transform. For example, at $s = 3$ the correct value is $2/15$, while the proposed value is $1/36$. Squaring a time function is not transformed by squaring its transform.

Step 3: Obtain a bound from the integral. The inequality $0 \leq \sin^2(3t) \leq 9t^2$ yields

$$0 < F(s) \leq 9 \int_0^\infty t^2 e^{-st} dt = \boxed{18/s^3}.$$

The last integral is $2/s^3$, by two integrations by parts. Its boundary terms vanish for $s > 0$.

Step 4: Recover the leading scale independently. With $u = st$,

$$s^3 F(s) = \int_0^\infty e^{-u} s^2 \sin^2(3u/s) du.$$

The integrand tends to $9u^2 e^{-u}$ and is bounded by that integrable function. Dominated convergence therefore gives $\boxed{s^3 F(s) \rightarrow 18}$. This reflects the local behavior $\sin^2(3t) \sim 9t^2$. In contrast, multiplying the proposed expression by s^3 gives a limit of zero. The proposed rule even predicts the wrong leading power of $1/s$.

Question 2

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Find the transform of $f(t) = t^2 e^{-2t} \cos 3t$ by combining exponential shifting with differentiation in the transform parameter. You may use $C(p) = \mathcal{L}\{\cos 3t\}(p) = p/(p^2 + 9)$ for $p > 0$.

Tasks

1. Justify the identity $\mathcal{L}\{t^2 g(t)\}(s) = G''(s)$ in this example, including the convergence condition needed for differentiation.
2. Compute and simplify the transform explicitly.
3. Find all its zeros and determine its sign within the convergence interval. Explain why a zero transform value need not mean that the original function is zero.
4. The simplified expression also has value zero at $s = -2$. Does that value represent an ordinary Laplace integral? Check the boundary directly and determine whether any smaller real parameter can work.

Question 2 – Solution

Strategy. Account for the exponential shift first, then differentiate the correct parameter expression and retain its convergence domain.

Step 1: Justify parameter differentiation. The factor e^{-2t} changes the effective parameter to $p = s + 2$. For any $p_0 > 0$, a sufficiently small neighborhood has $p \geq p_0/2$. Differentiating the kernel twice produces $t^2 e^{-pt} \cos 3t$, dominated by $t^2 e^{-p_0 t/2}$, an integrable function. The first derivative is justified similarly. Thus, for $s > -2$, the desired transform is $C''(s + 2)$.

Step 2: Differentiate and simplify. Direct calculation gives

$$C'(p) = \frac{9 - p^2}{(p^2 + 9)^2}, \quad C''(p) = \frac{2p(p^2 - 27)}{(p^2 + 9)^3}.$$

Therefore

$$F(s) = \frac{2(s + 2)[(s + 2)^2 - 27]}{[(s + 2)^2 + 9]^3}, \quad s > -2.$$

The exponential times a polynomial gives absolute convergence on this interval.

Step 3: Interpret its zero and sign. For $p > 0$, the denominator and $2p$ are positive. Hence the only zero in the convergence interval is $s = 3\sqrt{3} - 2$. The transform is negative below that value and positive above it. At the zero, weighted positive and negative portions of $t^2 e^{-2t} \cos 3t$ cancel; the function itself is certainly not identically zero.

Step 4: Test the boundary rather than the rational expression. At $s = -2$, the integral becomes $\int_0^R t^2 \cos 3t \, dt$. Integration by parts gives

$$\frac{R^2 \sin 3R}{3} + \frac{2R \cos 3R}{9} - \frac{2 \sin 3R}{27}.$$

At $R_n = 2n\pi/3$, this equals $2R_n/9 \rightarrow \infty$. For $p = s + 2 < 0$, consider $[R_n, R_n + \pi/9]$, where $\cos 3t \geq 1/2$. The integral on that interval is at least $(\pi/18)R_n^2 e^{-pR_n}$ and does not tend to zero, violating the Cauchy criterion. Thus no real $s \leq -2$ works; the algebraic boundary value is not a transform value.

Question 3

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $g(t) = e^{-t} \sin 2t$, with $G(s) = 2/[(s+1)^2 + 4]$ for $s > -1$. Define the compressed signals $f(t) = g(3t)$ and $h(t) = 3g(3t)$.

Tasks

1. Derive the time-scaling identity for $g(at)$, $a > 0$, by substitution in the defining integral. Track the convergence parameter.
2. Find the transforms of f and h , and their full real convergence intervals. Explain why replacing $G(s)$ by $G(3s)$ gives the wrong scaling.
3. Compute the signed total integrals of g, f, h . Which compression preserves the total integral?
4. Use parameter differentiation to derive the first-moment identity. Compute $\int_0^\infty t f(t) dt$ and $\int_0^\infty t h(t) dt$. Explain their different scaling factors.

Question 3 – Solution

Strategy. Time compression changes both the exponential weight and the integration measure; amplitude compensation affects area separately.

Step 1: Derive the scaling rule. For $a > 0$, substituting $u = at$ gives

$$\int_0^\infty e^{-st} g(at) dt = \frac{1}{a} \int_0^\infty e^{-(s/a)u} g(u) du = \boxed{\frac{1}{a} G(s/a)}.$$

The effective parameter is s/a , so $s/a > -1$ here. Both the reciprocal parameter scaling and the factor $1/a$ are essential.

Step 2: Apply the rule to both signals. For $a = 3$,

$$\boxed{F(s) = \frac{6}{(s+3)^2 + 36}, \quad H(s) = \frac{18}{(s+3)^2 + 36}, \quad s > -3.}$$

These are the transforms of $e^{-3t} \sin 6t$ and three times that signal. At the boundary the undamped sine integral fails to converge; below it the exponentially growing sine fails the Cauchy criterion on positive half-wave subintervals. Thus $s > -3$ is exact. $G(3s)$ would stretch the parameter in the opposite direction and omit the measure factor.

Step 3: Compare signed areas. Since zero lies in the convergence intervals, substitution of $s = 0$ gives

$$\int_0^\infty g = \frac{2}{5}, \quad \int_0^\infty f = \frac{2}{15}, \quad \int_0^\infty h = \frac{2}{5}.$$

Compression alone reduces signed area by three; multiplying the amplitude by three restores it. These are signed integrals, not integrals of absolute value.

Step 4: Track first moments. Exponential decay justifies differentiation near $s = 0$, so $G'(s) = -\int_0^\infty te^{-st} g(t) dt$. As $G'(0) = -4/25$, the original first moment is $4/25$. Differentiating $F(s) = G(s/3)/3$ and $H(s) = G(s/3)$ yields

$$\boxed{\int_0^\infty tf(t) dt = \frac{4}{225}, \quad \int_0^\infty th(t) dt = \frac{4}{75}.$$

For f , one factor $1/3$ comes from dt and another from $t = u/3$. The amplitude factor in h cancels only one of them.

Question 4

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $g(t) = e^{-t} \cos t$ and $y(t) = \int_0^t g(u) du$. The transform of g is $G(s) = (s+1)/[(s+1)^2 + 1]$ for $s > -1$.

Tasks

1. Derive the transform rule for y by integration by parts, showing where the zero initial value and the boundary at infinity enter.
2. Find an elementary expression for $y(t)$, its transform $Y(s)$, and the exact real convergence interval of Y .
3. Set $z(t) = y(t) - 1/2$. Find its transform and convergence interval, including its value at $s = 0$. Explain why cancellation can enlarge the interval.
4. Find the first maximum of y and its height. Does integration of this exponentially decaying input produce an exponentially decaying or monotone cumulative response?

Question 4 – Solution

Strategy. A cumulative integral may retain a nonzero constant even when its input decays. Its transform domain must be checked anew.

Step 1: Derive the cumulative rule. Here $y(0) = 0$, $y' = g$, and y is bounded because g is absolutely integrable. For $s > 0$, integration by parts gives

$$G(s) = [e^{-st}y(t)]_0^\infty + s \int_0^\infty e^{-st}y(t) dt = sY(s).$$

The initial term is zero and the upper term vanishes because y is bounded. Thus $Y = G/s$ on this half-line.

Step 2: Determine the response and its domain. An antiderivative gives

$$y(t) = \frac{1 + e^{-t}(\sin t - \cos t)}{2}, \quad Y(s) = \frac{s+1}{s[(s+1)^2 + 1]}, \quad s > 0.$$

The formula satisfies $y(0) = 0$ and $y' = g$. Since $y(t) \rightarrow 1/2$, it is eventually bounded below by a positive constant. Its transform therefore diverges for every $s \leq 0$. The cumulative response has a smaller convergence interval than the input.

Step 3: Remove the persistent constant. Now $z = e^{-t}(\sin t - \cos t)/2$. Transforming these two terms directly gives

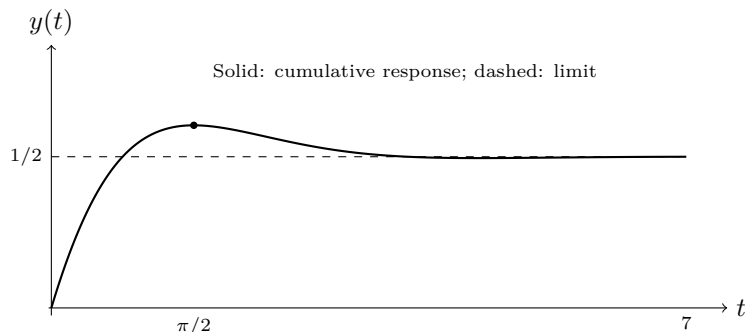
$$Z(s) = -\frac{s}{2[(s+1)^2 + 1]}, \quad s > -1.$$

It converges absolutely there; at and below -1 the nondecaying or growing trigonometric tail prevents ordinary convergence. In particular, $Z(0) = 0$: its signed areas cancel. The identity $Z = Y - 1/(2s)$ is initially valid only for $s > 0$. The wider domain follows from the actual centered function, not by subtracting divergent integrals at $s = 0$.

Step 4: Locate the overshoot. Since $y' = e^{-t} \cos t$, the first positive turning time is $t = \pi/2$, with derivative changing from positive to negative. Its height is

$$y(\pi/2) = \frac{1}{2}(1 + e^{-\pi/2}) > \frac{1}{2}.$$

The response overshoots and oscillates about its limiting constant, so it is neither monotone nor decaying to zero. Its centered part does decay exponentially.



Question 5

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let $f(t) = e^{-t}(1+t)$. A student claims that the second derivative has transform $s^2 F(s)$, without any initial terms.

Tasks

1. Derive the first- and second-derivative transform formulas using integration by parts on a finite interval before taking the limit.
2. Compute $F(s)$, $f'(t)$ and $f''(t)$, and obtain their transforms explicitly.
3. Identify the omitted term in the student's claim and verify the corrected result independently from the expression for f'' .
4. Evaluate the transforms of f' and f'' at $s = 0$ and check them using endpoint differences. Explain how a nonzero second derivative can have zero total signed integral.

Question 5 – Solution

Strategy. Differentiation in time creates boundary terms. Keep them until the initial data and exponential decay have been used.

Step 1: Retain the boundaries. Integration by parts on $[0, R]$ gives

$$\int_0^R e^{-st} f'(t) dt = e^{-sR} f(R) - f(0) + s \int_0^R e^{-st} f(t) dt.$$

Repeating for f'' and then taking $R \rightarrow \infty$ yields

$$\mathcal{L}\{f'\} = sF - f(0), \quad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0),$$

provided the upper boundary terms vanish. Here they do for every $s > -1$, since f, f' are polynomials times e^{-t} .

Step 2: Compute the functions and transforms. Using elementary exponential integrals,

$$F(s) = \frac{1}{s+1} + \frac{1}{(s+1)^2} = \frac{s+2}{(s+1)^2}.$$

Direct differentiation gives $f' = -te^{-t}$ and $f'' = (t-1)e^{-t}$, with $f(0) = 1$ and $f'(0) = 0$. Hence

$$\boxed{\mathcal{L}\{f'\} = -\frac{1}{(s+1)^2}, \quad \mathcal{L}\{f''\} = -\frac{s}{(s+1)^2}, \quad s > -1.}$$

These intervals are exact: at or below -1 , the resulting polynomial or growing exponential tails do not have convergent improper integrals.

Step 3: Repair and check the claim. The omitted initial contribution is $-sf(0) - f'(0) = -s$. Thus the corrected expression is $s^2F - s = -s/(s+1)^2$. Independently, transforming $(t-1)e^{-t}$ gives

$$\frac{1}{(s+1)^2} - \frac{1}{s+1} = -\frac{s}{(s+1)^2}.$$

This agreement verifies both the boundary terms and the algebra.

Step 4: Check zero-parameter integrals. At zero, the first-derivative transform equals -1 , consistent with $\int_0^\infty f' = f(\infty) - f(0) = 0 - 1$. The second-derivative transform is zero, consistent with $\int_0^\infty f'' = f'(\infty) - f'(0) = 0$. The function f'' is negative before $t = 1$ and positive afterward; its two signed contributions cancel. Zero integral is not the same as a zero function.

Question 6

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

For $t > 0$, define

$$h(t) = \frac{e^{-t} - e^{-4t}}{t},$$

and extend it continuously to $t = 0$. This example concerns division by time; do not assume a transform rule for dividing an arbitrary function by t .

Tasks

1. Find $h(0)$ and express $h(t)$ as an integral of e^{-ut} over a finite interval in u .
2. Use that representation to derive the transform $H(s)$ and justify the order of integration.
3. Find the exact real convergence interval. Explain why transforming e^{-t}/t and e^{-4t}/t separately is invalid.
4. Find $H'(s)$ and $\lim_{s \rightarrow \infty} sH(s)$. Relate them to positivity of h and its value at zero.

Question 6 – Solution

Strategy. Cancellation removes the origin singularity. Preserve that cancellation by integrating over an exponential parameter.

Step 1: Remove the apparent singularity. Expansion at zero, or l'Hopital's rule, gives $\boxed{h(0) = 3}$. Direct integration in u gives

$$h(t) = \int_1^4 e^{-ut} du,$$

including at $t = 0$ by continuity. In particular, h is positive and bounded above by three.

Step 2: Derive the logarithmic transform. For $s > -1$, $s + u > 0$ throughout $[1, 4]$. The absolute double integral is finite, as seen by integrating first in t :

$$H(s) = \int_1^4 \int_0^\infty e^{-(s+u)t} dt du = \int_1^4 \frac{du}{s+u} = \boxed{\ln \frac{s+4}{s+1}}, \quad s > -1.$$

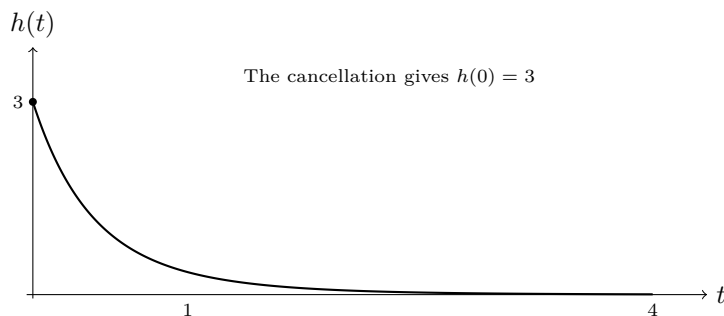
The finite value justifies interchange; both logarithm arguments are positive on this domain.

Step 3: Check the domain and invalid separation. At $s = -1$, the integrand is $(1 - e^{-3t})/t$, whose tail is at least $1/(2t)$ for sufficiently large t . It diverges. For smaller s , a larger positive exponential weight makes divergence persist. Thus $s > -1$ is the full real convergence interval. Each separate term e^{-t}/t or e^{-4t}/t has a nonintegrable origin singularity for every s . Their divergent transforms cannot be subtracted; it is the difference of the functions that is transformable.

Step 4: Interpret derivative and initial scale. Differentiating the logarithm gives

$$\boxed{H'(s) = \frac{1}{s+4} - \frac{1}{s+1} = -\frac{3}{(s+1)(s+4)} < 0.}$$

This agrees with $H'(s) = -\int_0^\infty te^{-st}h(t) dt$, since $h > 0$. Also $H(s) = \ln[1 + 3/(s+1)]$, so $\ln(1+x)/x \rightarrow 1$ implies $\boxed{sH(s) \rightarrow 3 = h(0)}$. The graph displays the continuous signal; its apparent division-by-zero point has been correctly filled in.



Question 7

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

An unknown signal belongs to the family $f(t) = e^{-at} \cos bt$, with $a \geq 0$ and $b > 0$. Its transform satisfies the exact observations

$$\lim_{s \rightarrow \infty} s[sF(s) - 1] = -2, \quad \lim_{s \rightarrow \infty} s^3[F(s) - 1/s + 2/s^2] = -5.$$

Tasks

1. Derive the transform in terms of a, b and expand it through the term in s^{-3} .
2. Recover a, b from the observations and state the resulting transform with its convergence interval.
3. Compute $f(0)$, $f'(0)$ and $f''(0)$ directly. Explain how the expansion coefficients encode these initial quantities.
4. For $q(t) = t^3 e^{-t}$, compute its transform and show that $f + \lambda q$ has the same two observed limits for every real λ . Explain the limit of the parameter recovery claim.

Question 7 – Solution

Strategy. Large transform parameters emphasize early time. Within a specified family, the first few coefficients can determine parameters, but they do not determine an arbitrary signal.

Step 1: Transform and expand. Exponential shifting in the cosine pair gives

$$F(s) = \frac{s+a}{(s+a)^2 + b^2}, \quad s > -a.$$

Expanding this rational expression for large positive s yields

$$F(s) = \frac{1}{s} - \frac{a}{s^2} + \frac{a^2 - b^2}{s^3} + O(s^{-4}).$$

For example, factor s^2 from the denominator and expand its reciprocal through order s^{-2} before multiplying by $1/s + a/s^2$.

Step 2: Recover the parameters. The first limit equals $-a$, so $a = 2$. With this value, the second equals $a^2 - b^2 = -5$, giving $b^2 = 9$. Since $b > 0$,

$$a = 2, \quad b = 3, \quad F(s) = \frac{s+2}{(s+2)^2 + 9}, \quad s > -2.$$

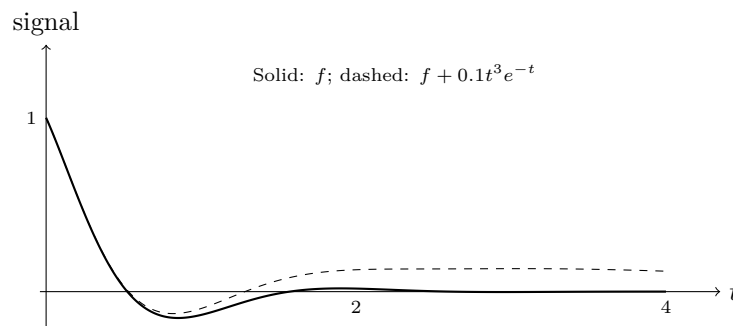
The boundary has an undamped cosine integral that does not converge; smaller parameters produce growing oscillations. Thus the interval is exact.

Step 3: Read the initial information. Direct differentiation gives

$$f(0) = 1, \quad f'(0) = -a = -2, \quad f''(0) = a^2 - b^2 = -5.$$

These are precisely the coefficients of $1/s$, $1/s^2$ and $1/s^3$ in the expansion. Repeated integration by parts explains the same pattern when the needed derivative bounds hold.

Step 4: Show what the measurements cannot identify. Three integrations by parts give $\mathcal{L}\{q\}(s) = 6/(s+1)^4$ for $s > -1$. Thus $f + \lambda q$ has transform $F(s) + 6\lambda/(s+1)^4$ on their common domain. The added term is $O(s^{-4})$, so it changes neither observed limit. Also $q(0) = q'(0) = q''(0) = 0$. Distinct λ therefore give different signals with the same recorded information. The parameters are uniquely recovered only within the stated exponential-cosine family.



Question 8

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Suppose f is continuous, bounded and nonnegative on $[0, \infty)$. Consider the candidate expression

$$R(s) = \frac{s}{(s+1)^2}, \quad s > 0.$$

Tasks

1. Derive the sign requirements on $F(s)$, $F'(s)$ and $F''(s)$ imposed by these hypotheses. State when $F(s) > 0$.
2. Although $R(s) > 0$ for every $s > 0$, prove that it cannot be the transform of such a nonnegative f .
3. Construct a continuous bounded sign-changing function g whose transform is R , using basic exponential pairs. Verify its sign change and full real convergence interval.
4. Compute $\int_0^\infty g(t) dt$ and explain why positivity of a transform throughout $s > 0$ does not imply positivity of its original function.

Question 8 – Solution

Strategy. Positivity of the transform is only one consequence of a nonnegative signal; parameter derivatives supply stronger necessary tests.

Step 1: Derive the necessary signs. If $0 \leq f \leq M$, then the kernels obtained by differentiating once or twice are dominated locally in $s > 0$ by constant multiples of $te^{-s_0 t/2}$ and $t^2 e^{-s_0 t/2}$. Differentiation under the integral is therefore justified, and

$$F(s) \geq 0, \quad F'(s) = - \int_0^\infty t e^{-st} f(t) dt \leq 0,$$

$$F''(s) = \int_0^\infty t^2 e^{-st} f(t) dt \geq 0.$$

If f is not identically zero, continuity gives a positive interval contribution and hence $F(s) > 0$ for every $s > 0$.

Step 2: Test the positive candidate. Although $R(s) > 0$,

$$R'(s) = \frac{1-s}{(s+1)^3} > 0 \quad (0 < s < 1).$$

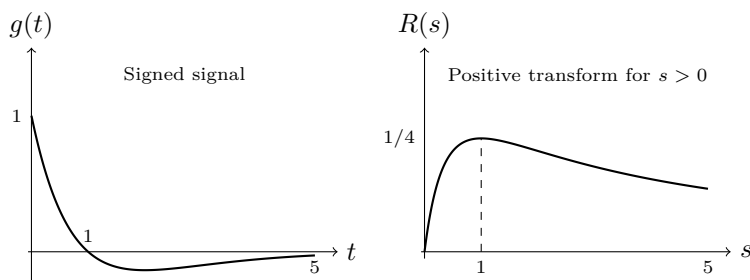
This contradicts the required nonpositive derivative. Thus no continuous bounded nonnegative signal can have this transform.

Step 3: Find a valid signed signal. Rewrite $R = 1/(s+1) - 1/(s+1)^2$. The basic exponential integrals give

$$g(t) = e^{-t}(1-t), \quad \mathcal{L}\{g\}(s) = R(s), \quad s > -1.$$

The function is positive for $0 \leq t < 1$, zero at one, and negative for $t > 1$. It is continuous and bounded. At or below $s = -1$, its eventually negative polynomial or growing tail has no finite improper integral, so the domain is exact.

Step 4: Explain the weighted cancellation. At $s = 0$, the integral exists and equals $R(0) = 0$, also verified by $\int e^{-t} = 1$ and $\int t e^{-t} = 1$. For positive s , the extra weight suppresses the late negative portion more strongly, producing a positive transform. Consequently $F > 0$ alone is insufficient to infer $f \geq 0$. The two plots show a sign-changing signal and its everywhere-positive transform on $s > 0$.



Question 9

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

A nonnegative density on $t \geq 0$ has the form $f(t) = cte^{-at}$ with $a > 0$, $c > 0$ and $\int_0^\infty f(t) dt = 1$. Define its mean by $\mu = \int_0^\infty tf(t) dt$ and variance by $\sigma^2 = \int_0^\infty t^2 f(t) dt - \mu^2$. An exact measurement gives $\mu = 4$.

Tasks

1. Determine the relation between a, c from normalization and compute the transform.
2. Derive the first two moment identities using parameter differentiation. Recover a, c and the variance from the measured mean.
3. Could this same family have mean 4 and variance 4? Give an exact consistency relation for arbitrary positive mean.
4. Find the location of the density maximum for the recovered parameters and compare it with the mean. Explain why those locations differ.

Question 9 – Solution

Strategy. Normalization fixes one parameter, while derivatives of the transform at zero recover moments. A density's peak is not its mean.

Step 1: Normalize and transform. Integration by parts gives $\int_0^\infty te^{-at} dt = 1/a^2$. Thus $c = a^2$, and exponential shifting yields

$$F(s) = \frac{a^2}{(s+a)^2}, \quad s > -a.$$

The nonnegative polynomial-exponential tail diverges at or below $-a$. In particular, $F(0) = 1$, as normalization requires.

Step 2: Extract moments and parameters. Since zero lies strictly inside the convergence interval, exponential decay justifies differentiation near zero:

$$-F'(0) = \int_0^\infty tf(t) dt, \quad F''(0) = \int_0^\infty t^2 f(t) dt.$$

The rational expression gives $F'(s) = -2a^2/(s+a)^3$ and $F''(s) = 6a^2/(s+a)^4$. Hence

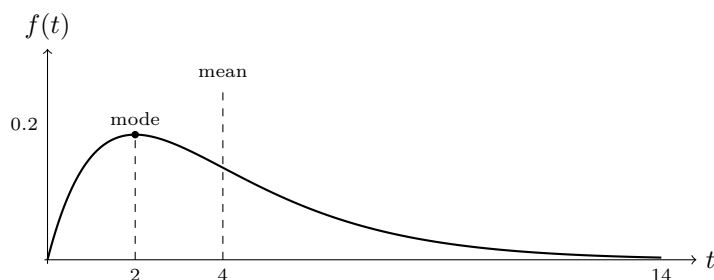
$$\mu = \frac{2}{a}, \quad \sigma^2 = \frac{6}{a^2} - \frac{4}{a^2} = \frac{2}{a^2}.$$

From $\mu = 4$, we obtain

$$a = \frac{1}{2}, \quad c = \frac{1}{4}, \quad \sigma^2 = 8, \quad F(s) = \frac{1}{(2s+1)^2}.$$

Step 3: Test consistency rather than refitting. Eliminating a gives $\sigma^2 = \mu^2/2$. Thus mean four forces variance eight, and the pair $(4, 4)$ is incompatible with this family. Changing c cannot repair it because normalization already fixes $c = a^2$.

Step 4: Locate the peak. For general a , $f'(t) = a^2e^{-at}(1-at)$, positive before $t = 1/a$ and negative afterward. The unique density maximum is therefore at $t = 1/a = 2$, while the mean is four. The long right tail contributes to the mean; the point of largest density measures a different feature. The graph marks both the mode and the mean for $f(t) = te^{-t/2}/4$.



Question 10

For real s , write $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t) dt$ where the integral converges.

Let f be twice continuously differentiable, with f, f', f'' of exponential order. Work at sufficiently large real s so that all transforms below converge and all exponential boundary terms vanish. You may use

$$\mathcal{L}\{f'\} = sF - f(0), \quad \mathcal{L}\{tf\} = -F', \quad \mathcal{L}\{f''\} = s^2F - sf(0) - f'(0).$$

Tasks

1. Compute $\mathcal{L}\{(tf)'\}$ and $\mathcal{L}\{tf'\}$ separately. Explain their difference using the product rule in time.
2. Verify both results explicitly for $f(t) = e^{-t}$.
3. Derive $\mathcal{L}\{t^2 f''\}$, carefully differentiating every s -dependent factor.
4. A student keeps only $s^2 F''$ in the last result. Identify the missing terms and verify the correct formula for $f = e^{-t}$ by transforming $t^2 e^{-t}$ directly.

Question 10 – Solution

Strategy. Multiplication by time and differentiation in time correspond to operations in s that must be applied in the correct order.

Step 1: Compare the two orders. Since $(tf)(0) = 0$, applying the time-derivative rule after time multiplication gives

$$\boxed{\mathcal{L}\{(tf)'\} = -sF'(s)}.$$

In the opposite order, parameter differentiation applies to the entire transform of f' :

$$\boxed{\mathcal{L}\{tf'\} = -\frac{d}{ds}[sF(s) - f(0)] = -F(s) - sF'(s)}.$$

Their difference is $F(s)$, exactly as required by $(tf)' - tf' = f$. The initial value $f(0)$ is constant in s and differentiates to zero.

Step 2: Check a concrete example. For $f = e^{-t}$, $F = 1/(s+1)$ and $F' = -1/(s+1)^2$. The formulas predict

$$\mathcal{L}\{(1-t)e^{-t}\} = \frac{s}{(s+1)^2}, \quad \mathcal{L}\{-te^{-t}\} = -\frac{1}{(s+1)^2}.$$

Direct use of the exponential and te^{-t} integrals verifies both, for $s > -1$. Their difference is $1/(s+1) = F$.

Step 3: Differentiate the complete expression twice. Multiplication by t^2 corresponds to two parameter derivatives with positive sign. Thus

$$\mathcal{L}\{t^2 f''\} = \frac{d^2}{ds^2}[s^2 F - sf(0) - f'(0)] = \boxed{2F + 4sF' + s^2 F''}.$$

The first derivative of $s^2 F$ is $2sF + s^2 F'$; differentiating again produces both product-rule contributions. The initial terms are at most linear in s , so their second derivatives vanish.

Step 4: Diagnose and verify. The proposed $s^2 F''$ omits $2F + 4sF'$. For $f = e^{-t}$, the correct result simplifies to

$$\frac{2}{s+1} - \frac{4s}{(s+1)^2} + \frac{2s^2}{(s+1)^3} = \boxed{\frac{2}{(s+1)^3}}.$$

Since $f'' = e^{-t}$, direct integration of $t^2 e^{-(s+1)t}$ gives this same value for $s > -1$. Differentiating only F while treating its prefactors as constants is the source of the error.