

Question 1

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = e^{2t}(1+t)$ for $t \geq 0$. A student obtains the expression $1/(s-2) + 1/(s-2)^2$ and claims that it gives a transform value at every real $s \neq 2$.

Tasks

1. Evaluate the truncated defining integral from zero to R , treating $s = 2$ separately.
2. Determine exactly which real s give convergence, and find $F(s)$ on that set.
3. Evaluate the student's expression at $s = 1$ and explain why it cannot be the Laplace integral there.
4. Prove directly from the integral, without differentiating the final formula, that F is positive and strictly decreasing throughout its convergence interval.

Question 1 – Solution

Strategy. Keep the upper endpoint finite until the parameter-dependent boundary terms have been examined.

Step 1: Evaluate the finite integral. Write $a = s - 2$. Integration by parts gives, for $a \neq 0$,

$$I_R = \int_0^R (1+t)e^{-at} dt = \frac{1}{a} + \frac{1}{a^2} - e^{-aR} \left[\frac{1+R}{a} + \frac{1}{a^2} \right].$$

At $a = 0$, direct integration gives $I_R = R + R^2/2$. The formula with denominators cannot be used at that parameter.

Step 2: Determine the convergence set. For $a > 0$, both e^{-aR} and Re^{-aR} tend to zero. Therefore

$$F(s) = \frac{1}{s-2} + \frac{1}{(s-2)^2}, \quad s > 2.$$

For $a = 0$, the displayed polynomial diverges. For $a < 0$, the integrand $(1+t)e^{|a|t}$ is at least one, so $I_R \geq R \rightarrow \infty$. Thus $s > 2$ is the entire real convergence set, and convergence there is absolute.

Step 3: Reject an algebraic continuation as an integral value. At $s = 1$, the rational expression equals $-1 + 1 = 0$. But the actual integral is

$$I_R = \int_0^R (1+t)e^t dt = Re^R \rightarrow \infty.$$

The discarded boundary term grows instead of vanishing. A finite value of the algebraic expression outside the convergence interval does not assign a value to the defining improper integral.

Step 4: Prove order properties from the kernel. For $s > 2$, the integrand is positive, so $F(s) > 0$. If $s_2 > s_1 > 2$, then for every $t > 0$,

$$0 < e^{-s_2 t} f(t) < e^{-s_1 t} f(t).$$

Both integrals converge, and their difference is strictly positive on any fixed interval of positive length inside $(0, \infty)$. Hence $F(s_2) < F(s_1)$. This argument shows why positivity and monotonicity hold, independently of the rational simplification.

Question 2

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Consider the finite-duration triangular signal

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2 - t, & 1 < t \leq 2, \\ 0, & t > 2. \end{cases}$$

Tasks

1. Compute $F(s)$ directly by integrating over the two nonzero pieces, first for $s \neq 0$.
2. Determine its full real convergence set and compute $F(0)$ directly. Is the apparent singularity at zero a true divergence?
3. Prove $F(-s) = e^{2s} F(s)$ by a change of variable, using the symmetry of the triangle.
4. If the value at $t = 1$ alone is changed to 100, does the transform change? Explain what this example says about recovering individual point values from an integral transform.

Question 2 – Solution

Strategy. Finite support guarantees convergence for every real parameter. Apparent poles in a simplified expression must be checked against the original integral.

Step 1: Integrate both pieces. For $s \neq 0$, integration by parts gives

$$\int_0^1 te^{-st} dt = \frac{1 - (1+s)e^{-s}}{s^2}, \quad \int_1^2 (2-t)e^{-st} dt = \frac{(s-1)e^{-s} + e^{-2s}}{s^2}.$$

Adding yields

$$F(s) = \frac{(1 - e^{-s})^2}{s^2} \quad (s \neq 0).$$

Step 2: Restore the missing parameter value. For any fixed real s , the integrand is bounded and piecewise continuous on $[0, 2]$, and vanishes afterward. Thus the real convergence set is all of $\mathbb{R}$. At zero, the integral is the triangle's area:

$$F(0) = 1.$$

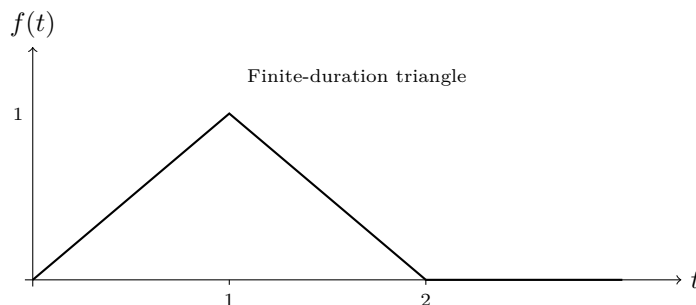
Also $(1 - e^{-s})/s \rightarrow 1$ as $s \rightarrow 0$, so the simplified formula has a removable singularity, not divergent integral behavior.

Step 3: Use reflection symmetry. On $[0, 2]$, $f(2-u) = f(u)$. Substituting $u = 2-t$ gives

$$F(-s) = \int_0^2 e^{st} f(t) dt = e^{2s} \int_0^2 e^{-su} f(2-u) du = e^{2s} F(s).$$

This holds at zero as well. Equivalently, $e^s F(s)$ is an even function of s , expressing symmetry about the triangle's midpoint.

Step 4: Distinguish a point from an interval. Changing one finite point value does not change any of these Riemann integrals, so the transform is unchanged. The modified function is different at one point but agrees elsewhere. Therefore the transform alone cannot recover arbitrary isolated point assignments; this observation does not contradict uniqueness results that impose continuity.



Question 3

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin 3t$ for $t \geq 0$. Its truncated transform is $F_R(s) = \int_0^R e^{-st} \sin 3t dt$.

Tasks

1. Evaluate $F_R(s)$ for real s , retaining its upper-endpoint contribution.
2. Find the exact real convergence interval for $F(s)$ and determine whether convergence there is absolute.
3. At $s = 0$, exhibit two sequences of upper endpoints giving different values of $F_R(0)$.
4. Compare $\lim_{s \rightarrow 0^+} F(s)$ with the existence of $F(0)$. Why does a finite parameter limit not justify interchanging the two limits $R \rightarrow \infty$ and $s \rightarrow 0^+$?

Question 3 – Solution

Strategy. Damping by e^{-st} can create convergence that disappears at the boundary parameter.

Step 1: Keep the endpoint term. Two integrations by parts, or differentiation of the resulting antiderivative, give

$$F_R(s) = \frac{3 - e^{-sR}[s \sin 3R + 3 \cos 3R]}{s^2 + 9}.$$

There is no real zero of the denominator, but that alone says nothing about the limit as $R \rightarrow \infty$.

Step 2: Classify convergence. If $s > 0$, the endpoint term vanishes and

$$F(s) = \frac{3}{s^2 + 9}, \quad s > 0.$$

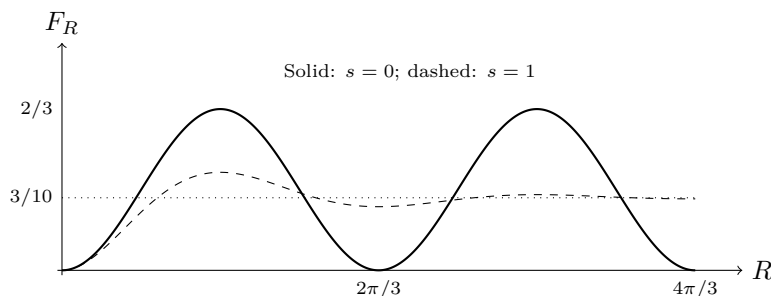
Absolute convergence follows from $|e^{-st} \sin 3t| \leq e^{-st}$. If $s < 0$, take $R_n = 2n\pi/3$. Then $F_{R_n}(s) = 3(1 - e^{|s|R_n})/(s^2 + 9) \rightarrow -\infty$, so the improper integral cannot converge.

Step 3: Test the boundary directly. At $s = 0$, $F_R(0) = (1 - \cos 3R)/3$. For

$$R_n = 2n\pi/3, \quad F_{R_n}(0) = 0; \quad Q_n = (2n+1)\pi/3, \quad F_{Q_n}(0) = 2/3.$$

The two sequences tend to infinity but have different integral values. Thus $F(0)$ does not exist, and the full real convergence interval is $(0, \infty)$.

Step 4: Separate the two limiting procedures. The damped transforms satisfy $\lim_{s \rightarrow 0^+} F(s) = 1/3$. In contrast, setting $s = 0$ first produces an oscillating truncated integral with no limit. For fixed R , the limit in s exists, but convergence in R is not uniform as damping disappears. The value $1/3$ is a limit of damped integrals, not the value of the undamped improper integral. The graph compares accumulated integrals, not the original sine signal.



Question 4

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

For $t > 0$, let $f(t) = t^{-1/2}e^{-t}$; assign any finite value to $f(0)$. For the defining integral, treat zero as an improper endpoint as well as infinity. You may use $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2$.

Tasks

1. Analyze convergence near zero and near infinity separately, and find the exact real convergence set.
2. Evaluate $F(s)$ on that set by a substitution.
3. Repeat the endpoint analysis for $h(t) = t^{-1}e^{-t}$, $t > 0$. Does exponential decay guarantee a transform for this function?
4. Explain why the first example does not contradict a theorem giving piecewise continuity on each finite interval and exponential order as sufficient conditions for existence.

Question 4 – Solution

Strategy. An improper transform can fail at either endpoint. Rapid decay at infinity cannot repair a nonintegrable singularity at zero.

Step 1: Separate the endpoints. Write $a = s + 1$. Near zero, e^{-at} is bounded above and below by positive constants, and $\int_0^1 t^{-1/2} dt = 2$, so this endpoint is integrable for every fixed real s . At infinity, $a > 0$ gives exponential decay. If $a = 0$, the integral of $t^{-1/2}$ diverges; if $a < 0$, the integrand eventually exceeds one. Hence convergence occurs exactly for $s > -1$, and is absolute.

Step 2: Evaluate the convergent integral. For $a > 0$, use $u = \sqrt{at}$, so $t = u^2/a$ and $dt = 2u du/a$. Then

$$F(s) = \frac{2}{\sqrt{a}} \int_0^\infty e^{-u^2} du = \boxed{\frac{\sqrt{\pi}}{\sqrt{s+1}}, \quad s > -1.}$$

Both the substitution and the square root require the already established condition $a > 0$. An assigned value at the single point zero does not change the improper integral.

Step 3: Test the stronger singularity. For h , the integrand is $t^{-1}e^{-(s+1)t}$. On a sufficiently small interval $(0, \delta)$, its exponential factor is at least $1/2$. Thus

$$\int_0^\delta t^{-1} e^{-(s+1)t} dt \geq \frac{1}{2} \int_0^\delta \frac{dt}{t} = \infty.$$

This holds for every real s . Consequently h has no transform under this definition, even for parameters that give fast decay at infinity.

Step 4: Interpret the sufficient theorem. The first function is unbounded as $t \rightarrow 0^+$, so it does not satisfy the usual finite-limit requirement for piecewise continuity on $[0, A]$. Yet its endpoint singularity is integrable, and the direct calculation establishes existence for $s > -1$. A sufficient theorem guarantees existence when its hypotheses hold; failure of a hypothesis does not prove nonexistence. Here direct endpoint analysis gives the sharper conclusion.

Question 5

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let $f(t) = \sin t/t$ for $t > 0$, with $f(0) = 1$. In this problem examine parameters $s \geq 0$. You may derive the positive-parameter transform using $\sin t/t = \int_0^1 \cos(ut) du$.

Tasks

1. For $s > 0$, justify interchanging the two integrations and evaluate $F(s)$.
2. For $s \geq 0$ and $R > 0$, prove the tail bound $|\int_R^\infty e^{-st} \sin t/t dt| \leq 2/R$, including existence of the tail at $s = 0$.
3. Use that uniform tail bound and finite-interval convergence as $s \rightarrow 0^+$ to determine $F(0)$.
4. Determine whether the integral at $s = 0$ converges absolutely, and contrast it with the positive-parameter integrals.

Question 5 – Solution

Strategy. A uniform tail estimate can justify passage to the boundary parameter; absolute and conditional convergence must still be distinguished.

Step 1: Evaluate for positive damping. For $s > 0$, the double integral of the absolute integrand is at most $\int_0^\infty e^{-st} dt = 1/s$, so interchange is justified. Integration by parts gives $\int_0^\infty e^{-st} \cos(ut) dt = s/(s^2 + u^2)$. Therefore

$$F(s) = \int_0^1 \frac{s}{s^2 + u^2} du = \boxed{\arctan(1/s), \quad s > 0.}$$

Step 2: Control the tail uniformly. Set $w(t) = e^{-st}/t$ for $t \geq R$. For every $s \geq 0$, w decreases to zero and $w' \leq 0$. Integration by parts on $[R, L]$ gives

$$\int_R^L w(t) \sin t dt = [-w(t) \cos t]_R^L + \int_R^L w'(t) \cos t dt.$$

The integral of $|w'|$ is $w(R) - w(L)$, so the right side has magnitude at most $2w(R) \leq 2/R$. Applying the same bound with any larger lower endpoint proves the Cauchy criterion as $L \rightarrow \infty$, including at $s = 0$. The bound persists for the improper tail.

Step 3: Pass to the boundary parameter. On each fixed $[0, R]$, $e^{-st}f(t) \rightarrow f(t)$ uniformly as $s \rightarrow 0^+$, since f is continuous and bounded there. The tails at both s and zero have magnitude at most $2/R$. Thus

$$|F(s) - F(0)| \leq \left| \int_0^R (e^{-st} - 1)f(t) dt \right| + 4/R.$$

First choose large R , then let $s \rightarrow 0^+$. This proves continuity at the boundary and hence $\boxed{F(0) = \pi/2}$.

Step 4: Test absolute convergence. On $[n\pi + \pi/6, n\pi + 5\pi/6]$, $|\sin t| \geq 1/2$ and $t \leq (n+1)\pi$. Each such interval contributes at least $1/[3(n+1)]$ to $\int |\sin t|/t dt$. The harmonic series diverges, so convergence at zero is conditional. For $s > 0$, $|\sin t/t| \leq 1$ gives absolute convergence by comparison with e^{-st} . The boundary value here exists as an ordinary improper integral, not merely as a limit of damped values.

Question 6

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Compare $f_+(t) = e^{t^2}$ and $f_-(t) = e^{-t^2}$ for $t \geq 0$, and write their transforms as F_+ and F_- .

Tasks

1. Determine the real convergence set of each transform directly from the defining integral.
2. Complete the square to express $F_-(s)$ through a Gaussian tail for every real s . No special-function notation is required.
3. For a truncation at T with $T + s/2 > 0$, prove an explicit upper bound for the omitted tail of $F_-(s)$.
4. At $s = -2$, find the smallest integer $T > 1$ for which your bound certifies an error at most 10^{-4} . Explain why negative s does not destroy convergence for f_- .

Question 6 – Solution

Strategy. Compare the quadratic growth or decay with the linear term supplied by the Laplace weight, and use the completed square to bound numerical truncation.

Step 1: Classify both integrals. For any real s , $t^2 - st \geq 0$ when $t \geq \max\{s, 0\}$, so $e^{t^2-st} \geq 1$ on a half-line. Hence F_+ diverges for every real s . For F_- ,

$$-t^2 - st = -(t + s/2)^2 + s^2/4.$$

A shifted Gaussian has an integrable tail and no finite-endpoint singularity, so F_- converges absolutely for every real s .

Step 2: Express the exact transform. Substitution $u = t + s/2$ gives

$$F_-(s) = e^{s^2/4} \int_{s/2}^{\infty} e^{-u^2} du, \quad s \in \mathbb{R}.$$

The lower endpoint is allowed to be negative; the finite portion of the integral then remains harmless. This definite integral is an exact answer.

Step 3: Bound the omitted tail. Let $a = T + s/2 > 0$. Since $1 \leq u/a$ for $u \geq a$,

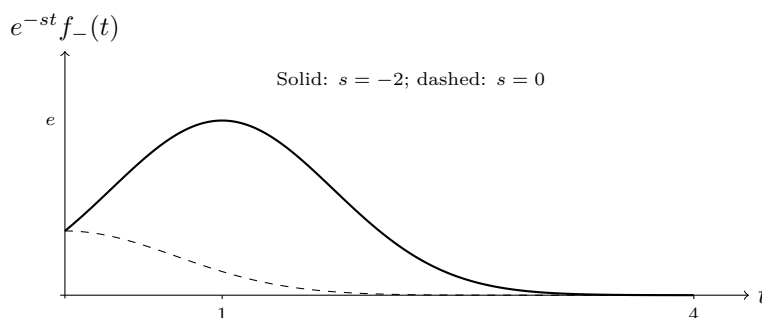
$$\int_a^{\infty} e^{-u^2} du \leq \frac{1}{a} \int_a^{\infty} u e^{-u^2} du = \frac{e^{-a^2}}{2a}.$$

Consequently the positive truncation error satisfies

$$\int_T^{\infty} e^{-t^2-st} dt \leq \frac{e^{-T^2-sT}}{2T+s}.$$

The condition $2T + s > 0$ is part of the bound and cannot be dropped.

Step 4: Certify a cutoff. For $s = -2$, the bound is $e^{-T^2+2T}/(2T-2)$, decreasing for $T > 1$. At $T = 3$ it is $e^{-3}/4 \approx 0.01245$, while at $T = 4$ it is $e^{-8}/6 \approx 5.59 \times 10^{-5}$. Thus the smallest integer cutoff certified by this bound is $\boxed{T = 4}$. Negative s makes the weight grow exponentially, but the quadratic decay still dominates. The graph shows the weighted integrands, whose areas give the transforms.



Question 7

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

For each integer $n \geq 1$, let $w_n = e^{-2n^2}$. Define a nonnegative signal by

$$f(t) = \begin{cases} e^{n^2}, & n \leq t \leq n + w_n \text{ for some } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

A function is of exponential order if $|f(t)| \leq M e^{at}$ for all sufficiently large t , for some constants $M > 0$ and real a .

Tasks

1. Explain why f is piecewise continuous on every finite interval but is not of exponential order.
2. Prove that its Laplace integral nevertheless converges absolutely for every real s .
3. Express $F(s)$ as a convergent series, treating $s = 0$ separately.
4. Explain how the spike widths resolve the apparent conflict between arbitrarily large heights and convergence, and why the usual exponential-order theorem is not contradicted.

Question 7 – Solution

Strategy. The integral depends on weighted areas, not just pointwise heights. Quantify the widths before invoking a growth criterion.

Step 1: Examine local regularity and height. Since $w_n < 1$, the spike intervals are disjoint. Any bounded time interval meets only finitely many spikes, each with finite height and finite one-sided limits. Thus f is piecewise continuous locally. If an exponential-order bound held, evaluating at $t = n$ for large integers would give $e^{n^2} \leq Me^{an}$, or $n^2 - an \leq \ln M$. The left side tends to infinity, a contradiction.

Step 2: Estimate each weighted area. Fix real s . On the n th spike, $t \leq n + 1$ and $e^{-st} \leq e^{|s|(n+1)}$. Its contribution is therefore at most

$$e^{n^2} w_n e^{|s|(n+1)} = e^{-n^2 + |s|(n+1)}.$$

For sufficiently large n , $|s|(n+1) \leq n^2/2$, so these terms are at most $e^{-n^2/2} \leq e^{-n/2}$. The comparison series converges. Since the integrand is nonnegative, the bound proves absolute convergence for every real s .

Step 3: Sum the exact contributions. Split finite integrals into their finitely many spike intervals, then pass to the limit using the convergent positive series. For $s \neq 0$,

$$F(s) = \sum_{n=1}^{\infty} e^{n^2 - sn} \frac{1 - e^{-sw_n}}{s}.$$

The quotient is positive even for negative s . At $s = 0$, each contribution is simply height times width, giving

$$F(0) = \sum_{n=1}^{\infty} e^{-n^2}.$$

No closed form for this convergent series is required.

Step 4: Interpret the theorem correctly. The heights grow faster than every exponential, but the widths shrink fast enough that their areas are e^{-n^2} . Any fixed exponential weight still leaves a summable sequence of areas. The common existence theorem provides sufficient conditions; exponential order is not necessary for an individual function to have a Laplace transform. This example establishes that distinction by direct estimates, rather than by trying to apply a theorem whose hypothesis fails.

Question 8

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

An unknown signal belongs to the family $f(t) = 0$ for $0 \leq t < a$ and $f(t) = A$ for $t \geq a$, where $A > 0$ and $a \geq 0$. Exact measurements are

$$F(1) = 2/e, \quad F(2) = e^{-2}.$$

Tasks

1. Derive the transform and its real convergence set directly, then recover A and a from the measurements.
2. For arbitrary positive measurements $M_1 = F(1)$ and $M_2 = F(2)$, give a necessary and sufficient consistency condition for this family and explicit recovery formulas.
3. For $h(t) = e^{-t}(t^2 - 5t/3 + 1/3)$, compute its transform at $s > 0$ directly by integration by parts and show that it vanishes at both measured parameters.
4. Use h to explain precisely why the two measurements identify the signal within the stated family but do not identify an arbitrary real-valued signal.

Question 8 – Solution

Strategy. Distinguish uniqueness of a few model parameters from uniqueness among all possible functions.

Step 1: Integrate and calibrate. For $s > 0$,

$$F(s) = A \int_a^\infty e^{-st} dt = \frac{Ae^{-as}}{s}.$$

For $s \leq 0$, the positive tail integral diverges. Dividing the two measured equations gives $2F(2)/F(1) = e^{-a} = e^{-1}$, so $\boxed{a = 1, A = 2}$. These values reproduce both measurements.

Step 2: Characterize all consistent data. For positive M_1, M_2 , the family requires $2M_2/M_1 = e^{-a} \in (0, 1]$. Thus consistency is exactly $\boxed{2M_2 \leq M_1}$. In that case the unique parameters are

$$\boxed{a = \ln \frac{M_1}{2M_2}, \quad A = \frac{M_1^2}{2M_2}.$$

They satisfy $a \geq 0, A > 0$ and both equations, proving sufficiency as well as necessity. Equality corresponds to onset at zero.

Step 3: Construct an invisible perturbation. Set $b = s + 1 > 0$. Integration by parts gives the three integrals of $1, t, t^2$ against e^{-bt} as $1/b, 1/b^2, 2/b^3$. Therefore

$$H(s) = \frac{2}{(s+1)^3} - \frac{5}{3(s+1)^2} + \frac{1}{3(s+1)} = \boxed{\frac{(s-1)(s-2)}{3(s+1)^3}}.$$

The exponential boundary terms vanish, justifying those integrations. In particular, $H(1) = H(2) = 0$, although h is not the zero function.

Step 4: State the identification limit. For every real λ , the signal $f + \lambda h$ has a convergent transform for $s > 0$. Splitting its defining integral gives $F(s) + \lambda H(s)$, so both recorded measurements are unchanged. Distinct λ give distinct real-valued signals; for nonzero λ they are not members of the original two-parameter family. The recovery formulas establish uniqueness only under that family assumption, not among arbitrary signals.

Question 9

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

Let f have period two, with $f(t) = 1$ for $0 \leq t < 1$ and $f(t) = 0$ for $1 \leq t < 2$, repeated for all $t \geq 0$. Use the definition rather than a precomputed periodic-transform formula.

Tasks

1. Split the integral into complete periods, sum the resulting series, and determine the exact real convergence set.
2. Simplify the transform on its convergence interval.
3. Compute $\lim_{s \rightarrow 0^+} sF(s)$ and $\lim_{s \rightarrow \infty} sF(s)$. Relate these to the period average and to the signal just after zero.
4. If the defining integral is truncated after N complete periods, find its exact relative error for $s > 0$. At $s = 0.2$, find the smallest integer N making that error at most 10^{-3} .

Question 9 – Solution

Strategy. A time translation by one period multiplies the weighted area by a fixed factor, producing a geometric series.

Step 1: Build the series and its domain. For $s > 0$, the n th nonzero interval contributes

$$\int_{2n}^{2n+1} e^{-st} dt = e^{-2ns} \frac{1 - e^{-s}}{s}.$$

The ratio e^{-2s} lies in $(0, 1)$, so the sum converges. At $s = 0$, each interval contributes one and the sum diverges. For $s < 0$, $e^{-st} \geq 1$ on each such interval, so it also diverges. The real convergence set is exactly $(0, \infty)$, with absolute convergence since $f \geq 0$.

Step 2: Simplify the geometric sum. Summing gives

$$F(s) = \frac{1 - e^{-s}}{s(1 - e^{-2s})} = \boxed{\frac{1}{s(1 + e^{-s})}, \quad s > 0.}$$

The integral over only one period is the first term of this sum, not the transform of the endlessly repeated signal.

Step 3: Interpret the two normalized limits. The formula gives

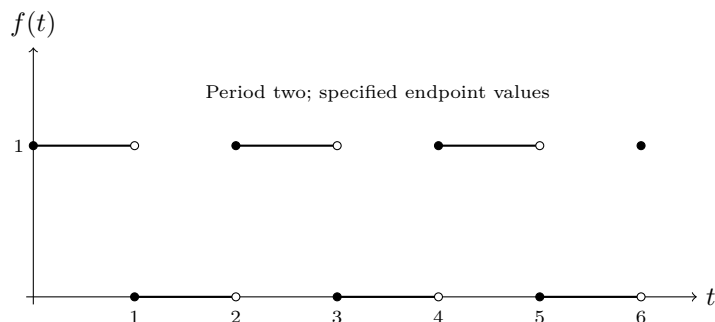
$$\boxed{\lim_{s \rightarrow 0^+} sF(s) = \frac{1}{2}, \quad \lim_{s \rightarrow \infty} sF(s) = 1.}$$

The first value is the period average $(1/2) \int_0^2 f(t) dt = 1/2$. The second agrees with $f(0^+) = 1$: large s weights short times most strongly. These normalized limits do not assert that $F(0)$ exists; it does not.

Step 4: Quantify finite-window error. Writing $F_N = \int_0^{2N} e^{-st} f(t) dt$, the finite geometric sum gives

$$F_N = F(s)(1 - e^{-2Ns}), \quad \boxed{\frac{F(s) - F_N}{F(s)} = e^{-2Ns}.}$$

At $s = 0.2$, the requirement is $e^{-0.4N} \leq 10^{-3}$, so $N \geq \ln(1000)/0.4 \approx 17.27$. The least integer is $\boxed{N = 18}$, corresponding to a cutoff time of 36. Endpoint assignments do not affect the integrals; the graph uses the values specified in the question.



Question 10

For real s , use $F(s) = \int_0^\infty e^{-st} f(t) dt$ whenever this ordinary improper integral converges.

A continuous signal on $[0, \infty)$ satisfies $0 \leq f(t) \leq 1$, and its observed part is $f(t) = t$ on $[0, 1]$. Its future after one is unknown. Consider $s > 0$.

Tasks

1. Derive explicit lower and upper bounds for $F(s)$ using only this information.
2. Decide which bound is attained within the continuous signal class. Construct admissible signals approaching any unattained bound.
3. State the exact resulting range of possible values of $F(1)$, justifying that every value in that range is possible.
4. Find the width of the uncertainty interval for general $s > 0$, prove it decreases with s , and find the smallest positive integer s making the width at most 0.01.

Question 10 – Solution

Strategy. Positivity of the Laplace weight gives bounds; continuity at the last observed point decides whether an extremum is actually attained.

Step 1: Bound the unknown tail. The observed contribution is

$$L(s) = \int_0^1 t e^{-st} dt = \frac{1 - (1+s)e^{-s}}{s^2}.$$

The remaining integral lies between zero and $\int_1^\infty e^{-st} dt = e^{-s}/s$. Therefore

$$L(s) \leq F(s) \leq U(s), \quad \boxed{U(s) = \frac{1 - e^{-s}}{s^2}}.$$

All integrals converge absolutely because $0 \leq f \leq 1$ and $s > 0$.

Step 2: Resolve attainment. The upper bound is attained by $f(t) = \min\{t, 1\}$. The lower bound is not attained: continuity and $f(1) = 1$ force a strictly positive contribution on some interval immediately after one. For $\varepsilon > 0$, instead continue linearly from $(1, 1)$ to $(1 + \varepsilon, 0)$ and then stay zero. This continuous f_ε satisfies

$$0 < F_\varepsilon(s) - L(s) \leq \varepsilon e^{-s} \rightarrow 0.$$

Thus $L(s)$ is the infimum, while $U(s)$ is a maximum.

Step 3: Give the full range at one. Substitution gives the interval

$$\boxed{1 - 2/e < F(1) \leq 1 - 1/e}.$$

For any target strictly above the lower endpoint, choose ε small enough that $F_\varepsilon(1)$ is below it. A convex combination of f_ε and the upper-bound signal remains continuous, in $[0, 1]$, and agrees with the observed ramp. Splitting the defining integral shows its transform is the same convex combination. Choosing the weight appropriately reaches the target, proving no intermediate values are missing.

Step 4: Measure the uncertainty. The interval width is $W(s) = U(s) - L(s) = e^{-s}/s$, with

$$W'(s) = -e^{-s}(s+1)/s^2 < 0.$$

Since $W(3) = e^{-3}/3 \approx 0.01660$ and $W(4) = e^{-4}/4 \approx 0.00458$, the least positive integer meeting the tolerance is $\boxed{s = 4}$. Increasing s reduces sensitivity to the unknown late-time portion; it does not supply missing pointwise information about that future.

