

Question 1

Consider the initial-value problem on $\mathbb{R}$

$$y'' - 3y' + 2y = t^2, \quad y(0) = 0, \quad y'(0) = 0.$$

A student proposes the particular trial $y_p = At^2$ because the forcing contains only a quadratic term.

Tasks

1. Explain why the proposed trial is insufficient. Choose the smallest complete polynomial trial and justify its degree.
2. Determine all coefficients of that trial and verify its residual.
3. Find the full initial-value solution, checking both data.
4. Prove that the polynomial particular solution is unique among polynomials, even though particular solutions in general are not unique.

Question 1 – Solution

Strategy. Differentiation creates lower polynomial powers, so include them before matching every coefficient.

Step 1: Complete the trial. For At^2 , the residual is $2At^2 - 6At + 2A$, which cannot equal t^2 . Use $y_p = at^2 + bt + c$. The nonzero coefficient of y preserves the highest polynomial degree, so degree two is necessary and sufficient here.

Step 2: Match coefficients. Substitution yields

$$2at^2 + (2b - 6a)t + (2a - 3b + 2c) = t^2.$$

Thus $a = 1/2$, $b = 3/2$, $c = 7/4$, giving

$$y_p = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{7}{4}.$$

The displayed coefficient equations verify the residual identically.

Step 3: Fit the homogeneous correction. The characteristic roots are 1, 2, so $y = y_p + Ce^t + De^{2t}$. At zero,

$$C + D = -7/4, \quad C + 2D = -3/2.$$

Hence $C = -2$, $D = 1/4$, and

$$y = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{7}{4} - 2e^t + \frac{1}{4}e^{2t}.$$

Its value is $7/4 - 2 + 1/4 = 0$ and its slope is $3/2 - 2 + 1/2 = 0$. Adding homogeneous modes does not change the verified forcing.

Step 4: Prove polynomial uniqueness. The difference of two polynomial particular solutions would be a polynomial h with $h'' - 3h' + 2h = 0$. If h were nonzero with degree m and leading coefficient a_m , the leading term of this residual would be $2a_mt^m$, which cannot vanish. Thus $h = 0$. Nonpolynomial homogeneous additions still give infinitely many other particular solutions.

Question 2

On $\mathbb{R}$, solve

$$y'' - 4y' + 4y = e^{2t}(6t + 2), \quad y(0) = 1, \quad y'(0) = 0.$$

Three candidate trial forms are $e^{2t}(At + B)$, $te^{2t}(At + B)$ and $t^2e^{2t}(At + B)$.

Tasks

1. Find the homogeneous roots and explain the resonance multiplicity. Derive the identity obtained by writing $y = e^{2t}F(t)$.
2. Test the first two trial forms and explain exactly why neither can produce the forcing.
3. Use the third trial to find a particular solution and then solve the initial-value problem.
4. Explain why the trial omits the constant and linear powers of F , and why their omission loses no solution freedom. Verify the final equation and data.

Question 2 – Solution

Strategy. Shift the exponential factor out of the operator; the repeated characteristic root reveals the necessary power of t .

Step 1: Identify the repeated root. The characteristic polynomial is $(r - 2)^2$. Product differentiation gives

$$(e^{2t}F)'' - 4(e^{2t}F)' + 4e^{2t}F = e^{2t}F''.$$

Thus the transformed equation is $F'' = 6t + 2$.

Step 2: Reject incomplete shifts. The first trial has $F = At + B$, so $F'' = 0$. The second has $F = At^2 + Bt$, so $F'' = 2A$, which cannot supply the linear term $6t$. Multiplying the usual degree-one trial by only one power of t is still insufficient at a double root.

Step 3: Use the correct trial. With $F = At^3 + Bt^2$, coefficient matching gives $6At + 2B = 6t + 2$, hence $A = B = 1$. The complete solution is

$$y = e^{2t}(C + Dt + t^2 + t^3).$$

The initial value sets $C = 1$, and the slope at zero is $2C + D$, so $D = -2$. Therefore

$$y = e^{2t}(1 - 2t + t^2 + t^3).$$

Step 4: Account for every constant. Constant and linear terms in F have zero second derivative and correspond exactly to e^{2t} and te^{2t} , the homogeneous modes. Leaving them out of the particular trial avoids redundant coefficients; the constants C, D restore them in the general solution. For the final polynomial, $F'' = 2 + 6t$, so the equation holds. Also $F(0) = 1$ and $2F(0) + F'(0) = 2 - 2 = 0$, verifying both data.

Question 3

Consider $y'' + y = t \cos 2t$ on $\mathbb{R}$, with $y(0) = y'(0) = 0$. A proposed trial includes only the cosine term $(at + b) \cos 2t$.

Tasks

1. Explain why a polynomial sine component must also be included, and write the smallest complete trigonometric-polynomial trial.
2. Determine its coefficients by matching both sine and cosine terms.
3. Solve the zero-data initial-value problem and verify the data.
4. Explain why nonresonance with frequency 2 does not guarantee a bounded response to this forcing. Prove that the selected response is unbounded on $[0, \infty)$.

Question 3 – Solution

Strategy. Polynomial differentiation couples the sine and cosine components even when only one appears in the forcing.

Step 1: Complete the trial space. Use

$$y_p = (at + b) \cos 2t + (ct + d) \sin 2t.$$

Frequency 2 differs from the homogeneous frequency 1, so no extra power of t is needed for resonance. Nevertheless, differentiating the proposed cosine-only term twice produces $-4a \sin 2t$, which cannot be ignored.

Step 2: Match all terms. Applying $D^2 + 1$ to the complete trial gives

$$(-3at - 3b + 4c) \cos 2t + (-3ct - 3d - 4a) \sin 2t.$$

Thus $a = -1/3$, $c = 0$, $b = 0$, $d = 4/9$, and

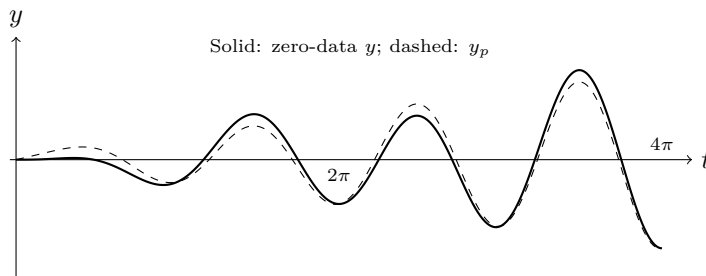
$$y_p = -\frac{1}{3}t \cos 2t + \frac{4}{9} \sin 2t.$$

Step 3: Fit the initial state. The particular data are $y_p(0) = 0$ and $y_p'(0) = -1/3 + 8/9 = 5/9$. The homogeneous correction is therefore $-(5/9) \sin t$, giving

$$y = -\frac{1}{3}t \cos 2t + \frac{4}{9} \sin 2t - \frac{5}{9} \sin t.$$

Its value and slope at zero vanish, and the correction leaves the verified forcing unchanged.

Step 4: Separate resonance from forcing growth. The forcing itself has a factor t and is unbounded. At $t_n = 2\pi n$, the solution is $y(t_n) = -t_n/3 \rightarrow -\infty$. At $s_n = \pi/2 + 2\pi n$, it is $y(s_n) = s_n/3 - 5/9 \rightarrow \infty$. Thus it is unbounded in both directions despite the absence of frequency resonance. Nonresonance alone is not a boundedness guarantee when the forcing amplitude grows.



Question 4

On $\mathbb{R}$, solve

$$y'' - y = (t + 1)e^t + 2 \cosh t, \quad y(0) = y'(0) = 0,$$

where $\cosh t = (e^t + e^{-t})/2$.

Tasks

1. Rewrite the forcing into distinct exponential families and identify every resonant component.
2. Choose one combined trial with no duplicate unknown coefficients. Explain why matching separate overlapping trials would introduce redundancy.
3. Determine a particular solution by coefficient matching.
4. Add the homogeneous correction, solve the initial-value problem, and verify both the equation and initial data.

Question 4 – Solution

Strategy. Combine overlapping forcing families before choosing the trial, then apply the resonance correction separately to each distinct exponential.

Step 1: Simplify the forcing. The right side becomes

$$(t+2)e^t + e^{-t}.$$

The characteristic roots of $y'' - y = 0$ are $1, -1$, both simple. Therefore each exponential family is resonant once.

Step 2: Choose a nonredundant trial. Use

$$y_p = e^t(at^2 + bt) + cte^{-t}.$$

A separate trial for the e^t inside $2 \cosh t$ would repeat the te^t term already in the first family's trial. Combining the forcing first avoids two coefficients multiplying the same function.

Step 3: Match coefficients. Product differentiation gives

$$(D^2 - 1)(e^t F) = e^t(F'' + 2F'), \quad (D^2 - 1)(e^{-t} G) = e^{-t}(G'' - 2G').$$

Thus the trial produces $e^t(4at + 2a + 2b) - 2ce^{-t}$. Matching gives $a = 1/4$, $b = 3/4$, $c = -1/2$, so

$$y_p = e^t(t^2/4 + 3t/4) - \frac{1}{2}te^{-t}.$$

Step 4: Fit and verify. The particular data are $(0, 1/4)$. With $y = y_p + Ce^t + De^{-t}$, the zero data require $C + D = 0$, $C - D = -1/4$, hence $C = -1/8$, $D = 1/8$. Therefore

$$y = e^t(t^2/4 + 3t/4 - 1/8) + e^{-t}(1/8 - t/2).$$

Its value is $-1/8 + 1/8 = 0$ and its slope is $1/4 - 1/8 - 1/8 = 0$. The two added exponentials are homogeneous, so the coefficient-matched forcing is preserved.

Question 5

For $t \in \mathbb{R}$, consider

$$y'' + 2y' + 5y = e^{-t} \cos 2t, \quad y(0) = y'(0) = 0.$$

Tasks

1. Find the homogeneous roots and choose a correct resonant trial. Derive a useful operator identity with $y = e^{-t}F$.
2. Determine the particular solution and the complete zero-data response.
3. Despite the resonance, prove that every solution tends to zero as $t \rightarrow \infty$.
4. Find a simple envelope for the zero-data response and prove the bound $|y(t)| \leq 1/(4e)$ for $t \geq 0$. Is equality actually attained by the response?

Question 5 – Solution

Strategy. Resonance adds a polynomial factor, but the sign of the exponential rate still controls long-time decay.

Step 1: Identify the resonance. The roots are $-1 \pm 2i$. Thus the exponential-trigonometric forcing matches a simple complex root pair, and a suitable trial is $te^{-t}(A \cos 2t + B \sin 2t)$. Differentiation gives

$$(e^{-t}F)'' + 2(e^{-t}F)' + 5e^{-t}F = e^{-t}(F'' + 4F).$$

Step 2: Match and fit. For $F = t(A \cos 2t + B \sin 2t)$, the transformed residual is $-4A \sin 2t + 4B \cos 2t$. Hence $A = 0$, $B = 1/4$. The particular solution and its first derivative vanish at zero, so

$$y_0 = \frac{1}{4}te^{-t} \sin 2t.$$

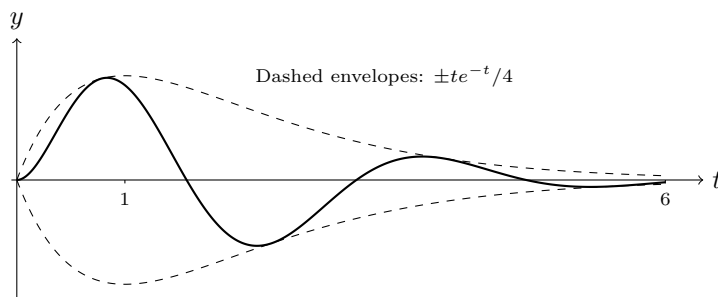
The full family is $e^{-t}[C \cos 2t + D \sin 2t + (t/4) \sin 2t]$. Zero initial data force $C = D = 0$.

Step 3: Prove decay for every solution. For fixed C, D and $t \geq 0$, its absolute value is at most $e^{-t}(|C| + |D| + t/4)$, which tends to zero. The resonant factor t does not overcome the negative exponential rate.

Step 4: Distinguish an envelope bound from an attained maximum. We have $|y_0(t)| \leq te^{-t}/4$. The envelope attains its unique maximum at $t = 1$, since its derivative is $e^{-t}(1 - t)/4$. Thus

$$|y_0(t)| \leq 1/(4e).$$

Equality would require both $t = 1$ and $|\sin 2t| = 1$. But $|\sin 2| < 1$, so the response never attains this bound. It is a valid envelope estimate, not the exact largest displacement.



Question 6

Let $F > 0$ and $\omega \geq 0$. Consider $y'' + 2y' + 5y = F \cos(\omega t)$ on $\mathbb{R}$. For $\omega > 0$, seek a particular response $y_p = A \cos \omega t + B \sin \omega t$ and define its amplitude $R = \sqrt{A^2 + B^2}$. At $\omega = 0$, define R as the absolute value of the constant particular response.

Tasks

1. Derive A, B for $\omega > 0$ and find the constant response at zero frequency.
2. Find the frequency maximizing R on $\omega \geq 0$ and the maximum amplitude.
3. At $\omega = \sqrt{5}$, describe the phase of the response relative to the forcing and compare its amplitude with the maximum.
4. For $\omega > 0$, prove that this is the unique $2\pi/\omega$ -periodic solution and that every other solution approaches it as $t \rightarrow \infty$.

Question 6 – Solution

Strategy. Solve the two coefficient equations, then minimize the amplitude denominator instead of differentiating a complicated square root.

Step 1: Match the harmonic components. The equations are

$$(5 - \omega^2)A + 2\omega B = F, \quad -2\omega A + (5 - \omega^2)B = 0.$$

Writing $\Delta = (5 - \omega^2)^2 + 4\omega^2 > 0$, we obtain

$$A = F(5 - \omega^2)/\Delta, \quad B = 2F\omega/\Delta.$$

At $\omega = 0$, use the constant particular solution $F/5$.

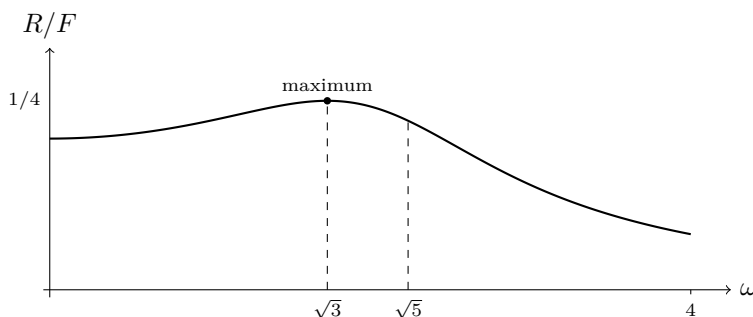
Step 2: Maximize the amplitude. For all $\omega \geq 0$, including this constant-response convention,

$$R = \frac{F}{\sqrt{\omega^4 - 6\omega^2 + 25}} = \frac{F}{\sqrt{(\omega^2 - 3)^2 + 16}}.$$

The denominator is minimized exactly at $\omega = \sqrt{3}$, giving $R_{\max} = F/4$.

Step 3: Locate the quarter-cycle phase lag. At $\omega = \sqrt{5}$, $A = 0$ and $B = F/(2\sqrt{5})$, so $y_p = [F/(2\sqrt{5})] \sin(\sqrt{5}t)$. This is a cosine delayed by phase $\pi/2$. Its amplitude $F/(2\sqrt{5})$ is smaller than $F/4$; the quarter-cycle phase frequency is not the amplitude-maximizing frequency.

Step 4: Prove the steady response is unique. Every difference from y_p is $e^{-t}(C \cos 2t + D \sin 2t)$ and tends to zero forward. If a second solution had the same period $T = 2\pi/\omega$, this difference would be periodic too. For any fixed t , its value equals its value at $t + nT$, whose limit is zero. Hence the difference is identically zero, proving uniqueness of the periodic response.



Question 7

For $\omega > 0$, let y_ω solve

$$y'' + y = \cos(\omega t), \quad y(0) = y'(0) = 0.$$

Tasks

1. Use undetermined coefficients to find y_ω for $\omega \neq 1$, including the homogeneous correction.
2. At $\omega = 1$, choose a resonant trial and solve the same initial-value problem directly.
3. For every fixed t , compute $\lim_{\omega \rightarrow 1} y_\omega(t)$. Explain why the divergent particular coefficient for $\omega \neq 1$ does not imply a divergent fixed-time limit of the full response.
4. Compare boundedness on $[0, \infty)$ in the resonant and nonresonant cases. Can the convergence as $\omega \rightarrow 1$ be uniform on that entire half-line?

Question 7 – Solution

Strategy. Fit the initial data before taking the frequency limit; the homogeneous correction cancels the divergent constant part.

Step 1: Solve off resonance. The trial $A \cos \omega t$ gives $A(1 - \omega^2) = 1$. Fitting zero data adds $-\cos t/(1 - \omega^2)$, so

$$y_\omega(t) = \frac{\cos(\omega t) - \cos t}{1 - \omega^2}, \quad \omega \neq 1.$$

Both the value and slope at zero vanish.

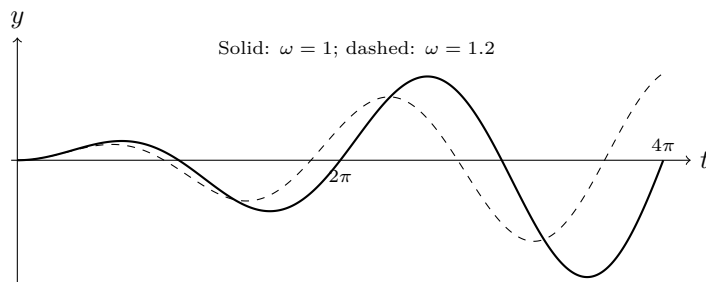
Step 2: Solve at resonance. Use $t(A \cos t + B \sin t)$. Applying $D^2 + 1$ gives $-2A \sin t + 2B \cos t$, hence $A = 0$, $B = 1/2$. Its data are already zero, so $y_1 = (t/2) \sin t$.

Step 3: Take the fixed-time limit. Differentiating numerator and denominator with respect to ω gives

$$\lim_{\omega \rightarrow 1} y_\omega(t) = \frac{-t \sin t}{-2} = \frac{1}{2} t \sin t.$$

This includes $t = 0$. The particular coefficient and the cosine correction separately diverge, but their full numerator vanishes at the same rate as the denominator. They must be combined before taking the limit.

Step 4: Compare long-time behavior. For fixed $\omega \neq 1$, $|y_\omega| \leq 2/|1 - \omega^2|$ on the whole half-line. The resonant response is unbounded, for example at $t = \pi/2 + 2\pi n$. Consequently, for each fixed nonresonant ω , the difference $y_\omega - y_1$ is unbounded on that half-line. Uniform convergence there is impossible, although the fixed-time limit exists. A finite-window plot cannot replace this distinction.



Question 8

Consider the standard finite polynomial–exponential–trigonometric version of undetermined coefficients for constant-coefficient equations. Then examine $y'' + y = 1/(1+t)$ on $I = (-1, \infty)$.

Tasks

1. Which forcings fit the standard finite trial families: $e^{-t}(t^2 + 1)$, $t \sin 3t$, $\cos^2 t$, e^{t^2} , and $1/(1+t)$? Explain any algebraic rewriting needed.
2. Test the rational trial $y_p = A/(1+t)$ for the stated equation and prove that no constant A works.
3. Prove that no finite sum $\sum_{k=1}^N a_k(1+t)^{-k}$ with real coefficients can be a particular solution.
4. Does the failure of these trials imply that an initial-value solution does not exist? State the existence and uniqueness conclusion for any initial point in I .

Question 8 – Solution

Strategy. The standard method depends on closure in a finite differentiation family; a plausible-looking trial outside that family need not close.

Step 1: Classify the forcings. The first two belong to the standard families. The third does too after $\cos^2 t = (1 + \cos 2t)/2$. The functions e^{t^2} and $1/(1+t)$ do not: successive derivatives introduce polynomial factors of ever-increasing degree in the first case and poles of ever-increasing order in the second. The usual finite constant-coefficient matching prescription does not apply to them.

Step 2: Test one inverse power. Put $x = 1 + t > 0$. For $y_p = A/x$,

$$y_p'' + y_p = A/x + 2A/x^3.$$

Equality to $1/x$ would require $A(x^2 + 2) = x^2$ for all $x > 0$. Matching x^2 forces $A = 1$, while matching the constant term forces $2A = 0$, a contradiction.

Step 3: Exclude any finite inverse-power sum. If such a sum were nonzero, choose its largest index N with $a_N \neq 0$. Its second derivative contains

$$a_N N(N+1)x^{-N-2}.$$

No other differentiated term, undifferentiated term or forcing term has that most singular power. Equivalently, multiply the proposed equation by x^{N+2} and let $x \downarrow 0$: the left side tends to $a_N N(N+1) \neq 0$, while the right side tends to zero. The zero sum also fails to give the nonzero forcing.

Step 4: Separate method failure from nonexistence. On I , the normalized coefficients and the forcing $1/(1+t)$ are continuous. For every $t_0 \in I$ and every finite value and slope at t_0 , the regular linear theorem gives exactly one solution on I . A failed finite ansatz is a limitation of the trial class, not a proof that the differential equation lacks solutions.

Question 9

An unknown forcing in $y'' - 3y' + 2y = Ae^{3t} + Bt$ has two real constants A, B . An exact response on $\mathbb{R}$ has measured data

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \quad y^{(3)}(0) = 0.$$

Tasks

1. Determine A, B using the equation and its derivative at zero before solving for y .
2. Choose a complete nonresonant trial and compute a particular solution for the identified forcing.
3. Fit the homogeneous correction and verify all four measured quantities.
4. Explain the distinct roles of the four measurements. Once A, B are fixed, are four freely prescribed initial quantities available for this second-order equation?

Question 9 – Solution

Strategy. Use acceleration and its derivative to identify the forcing, then use value and slope to select the trajectory.

Step 1: Identify the forcing parameters. At zero, the equation gives $1 = A$. Differentiating once gives $y^{(3)} - 3y'' + 2y' = 3Ae^{3t} + B$, so at zero $-3 = 3A + B$. Therefore

$$A = 1, \quad B = -6.$$

Step 2: Match a particular solution. Neither 3 nor 0 is a root of $(r-1)(r-2)$, so use $y_p = ce^{3t} + dt + k$. The residual is $2ce^{3t} + 2dt + (2k - 3d)$. Matching $e^{3t} - 6t$ gives

$$y_p = \frac{1}{2}e^{3t} - 3t - \frac{9}{2}.$$

Step 3: Fit and verify. Write $y = y_p + Ce^t + De^{2t}$. Since $y_p(0) = -4$ and $y'_p(0) = -3/2$, the zero data give $C + D = 4$, $C + 2D = 3/2$. Hence

$$y = \frac{1}{2}e^{3t} + \frac{13}{2}e^t - \frac{5}{2}e^{2t} - 3t - \frac{9}{2}.$$

At zero its value and slope are $1/2 + 13/2 - 5/2 - 9/2 = 0$ and $3/2 + 13/2 - 5 - 3 = 0$. Its second and third derivatives there are $9/2 + 13/2 - 10 = 1$ and $27/2 + 13/2 - 20 = 0$. The coefficient matching already verifies the forcing everywhere.

Step 4: Interpret the information count. Before calibration there are two unknown forcing parameters and two trajectory constants. The four measurements determine those four quantities here. Once the forcing is fixed, only the value and slope are freely assignable: the equation and its derivative determine the higher derivatives. Arbitrary additional acceleration measurements would be compatibility tests, not extra solution freedoms.

Question 10

For a real parameter c , consider on $\mathbb{R}$

$$y'' - y = e^{-t} + ce^{-2t}, \quad y(0) = y'(0) = 0.$$

Both forcing components tend to zero as $t \rightarrow \infty$.

Tasks

1. Choose resonant and nonresonant trials as appropriate and find the full zero-data solution in terms of c .
2. Find the unique value of c making this solution bounded on $[0, \infty)$.
3. For that tuned value, prove the sign of the solution for every $t > 0$ and find its limiting value. You may use the strict inequality $e^{-t} > 1 - t$ for $t > 0$.
4. If the tuned parameter is replaced by $c + \delta$ with $\delta \neq 0$, determine the leading long-time term and its sign. Explain why decaying forcing alone does not guarantee a decaying response.

Question 10 – Solution

Strategy. Determine which forcing combination cancels the growing homogeneous mode after the initial data are fitted.

Step 1: Match and fit. The root -1 is simple, while -2 is not a root. Use $y_p = ate^{-t} + be^{-2t}$. Since $(D^2 - 1)(te^{-t}) = -2e^{-t}$ and $(D^2 - 1)e^{-2t} = 3e^{-2t}$, we obtain $a = -1/2$, $b = c/3$. Fitting the zero data gives

$$y = \left(\frac{1}{4} + \frac{c}{6}\right)e^t + \left(-\frac{1}{4} - \frac{c}{2}\right)e^{-t} - \frac{1}{2}te^{-t} + \frac{c}{3}e^{-2t}.$$

The value is zero because the constant coefficients sum to zero; the slope is zero because the derivative contributions sum to zero.

Step 2: Cancel the unstable mode. Every term except the e^t term tends to zero. Thus boundedness holds exactly when $1/4 + c/6 = 0$, giving $c = -3/2$. The tuned response is

$$y_* = \frac{1}{2}e^{-t}(1 - t - e^{-t}).$$

Step 3: Prove sign and decay. For $t > 0$, the supplied strict exponential inequality gives $1 - t - e^{-t} < 0$, so $y_* < 0$. All its terms decay, hence $y_* \rightarrow 0$ from below. It has zero initial value and slope despite remaining strictly negative afterward.

Step 4: Examine imperfect cancellation. Replacing the tuned value by $-3/2 + \delta$ makes the growing-mode coefficient $\delta/6$. More explicitly, the change in solution is

$$\delta[e^t/6 - e^{-t}/2 + e^{-2t}/3].$$

Consequently $y(t)/e^t \rightarrow \delta/6$: the response tends to positive infinity for $\delta > 0$ and negative infinity for $\delta < 0$. The forcing still decays; the initial-data adjustment excites a growing homogeneous component unless the cancellation is exact.

