

Question 1

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A draft transform table contains these three proposed pairs, with $a > 0$:

$$t^3 e^{-2t} \longleftrightarrow \frac{1}{(s+2)^4}, \quad e^{-at} \sin(3t) \longleftrightarrow \frac{s+a}{(s+a)^2+9}, \quad f(2t) \longleftrightarrow F(s/2).$$

In the third pair $F = \mathcal{L}\{f\}$, where f is continuous and of exponential order.

Tasks

1. Identify and correct every error. Derive the power entry by integration by parts and the scaling entry by substitution.
2. State the exact real convergence domains for the first two corrected entries. State the sufficient transformed domain for the third if F converges absolutely for $s > c$.
3. Use the corrected rules to find the transform of $t^3 e^{-2t} + e^{-at} \sin(3t)$, without a fresh integration.
4. Check your answer independently using the leading large- s behavior and the value at $s = 0$. Explain what each check can and cannot establish.

Question 1 – Solution

Strategy. A table entry includes its normalization and domain, not only its denominator.

Step 1: Repair the entries. For $p > 0$, integration by parts gives $I_n(p) = nI_{n-1}(p)/p$, $I_0(p) = 1/p$, with vanishing boundary terms. Thus $I_3(p) = 6/p^4$. The sine entry has frequency in its numerator. Substitution $u = 2t$ in the transform supplies the missing Jacobian. The corrected pairs are

$$t^3 e^{-2t} \leftrightarrow \frac{6}{(s+2)^4}, \quad e^{-at} \sin 3t \leftrightarrow \frac{3}{(s+a)^2 + 9}, \quad f(2t) \leftrightarrow \frac{1}{2}F(s/2).$$

The proposed sine numerator instead represents $e^{-at} \cos 3t$.

Step 2: Include convergence information. The first domain is $s > -2$. At its boundary the integrand is t^3 . The second is $s > -a$; at the boundary the sine integral has no improper limit, and below it the oscillations grow. For the scaled general function, $s > 2c$ ensures absolute convergence. This is sufficient and need not be exact if the supplied c is merely an exponential-order bound.

Step 3: Combine the corrected entries. Linearity gives

$$Y(s) = \frac{6}{(s+2)^4} + \frac{3}{(s+a)^2 + 9}, \quad s > \max(-2, -a).$$

The polynomial-exponential and sinusoidal tails cannot cancel one another.

Step 4: Make independent checks. Near zero, the sum is $3t + O(t^2)$, so its leading transform behavior is $3/s^2$, consistent with the displayed expression. This would immediately reject a cosine numerator, which would introduce a $1/s$ term. Direct areas are $6/2^4 = 3/8$ and $3/(a^2 + 9)$, giving $Y(0) = 3/8 + 3/(a^2 + 9)$. This catches the missing factorial. These checks are necessary consistency tests, not substitutes for the integration and operational derivations: many different transforms can share one asymptotic term and one value.

Question 2

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

For $a > 0$, define the delay and time-multiplication operations

$$(D_a f)(t) = H_a(t)f(t-a), \quad (Mf)(t) = tf(t).$$

Take $f(t) = e^{-t}$, and compare $p = D_a(Mf)$ with $q = M(D_a f)$.

Tasks

1. Derive the transform rules for D_a and M from their defining integrals. State a domain where the derivations are justified for this f .
2. Find p, q and their transforms. Explain exactly why applying the two operations in opposite orders changes the answer.
3. Prove the general identity $MD_a f - D_a Mf = aD_a f$ whenever the expressions are defined, and verify its transform.
4. For $a = 2$, invert $e^{-2s}/(s+1)^2$ and $-\frac{d}{ds}[e^{-2s}/(s+1)]$. Compute the area of each inverse and use it to distinguish them.

Question 2 – Solution

Strategy. A delayed clock measures $t - a$; multiplication after the delay still uses the original clock t .

Step 1: Derive the rules. The substitution $u = t - a$ gives $\mathcal{L}\{D_a f\} = e^{-as}F(s)$. Differentiating the transform integral with respect to s gives $\mathcal{L}\{Mf\} = -F'(s)$. For $f = e^{-t}$ and its delayed polynomial multiples, $s > -1$ makes these integrals absolutely convergent and permits this differentiation, locally dominated by an integrable exponential times a polynomial.

Step 2: Keep the clocks explicit. Here

$$p = H_a(t)(t - a)e^{-(t-a)}, \quad q = H_a(t)te^{-(t-a)},$$

so

$$P = \frac{e^{-as}}{(s+1)^2}, \quad Q = -\frac{d}{ds} \frac{e^{-as}}{s+1} = e^{-as} \left[\frac{a}{s+1} + \frac{1}{(s+1)^2} \right].$$

The extra term comes from differentiating the delay factor itself.

Step 3: Prove the operation identity. For $t < a$ both sides vanish. For $t \geq a$, subtraction gives $tf(t-a) - (t-a)f(t-a) = af(t-a)$. Therefore

$$\boxed{MD_a f - D_a Mf = aD_a f.}$$

In transform language, $-(e^{-as}F)' - e^{-as}(-F') = ae^{-as}F$, the same identity by the product rule.

Step 4: Invert and distinguish the two expressions. For $a = 2$ the first inverse is $H_2(t)(t-2)e^{-(t-2)}$, while the second is $H_2(t)te^{-(t-2)}$. Their areas, after putting $u = t - 2$, are

$$\int_0^\infty ue^{-u} du = \boxed{1}, \quad \int_0^\infty (u+2)e^{-u} du = \boxed{3}.$$

Their transforms at zero have these same values. The first starts continuously from zero at the delay; the second has right-hand value 2 there. Isolated endpoint conventions do not change either area.

Question 3

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A short table gives only $\mathcal{L}\{\sin(at)\} = a/(s^2 + a^2)$, where $a > 0$. You need inverses of

$$A_a(s) = \frac{s}{(s^2 + a^2)^2}, \quad B_a(s) = \frac{1}{(s^2 + a^2)^2}.$$

Tasks

1. Derive the inverse of A_a by differentiating a transform with respect to s .
2. Derive the inverse of B_a by differentiating the given sine entry with respect to a . Explain why the parameter differentiation is justified for real $s > 0$.
3. Check the first nonzero Taylor term at $t = 0$ against the leading large- s term of each transform.
4. Find both pointwise limits as $a \rightarrow 0^+$ and explain why substituting $a = 0$ into an unsimplified inverse formula is invalid.

Question 3 – Solution

Strategy. Generate missing table rows through operations whose normalization can be checked independently.

Step 1: Differentiate with respect to s . Since $-\frac{d}{ds}[a/(s^2 + a^2)] = 2as/(s^2 + a^2)^2$,

$$\boxed{\mathcal{L}^{-1}\{A_a\} = \frac{t \sin(at)}{2a}.$$

The factor $2a$ must be divided out.

Step 2: Differentiate with respect to a . The parameter derivative gives $\mathcal{L}\{t \cos(at)\} = (s^2 - a^2)/(s^2 + a^2)^2$. For fixed $s > 0$, te^{-st} dominates the differentiated integrand uniformly in a , justifying differentiation under the integral. Subtracting this row from $\mathcal{L}\{\sin(at)/a\}$ yields

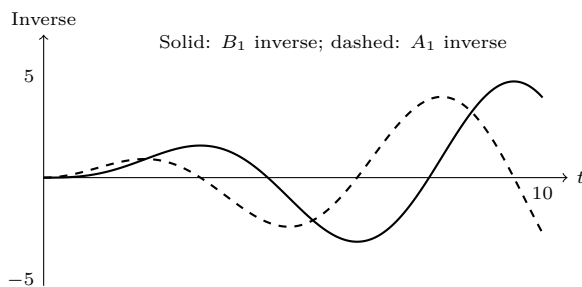
$$\frac{1}{s^2 + a^2} - \frac{s^2 - a^2}{(s^2 + a^2)^2} = \frac{2a^2}{(s^2 + a^2)^2}.$$

Thus

$$\boxed{\mathcal{L}^{-1}\{B_a\} = \frac{\sin(at) - at \cos(at)}{2a^3}.$$

Step 3: Check the onset powers. The inverses start as $t^2/2 + O(t^4)$ and $t^3/6 + O(t^5)$. Their leading transforms are respectively $1/s^3$ and $1/s^4$, matching A_a and B_a . This checks both factorials and the cancellation of the linear term in the second numerator.

Step 4: Remove the apparent parameter singularities. At each fixed t , the Taylor expansions give limits $t^2/2$ and $t^3/6$. Their transforms are $1/s^3$ and $1/s^4$, agreeing with the limits of the rational functions for $s > 0$. Directly inserting $a = 0$ into the inverse quotients produces $0/0$; the limits require cancellation first. The figure uses $a = 1$ and shows that the repeated denominator can produce oscillations with increasing size, unlike a single sine entry.



Question 4

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

An unknown function has transform

$$F(s) = \frac{As + B}{s^2 + 4}, \quad s > 0,$$

where A, B are real. Measurements give $f(0) = 1$ and $f'(0) = -2$.

Tasks

1. Invert the general transform by matching numerators to table entries, then recover A, B from the data.
2. For the recovered function, find its amplitude, all positive zeros, and the conserved quantity $f'(t)^2 + 4f(t)^2$.
3. Classify all real pairs (A, B) for which the inverse is nonnegative for every $t \geq 0$. Prove the classification.
4. A student argues that $\lim_{s \rightarrow 0^+} sF(s) = 0$ proves $\lim_{t \rightarrow \infty} f(t) = 0$. Refute the conclusion with two sequences of times and identify the missing hypothesis in that use of the final-value theorem.

Question 4 – Solution

Strategy. Matching a denominator is insufficient: the numerator determines the phase, initial state and sign changes.

Step 1: Match and use the data. The table gives $f = A \cos 2t + (B/2) \sin 2t$, so $f(0) = A$ and $f'(0) = B$. Therefore $A = 1$, $B = -2$ and

$$f(t) = \cos 2t - \sin 2t = \sqrt{2} \cos(2t + \pi/4).$$

Step 2: Determine the motion. The amplitude is $\sqrt{2}$. The positive zeros are $t = \pi/8 + k\pi/2$, $k = 0, 1, 2, \dots$. Direct differentiation gives $f'' + 4f = 0$, so $(f'^2 + 4f^2)' = 2f'(f'' + 4f) = 0$. Its initial value is $(-2)^2 + 4(1)^2 = \boxed{8}$.

Step 3: Prove the positivity classification. If $(A, B) \neq (0, 0)$, the amplitude $\sqrt{A^2 + B^2/4}$ is positive. The sinusoid attains both that amplitude and its negative at arbitrarily large nonnegative times. It cannot be nonnegative on the whole half-line. Only $\boxed{A = B = 0}$ works. An alternative proof uses zero integral over one period: a continuous nonnegative function with zero integral must vanish.

Step 4: Check the claimed final value. Although $sF(s) \rightarrow 0$ as $s \rightarrow 0^+$, the selected solution obeys $f(n\pi) = 1$ and $f(n\pi + \pi/2) = -1$. Hence it has no time limit. The usual rational final-value theorem requires the poles of $sF(s)$ to lie strictly in the left half-plane after cancellation. Here the uncanceled poles are $s = \pm 2i$, supporting persistent oscillations. An algebraic limit of the transform does not by itself establish a limit of the function.

Question 5

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A table contains the infinite-tail entry $te^{-t} \leftrightarrow 1/(s+1)^2$. A finite experiment instead uses

$$w(t) = \begin{cases} te^{-t}, & 0 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

A student subtracts $e^{-2s}/(s+1)^2$ to remove the tail.

Tasks

1. Explain why the proposed subtraction is incorrect, and express the actual tail as a delayed function with its internal clock made explicit.
2. Derive $W(s)$ from table operations and verify it by integration over the finite experiment.
3. Find the exact real transform domain and the removable value at $s = -1$. Explain why the infinite-tail table domain does not restrict this finite-window transform.
4. Find the area, the global maximum and the one-sided values at the cutoff. Draw the finite-window response accurately on the solution page.

Question 5 – Solution

Strategy. Removing an old tail requires its original age, not a fresh copy starting at zero.

Step 1: Identify the tail that must be removed. For $t = 2 + u$, the old signal is $e^{-2}(u + 2)e^{-u}$. Thus

$$w = te^{-t} - e^{-2}H_2(t)[(t - 2) + 2]e^{-(t-2)}.$$

The proposed subtraction instead removes $H_2(t)(t - 2)e^{-(t-2)}$, which has both the wrong age polynomial and the wrong exponential scale.

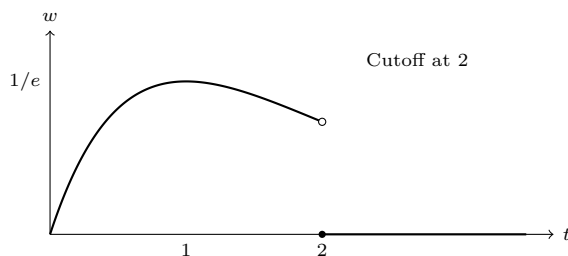
Step 2: Transform and check directly. For $s \neq -1$, the correct expression is

$$W(s) = \frac{1 - e^{-2(s+1)}[1 + 2(s+1)]}{(s+1)^2}.$$

Initially table operations justify it for $s > -1$. Independently put $p = s + 1$: $W = \int_0^2 te^{-pt} dt$. Integration by parts gives $[1 - e^{-2p}(1 + 2p)]/p^2$, agreeing with the table calculation and extending its formula to $p \neq 0$ of either sign.

Step 3: Remove the singularity and identify the domain. A bounded finite interval makes this transform finite for every real s . At $s = -1$ the defining integral is $\int_0^2 t dt = [2]$. The numerator's Taylor expansion is $2p^2 + O(p^3)$, so the apparent pole is removable. The two infinite-tail terms need only be transformed separately on their common domain; their difference has a larger actual domain.

Step 4: Record the geometric checks. The area is $W(0) = [1 - 3e^{-2}]$. Since $(te^{-t})' = (1 - t)e^{-t}$, the unique global maximum is $1/e$ at $t = 1$. At the cutoff, $w(2^-) = 2e^{-2}$ while $w(2) = w(2^+) = 0$. The open dot marks the excluded left-hand value and the filled dot the actual value at 2; no slanted segment bridges the jump.



Question 6

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Let p be the continuous periodic triangular signal of period 2 with

$$p(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2-t, & 1 \leq t < 2, \end{cases} \quad p(t+2) = p(t).$$

A standard table lists delayed ramps but does not list this periodic function.

Tasks

1. Derive the periodic-transform formula by splitting the half-line into cycles. Apply it to this signal.
2. Write a locally finite series of delayed ramps for p and use it to obtain the same transform independently.
3. Find the exact real transform domain. Compute $\lim_{s \rightarrow 0^+} sP(s)$ and interpret it using the mean over one period.
4. Explain why the cycle mean is not a pointwise final value, and determine the first two leading terms of $P(s)$ as $s \rightarrow 0^+$.

Question 6 – Solution

Strategy. A one-cycle transform and a geometric series supply the periodic row; delayed ramps independently encode its slope changes.

Step 1: Sum the cycles. For $s > 0$, boundedness gives absolute convergence. Splitting into intervals $[2n, 2n+2]$ and translating yields

$$P(s) = \frac{\int_0^2 e^{-st} p(t) dt}{1 - e^{-2s}} = \frac{(1 - e^{-s})^2}{s^2(1 - e^{-2s})} = \frac{1 - e^{-s}}{s^2(1 + e^{-s})}.$$

The one-cycle integral is $(1 - e^{-s})^2/s^2$, obtained by integration by parts on $[0, 1]$ and $[1, 2]$.

Step 2: Recover the slope changes. Write $R_a = (t - a)H_a(t)$. The series

$$p(t) = \sum_{n=0}^{\infty} [R_{2n} - 2R_{2n+1} + R_{2n+2}]$$

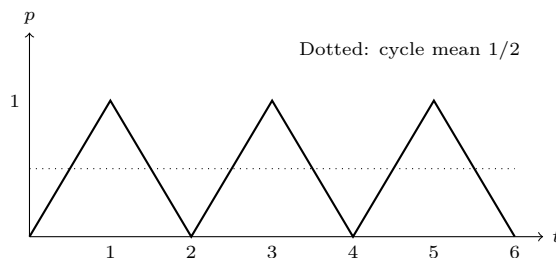
is locally finite; each bracket is a unit triangular pulse on $[2n, 2n+2]$. Its transform is $e^{-2ns}(1 - 2e^{-s} + e^{-2s})/s^2$. Summing for $s > 0$ gives the same P . These brackets, rather than the individual growing ramps across all cycles, also make absolute convergence clear.

Step 3: Check convergence and the mean. The signal is bounded, nonnegative and has area 1 per period. Thus the exact real domain is $s > 0$: at $s = 0$ the areas sum to infinity, and for $s < 0$ the weighted cycle contributions grow. Using $(1 - e^{-s})/(1 + e^{-s}) = \tanh(s/2)$, $\lim_{s \rightarrow 0^+} sP(s) = \boxed{1/2}$, the cycle mean $1/2$.

Step 4: Distinguish the limit notions. The samples $p(2n) = 0$ and $p(2n+1) = 1$ rule out a pointwise final value. Taylor expansion of the last quotient gives

$$P(s) = \frac{1}{2s} - \frac{s}{24} + O(s^3).$$

The leading pole records the mean, while the constant term vanishes. The figure displays continuous corners and equal successive cycles.



Question 7

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

For $t > 0$, let

$$g(t) = \frac{e^{-t} - e^{-3t}}{t}.$$

A rational-only table offers no obvious row for this function.

Tasks

1. Extend g continuously to zero and represent it as an integral of exponential functions over a finite parameter interval.
2. Use that representation to derive its transform, carefully justifying interchange and specifying a real domain.
3. Prove positivity and strict decrease on $[0, \infty)$, and compute the total area.
4. Find the exact real transform domain, including a proof of divergence at its boundary. Explain why an algebraically real continuation of a logarithm is not sufficient evidence of transform convergence.

Question 7 – Solution

Strategy. Integrating a familiar exponential row over its rate parameter generates a nonrational row.

Step 1: Remove the time singularity. Direct integration in a gives

$$g(t) = \int_1^3 e^{-at} da, \quad g(0) = 2.$$

This formula also proves continuity at zero without dividing by t .

Step 2: Integrate the table row. For real $s > -1$, the nonnegative double integral is finite, and interchange gives

$$G(s) = \int_1^3 \int_0^\infty e^{-(s+a)t} dt da = \int_1^3 \frac{da}{s+a} = \ln \frac{s+3}{s+1}.$$

Both logarithm arguments are positive in this domain. Finiteness follows also from $s+a \geq s+1 > 0$ on the finite parameter interval.

Step 3: Establish shape and area. The integrand is positive, so $g(t) > 0$. Parameter integration on a bounded interval permits differentiation at every $t \geq 0$:

$$g'(t) = - \int_1^3 a e^{-at} da < 0.$$

Thus the extension decreases strictly from 2, approaching zero. Since 0 lies in the transform domain, its area is $G(0) = \ln 3$.

Step 4: Check the boundary from the original integral. At $s = -1$, the transformed integrand is $(1 - e^{-2t})/t$ for $t > 0$. For $t \geq 1$ it is at least $(1 - e^{-2})/t$, whose integral diverges. For $s < -1$ the positive integrand is even larger on the tail. Therefore $s > -1$ is the exact real domain. For example, the logarithm of $(s+3)/(s+1)$ is real again when $s < -3$, but that algebraic expression cannot represent the divergent defining integral. The table entry must travel with its domain and the continuous value at zero.

Question 8

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Consider the transform $F(s) = s/(s+1)$. Allow a full-mass impulse at the origin, with $\mathcal{L}\{\delta(t)\} = 1$ and pre-impact data at 0^- . For ordinary candidates, restrict to functions continuous near zero and of exponential order. The input with transform F drives

$$y' + 2y = f, \quad y(0^-) = 0.$$

Tasks

1. Explain why F cannot be the transform of an ordinary candidate in the stated class. Recover its distributional inverse.
2. Find $Y(s)$ and the complete response for $t \geq 0$, including the jump at zero.
3. Find the positive zero, the global minimum, and the total response area. Explain how a nonzero response can have zero area.
4. Verify the post-impact equation and the equivalent ordinary restart. State the exact real transform domain of the response.

Question 8 – Solution

Strategy. Divide an improper rational expression before consulting ordinary-function rows. Its polynomial part may represent an impulse.

Step 1: Identify the input class. For an ordinary candidate continuous near zero and of exponential order, its Laplace transform tends to zero as $s \rightarrow \infty$: the bounded initial piece has integral $O(1/s)$ and the exponentially bounded tail tends to zero. But $s/(s+1) \rightarrow 1$. Polynomial division gives

$$F = 1 - \frac{1}{s+1}, \quad \boxed{f = \delta(t) - e^{-t}}.$$

The impulse supplies the nonvanishing constant term.

Step 2: Transform the equation and account for the jump. With the stated pre-impact convention, $(s+2)Y = F$, so

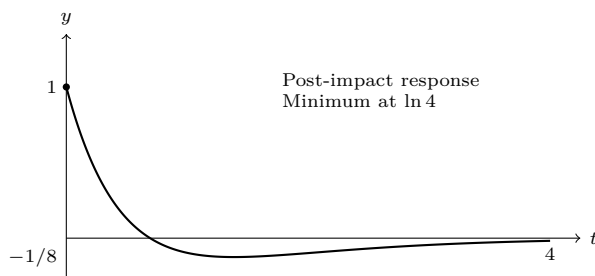
$$Y = \frac{s}{(s+1)(s+2)} = -\frac{1}{s+1} + \frac{2}{s+2}, \quad \boxed{y = 2e^{-2t} - e^{-t} \quad (t \geq 0)}.$$

Write $[y]_0 = y(0^+) - y(0^-)$. Integrating across zero gives $[y]_0 = 1$, consistent with $y(0^+) = 1$. The negative exponential input has no impulse mass.

Step 3: Find the sign changes and area. The unique positive zero is $t = \ln 2$. The derivative is $y' = e^{-t} - 4e^{-2t}$, negative before $\ln 4$ and positive afterward. Thus the global minimum is $\boxed{y(\ln 4) = -1/8}$.

The area is $2/2 - 1 = \boxed{0}$: positive and negative parts cancel. For comparison, the signed input area is $1 - 1 = 0$ when the full initial impulse is included.

Step 4: Verify the ordinary restart and convergence. For $t > 0$, direct substitution gives $y' + 2y = -e^{-t}$. The same response solves this ordinary equation with $y(0^+) = 1$. Adding another initial impulse to that restart would count it twice. The tail is asymptotic to $-e^{-t}$, so the exact real transform domain is $\boxed{s > -1}$. The figure shows only the post-impact branch on $t \geq 0$; the filled initial point records the prescribed right-hand value.



Question 9

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

A measured response is known to lie in the three-mode family

$$f(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

The available data are $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 2$.

Tasks

1. Recover the three coefficients and prove uniqueness within the stated family.
2. Construct $F(s)$ by table lookup and simplify it to one rational expression. Check its leading large- s term against the data.
3. Prove that the recovered function is positive for every $t > 0$ despite a negative coefficient. Find its unique global maximum.
4. Compute its area and exact real transform domain. Explain why the three measurements would not identify a unique response without the three-mode assumption.

Question 9 – Solution

Strategy. The assumed modes turn derivative data into a finite linear system. A negative coefficient need not make their sum negative.

Step 1: Solve and prove uniqueness. The data give

$$c_1 + c_2 + c_3 = 0, \quad -c_1 - 2c_2 - 3c_3 = 0, \quad c_1 + 4c_2 + 9c_3 = 2.$$

Adding the first two equations gives $c_2 = -2c_3$, and the first then gives $c_1 = c_3$. Substitution into the third gives $2c_3 = 2$. Every coefficient is forced, proving uniqueness: $(c_1, c_2, c_3) = (1, -2, 1)$.

Step 2: Assemble the rational transform. The table yields

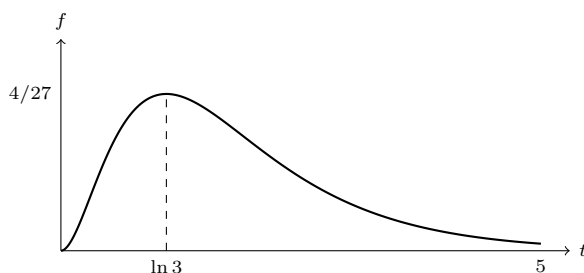
$$F = \frac{1}{s+1} - \frac{2}{s+2} + \frac{1}{s+3} = \frac{2}{(s+1)(s+2)(s+3)}.$$

Near zero, $f = t^2 + O(t^3)$, so its leading transform is $2/s^3$, matching the quotient. The $1/s$ and $1/s^2$ terms cancel exactly.

Step 3: Establish positivity and the peak. Factor the function as $f = e^{-t}(1 - e^{-t})^2$. It is positive for every $t > 0$ and zero at 0. Put $x = e^{-t} \in (0, 1]$. The function $x(1-x)^2$ has derivative $(1-x)(1-3x)$ with respect to x ; along increasing t its unique interior maximum is at $x = 1/3$. Therefore

$$t_{\max} = \ln 3, \quad f(t_{\max}) = 4/27.$$

Step 4: Check area, domain and identifiability. The area is $1 - 2/2 + 1/3 = 1/3$. The nonzero e^{-t} tail gives exact real domain $s > -1$. Without the mode restriction, adding any multiple of $t^3 e^{-t}$ would preserve all three measured derivatives but change the response. Thus the identification is unique within the stated model, not among all smooth functions.



Question 10

Use causal one-sided Laplace transforms. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Ordinary functions are zero for $t < 0$. Justify the table entries and operational rules you use; give exact expressions.

Use the chapter's table and operational rules to solve

$$y'' + 2y' + y = H_1(t)(t-1)e^{-(t-1)} + 2\delta(t-3), \quad y(0) = 1, \quad y'(0) = -1.$$

Ordinary force impulses preserve displacement and may jump velocity. All figures belong on the solution page.

Tasks

1. Find $Y(s)$, carefully retaining both initial terms and the two different forcing operations.
2. Invert the transform and give the response on the three intervals separated by 1 and 3.
3. Verify the initial state, the equation between switches, and the one-sided displacement and velocity changes at each switch.
4. Compute the total response area in two ways and determine the exact real transform domain. Explain why this finite list of table operations gives a unique solution.

Question 10 – Solution

Strategy. Separate the initial response, delayed ordinary forcing and impulse response before inversion.

Step 1: Transform with all data. The left side transforms to $(s^2Y - s + 1) + 2(sY - 1) + Y = (s + 1)^2Y - s - 1$. Consequently

$$Y = \frac{1}{s + 1} + \frac{e^{-s}}{(s + 1)^4} + \frac{2e^{-3s}}{(s + 1)^2}.$$

The ordinary forcing has transform $e^{-s}/(s + 1)^2$; the impulse has transform $2e^{-3s}$.

Step 2: Invert and make the intervals explicit. Define $v(\tau) = \tau^3 e^{-\tau}/6$ and $h(\tau) = \tau e^{-\tau}$ for $\tau \geq 0$. The factorial in the fourth-power table row gives

$$y(t) = \begin{cases} e^{-t}, & 0 \leq t < 1, \\ e^{-t} + v(t - 1), & 1 \leq t < 3, \\ e^{-t} + v(t - 1) + 2h(t - 3), & t \geq 3. \end{cases}$$

Step 3: Verify each kind of activation. The initial branch gives $(y(0), y'(0)) = (1, -1)$. For $D = d/d\tau$ and any polynomial P , $(D + 1)^2[e^{-\tau}P(\tau)] = e^{-\tau}P''(\tau)$. Thus v produces $\tau e^{-\tau}$ and h produces zero between switches. At 1, $v(0) = v'(0) = 0$, so displacement and velocity remain continuous. At 3, $h(0) = 0$, $h'(0) = 1$, so $[y]_3 = 0$ and $[y']_3 = 2$. Here $[z]_a = z(a^+) - z(a^-)$. These are exactly the required jump conditions.

Step 4: Check area, convergence and uniqueness. Direct areas are 1, $3!/6 = 1$, and 2, giving $\int_0^\infty y = 4$. Independently, integrating the equation, including the velocity jump, gives $1 - 2 + \int y = 1 + 2$, again yielding $\int y = 4$. The tail is a nonzero cubic polynomial times e^{-t} , so the exact real domain is $s > -1$. Linear IVP uniqueness on each ordinary interval, together with the forced matching and jump conditions, fixes the whole solution.

